MAT 566: Characteristic Classes

Problem Set 5

Due by Tuesday, 4/15, in class

(if you have not passed the orals yet)

Please do two out of 12-A, 14-B, 14-E, (vi), and (vii).

Problem (vi): Suppose $f\colon M \longrightarrow M$ is a diffeomorphism. A fixed point x of f (i.e. $f(x)=x$) is called *nondegenerate* if the isomorphism $\mathrm{d}_x f\colon T_xM \longrightarrow T_xM$ has no eigenvalues equal to 1.

(a) Show that a nondegenerate fixed point is necessarily isolated (in the set of fixed points).

(b) Suppose M is compact and f is homotopic to the identity.

- (b-i) If M is oriented and the Euler characteristic $\chi(M)$ is non-zero, show that f has at least one fixed point. If in addition all fixed points are nondegenerate, show that their number is at least $|\chi(M)|$.
- (b-ii) If $\chi(M)$ is odd, show that f has at least one fixed point.

Hint: This is a sequel to Problem (v).

Problem (vii): A homogeneous cubic polynomial in four variables,

$$f(X_0, X_1, X_2, X_3) = \sum_{i+j+k+l=3} a_{ijkl} X_0^i X_1^j X_2^k X_3^l,$$

naturally corresponds to an element

$$\tilde{f} \in \big(\mathrm{Sym}^3((\mathbb{C}^4)\big)^* \approx \mathrm{Sym}^3\big((\mathbb{C}^4)^*\big) \equiv (\mathbb{C}^4)^* \otimes (\mathbb{C}^4)^* \otimes (\mathbb{C}^4)^* / \sim, \quad \alpha\otimes\beta\otimes\gamma \sim \beta\otimes\alpha\otimes\gamma \sim \alpha\otimes\gamma\otimes\beta.$$

This is because each X_i defines a linear map $X_i : \mathbb{C}^4 \longrightarrow \mathbb{C}$, i.e. the projection onto the ith coordinate. Since every fiber of the tautological 2-plane bundle $\gamma_2 \longrightarrow \mathrm{Gr}_2\mathbb{C}^4$ is a linear subspace of $\mathbb{C}^4$, by restriction f induces a section

$$s_f \in \Gamma\big(\mathrm{Gr}_2\mathbb{C}^4; \mathrm{Sym}^3(\gamma_2^*)\big).$$

Note that the complex rank of the bundle $\mathrm{Sym}^3(\gamma_2^*) \longrightarrow \mathrm{Gr}_2\mathbb{C}^4$ equals to the dimension of $\mathrm{Gr}_2\mathbb{C}^4$. Thus, if s_f is transverse to the zero set (as is the case for a generic f), the set $s_f^{-1}(0)$ is finite and its signed cardinality is

$$^{\pm}\big|s_f^{-1}(0)\big| = \big\langle e\big(\mathrm{Sym}^3(\gamma_2^*)\big), \mathrm{Gr}_2\mathbb{C}^4 \big\rangle \in \mathbb{Z}.$$

In fact, s_f is a holomorphic section of a holomorphic vector bundle and thus all its zeros are positive.

A cubic surface Y in $\mathbb{C}P^3$ is the zero set of a homogeneous cubic polynomial in four variables, i.e.

$$Y \equiv Y_f = \Big\{[X_0, X_1, X_2, X_3] \in \mathbb{C}P^3 : \sum_{i+j+k+l=3} a_{ijkl} X_0^i X_1^j X_2^k X_3^l = 0\Big\} = \big(f^{-1}(0) - 0\big)/\mathbb{C}^*.$$

A projective line ℓ in $\mathbb{C}P^3$ corresponds to a 2-plane P in $\mathbb{C}^4$, i.e. an element of $\mathrm{Gr}_2\mathbb{C}^4$. Such a line ℓ lies in Y_f if and only if $f|_P$ vanishes identically, i.e. $P \in s_f^{-1}(0)$. Thus, if f is generic, the number of lines in Y_f is finite and is given by the euler class of $\mathrm{Sym}^3(\gamma_2^*)$.

(a) Formulate and prove a splitting principle for Chern classes.

(b) Use it to determine the number of lines that lie on a generic cubic surface in $\mathbb{C}P^3$.

(c) Determine the number of lines that lie on a generic quintic threefold in $\mathbb{C}P^4$.