MAT 566: Differential Topology

Problem Set 4

Due on Monday, 03/25, by 5pm in Math 3-111

Problem (v): If X is a manifold and Y is a compact oriented submanifold, the image of the fundamental homology class $\mu_Y = [Y]$ under the inclusion map into X determines a homology class in X, which will be denoted by $[Y]_X$. Throughout this problem, all manifolds will be assumed to be smooth, compact, and oriented; all vector bundles are smooth. If Y is a submanifold of X, let

$$\mathcal{N}_X Y \equiv TX|_Y / TY$$

be the normal bundle of Y in X.

(a) Suppose Y^k is a submanifold of X^n. Let

$$u'' \equiv u'|_X \in H^{n-k}(X; \mathbb{Z})$$

be the dual class to Y; see the bottom of p118. Show that u'' *is* the Poincare dual of $[Y]_X$, i.e. the Poincare duality isomorphism

$$\cap[X] \colon H^{n-k}(X; \mathbb{Z}) \longrightarrow H_k(X; \mathbb{Z}), \qquad \alpha \longrightarrow \alpha \cap [X],$$

takes u'' to $[Y]_X$, at least up to a sign in odd dimensions depending on conventions (see 11-C). Conclude that

$$e(\mathcal{N}_X Y) = \mathrm{PD}_X([Y]_X)\big|_Y \in H^{n-k}(Y; \mathbb{Z}).$$

(b) Suppose $V \longrightarrow X^n$ is an orientable vector bundle of rank k and $s \colon X \longrightarrow V$ is a smooth section which is transverse to the zero set. Show that $s^{-1}(0)$ is a smooth compact orientable submanifold of X and

$$\big[s^{-1}(0)\big]_X = \mathrm{PD}_X\big(e(V)\big) \in H_{n-k}(X; \mathbb{Z}),$$

provided $s^{-1}(0)$ is suitably oriented (describe how). If $k = n$, $s^{-1}(0)$ is a finite set and the image of $\big[s^{-1}(0)\big]_X$ under the natural homomorphism

$$H_0(X; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is the number of points in $s^{-1}(0)$ *counted with sign*. How is the sign of each point of $s^{-1}(0)$ determined?

(c) Suppose Y_1^k, Y_2^m are compact oriented submanifolds of X^n that intersect transversally. Show that $Y_1 \cap Y_2$ is a smooth compact orientable submanifold of X and

$$[Y_1 \cap Y_2]_X = \mathrm{PD}_X\big(\mathrm{PD}_X([Y_1]_X) \cup \mathrm{PD}_X([Y_2]_X)\big) \in H_{k+m-n}(X; \mathbb{Z}),$$

provided $Y_1 \cap Y_2$ is suitably oriented (describe how).