A Proof and a Modern "Proof"

Lemma. Every vector bundle E over a paracompact topological space X has a countable locally finite trivializing open cover.

Modern proof. Since X is paracompact, there is a locally finite trivializing open cover. Combine sets in this cover that are disjoint to form countably many sets. The details are straightforward, but very messy.

Milnor-Stasheff, p66. Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be a locally finite trivializing open cover. Since X is normal, there exists an open cover $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ such that $\overline{V_\alpha}\subset U_\alpha$ for every $\alpha\in\mathcal{A}$. Let $f_\alpha\colon X\longrightarrow\mathbb{R}$ be a continuous function such that $f_\alpha|_{V_\alpha}=1$ and $f_\alpha|_{X-U_\alpha}=0$. For each subset $S\subset\mathcal{A}$, define

$$U_S=\big\{x\in X:\ f_\alpha(x)>f_\beta(x)\ \forall\ \alpha\in S,\ \beta\in\mathcal{A}-S\big\}.$$

For each $k\in\mathbb{Z}^+$, let

$$U_k=\bigcup_{S\subset\mathcal{A},|S|=k}U_S\,.$$

Since $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a locally finite cover, so is $\{U_k\}_{k\in\mathbb{Z}^+}$. Since $U_{S_1}\cap U_{S_2}=\emptyset$ unless either $S_1\subset S_2$ or $S_2\subset S_1$, each U_k is a disjoint union of the sets U_S. If $\alpha\in\mathcal{A}$, then $U_S\subset V_\alpha\subset U_\alpha$. Thus, the restrictions $E|_{U_S}$ and $E|_{U_k}$ are trivial.