MAT 566: Differential Topology
Spring 2018

Presentation 3: Atiyah-Bott Equivariant Localization Theorem

Suppose M is a smooth compact oriented manifold of dimension n and and $V \longrightarrow M$ is an oriented vector bundle of rank n. Then, the number

$$\int_M e(V) \equiv \langle e(V), M \rangle \equiv \langle e(V), [M] \rangle$$

is an integer. In many cases, this number contains some geometric information. It is not always easy to compute though.

However, if M admits a torus action that lifts to V, this computation may become greatly simplified. The number $\langle e(V), M \rangle$ then equals the sum of certain rational functions over the fixed loci of the group action on M (that miraculously add up to the constant function of value $\langle e(V), M \rangle$). This sum is called a localization formula, as the entire computation is localized to the fixed loci. In some cases, the fixed loci may be points; in general, they will be smooth manifolds of possibly different dimensions.

You will need to do the following:

(1) Explain thoroughly what equivariant cohomology is and what the meaning of the previous paragraph is. It is sufficient to consider the torus case, so that the existence of the classifying space is not an issue.

(2) State and prove the Atiyah-Bott Equivariant Localization Theorem.

(3) Give a number of examples. These would mostly have to involve projective spaces and Grassmannians, since they have torus actions inherited from $\mathbb{C}^n$. If the group action has only isolated fixed points, what is their number? Give applications of this.

This is all described in my notes on mirror symmetry available on the course website. Chapter 1 of these notes details Atiyah-Bott's *The Moment Map and Equivariant Cohomology*, Sections 1-3. There is also some discussion of this in Chapter 4 of the book Mirror Symmetry by Kentaro Hori, et. al. The proof is a beautiful combination of a little algebraic topology (Thom Isomorphism) and commutative algebra. There is a separate proof due to Berline-Vergne, which is in French and more complicated.

You should meet with me before your presentation.