

Atlanta

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**Given four lines in space,  
how many lines meet all four?:  
The geometry, topology, and  
combinatorics  
of the Grassmannian**

**Ravi Vakil, Stanford University**

<http://math.stanford.edu/~vakil>

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## Behind this question:

- geometry
- topology
- representation theory
- physics (Gromov-Witten invariants; quantum cohomology)
- combinatorics
- Schubert calculus

**My goal today:** Introduce you to the **Grassmannian**, and show you how to think about it

## Setting conventions

Given coefficients of a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

how many zeros does it have?

**(1)** work over the complex numbers (even if you were interested in the real numbers); I'll still use my "real" intuition

**(2a)** count roots with multiplicity

**(2b)** look for solutions in  $\mathbb{CP}^1$

or **(2)** let the coefficients be chosen "generally" or "randomly"

## **Answering the question**

**Solution 1: the geometry of the quadric surface in three-space**

**Answering the question**

**Solution 2: virtual reality**

**Philosophy:** Why would you expect a finite number?

Answer: the “parameter space” of lines is 4-dimensional, and each “condition” costs one dimension. (We are talking about complex dimensions.)

How do you picture this four dimensional space?

Before defining the Grassmannian properly, we'll need to define **projective space**.

Three-space  $\mathbb{C}^3$  has a “nice compactification” called  $\mathbb{C}\mathbb{P}^3$ , by adding points “at infinity”.

The points of *complex projective 3-space*  $\mathbb{C}\mathbb{P}^3$  correspond to one-dimensional subspaces of  $\mathbb{C}^4$ ;  $[w; x; y; z]$  means the subspace generated by  $(w, x, y, z)$  ( $w, x, y, z$  not all 0!). Where is  $\mathbb{C}^3$  in this? Place a “screen” at  $(1, x, y, z)$ . Most lines meet this screen, at one point.

From the point of view of  $\mathbb{C}^3$ , points at  $\infty$  look “different”. From  $\mathbb{C}\mathbb{P}^3$ , they don't; the difference is an artifact of the choice of screen.

## The Grassmannian!

$G(2, 4)$  is the set of 2-dimensional subspaces of 4-space. Most of them meet our screen in a line, so this is a compactification of the space of lines in  $\mathbb{C}^3$ . For this reason, it is also written as  $\mathbb{G}(1, 3)$ , the set of  $\mathbb{C}\mathbb{P}^1$ 's in  $\mathbb{C}\mathbb{P}^3$ .

We have added some “lines at infinity”.

More generally,  $\mathbb{G}(k, n) = G(k + 1, n + 1)$ . Note that  $\mathbb{G}(0, n) = G(1, n + 1) = \mathbb{C}\mathbb{P}^n$ .



## Facts about the Grassmannian

We have already described  $\mathbb{G}(1, 3)$  as a manifold!

It is incredibly symmetric. Any two lines in space look the same.

Precisely:  $GL(4, \mathbb{C})$  acts transitively on the set of projective lines. So  $G(2, 4) = GL(4, \mathbb{C})/B$ , where  $B$  is the subgroup fixing one of the two-planes, say “stabilizer of  $w = x = 0$ .” Thus

$$G(2, 4) = GL(4, \mathbb{C}) / \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

It is compact.

## Back to the problem...

What is the (complex) codimension of the set of those lines meeting line  $L_1$ ? Answer: 1.

Hence the (complex) codimension of the set of those lines meeting  $L_1, L_2, L_3, L_4$  is 4, i.e. we expect a finite number of *points* of Grassmannian, i.e. we expect a finite number of *lines* meeting 4 general lines. *This is a question about the topology of the Grassmannian.*

More general type of question: How many  $\mathbb{C}P^k$ 's in  $\mathbb{C}P^n$ 's meeting various subspaces of various dimensions? **Fact:** If we can answer this question, we will have determined the full cohomology ring of the Grassmannian  $\mathbb{G}(k, n)$ .

## Abstract our solution to get something more general

Let's abstract our degeneration solution to the original solution. Instead of asking "how many lines meet  $L_1, L_2, L_3, L_4$ ," let's just ask about the set of lines meeting  $L_1$  and  $L_2$  (a two-dimensional subset of the Grassmannian).

Same argument: this can be deformed into the union of 2 things: the set of lines through a point, and the set of lines in a plane.

**Fact:** The set of lines meeting  $L_1$  and  $L_2$  can be deformed into the union of 2 things: the set of lines through a point, and the set of lines in a plane.

We can use this to answer various sorts of questions.

How many lines meet 2 lines and a point?

How many lines meet 2 lines and lie in a plane?

How many lines meet 4 general lines (the original question)?

## Punchline:

You can use this degeneration recipe to answer any question of this sort (how many  $k$ -planes in  $n$ -space meet various things in various ways); in other words, to determine the cohomology ring of the Grassmannian.

Even better: if you understand our one example (two lines in 3-space) well enough, you can solve all of these problems, in every dimension. (All technical complexities already arise here.)

## Interpretations of the magic ring

1. “universal cohomology ring of all Grassmannians” (any cohomology ring of any Grassmannian is a quotient of it)

2. ring of symmetric functions

3. “ring of partitions”

4. representation rings

⋮

## Final punchline

Multiplication in the magic ring is known as a “Littlewood-Richardson rule”. You can understand multiplication in the magic ring by knowing how to answer these questions in the Grassmannian. And you can answer these questions in the Grassmannian by knowing how to manipulate these “virtual reality” pictures in your head. And if you know how to solve this one problem (well enough), you can solve them all.

## Conclusion

The Grassmannian  $G(k, n)$ , parametrizing  $k$ -planes in  $n$ -space, naturally arises as an important tool in many fields. Questions like the one in the title are really questions about the geometry (and topology) of the Grassmannian. Although in some sense the Grassmannian is just about linear algebra, it has an incredibly rich and subtle structure.

I hope I gave you some insight as to how the Grassmannian works, how to do calculations on it using degenerations, and some inkling of how it connects to other parts of mathematics, via this magic ring.

Thank you!