Notes on Moment Maps

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Contents

Preface

in preparation

The argument in the present notes is a more detailed version of [1, §2], except for (D_k) and the part of (A_k) beyond ($\mathrm{A}_k^\star$); see Remark 1.1 below.

1 Introduction and Overview

For a smooth manifold X, we denote by $\mathrm{Diff}(X)$ the group of diffeomorphisms of X (with the product given by the composition of functions). For a symplectic manifold (X,ω), we denote by $\mathrm{Symp}(X,\omega)$ the group of symplectomorphisms of (X,ω), i.e. diffeomorphisms ψ of X such that $\psi^*\omega=\omega$. A smooth action of a Lie group G on a smooth manifold X (resp. symplectic manifold (X,ω)) is a group homomorphism

$$\psi\colon G\longrightarrow \mathrm{Diff}(X) \quad \big(\text{resp. } G\longrightarrow \mathrm{Symp}(X,\omega)\big), \qquad u\longrightarrow \psi_u\,, \tag{1.1}$$

such that the map

$$\Psi\colon G\times X\longrightarrow X, \qquad \Psi(u,x)=\psi_u(x),$$

is smooth. For each $v\in T_{\mathbb{1}}G$,

$$\zeta_v\equiv \mathrm{d}_{\mathbb{1}}\psi(v)\in\Gamma(X;TX), \qquad \zeta_v(x)=\mathrm{d}_{(\mathbb{1},x)}\Psi(v,0)\in T_xX \quad \forall\, x\in X, \tag{1.2}$$

is then a well-defined smooth vector field on X. For any action ψ as in (1.1), let

$$X^\psi\equiv\big\{x\in X\colon \psi_u(x)=x\ \forall\, u\in G\big\}$$

denote its fixed locus. If in addition $x\in X$, let

$$G_x(\psi)\equiv\big\{u\in G\colon \psi_u(x)=x\big\} \tag{1.3}$$

denote the stabilizer of x in G.

An action ψ in (1.1) is called effective if the group homomorphism ψ is injective. It is called irreducible if the associated vector space homomorphism

$$\mathrm{d}_{\mathbb{1}}\psi\colon T_{\mathbb{1}}G\longrightarrow\Gamma(X;TX), \qquad v\longrightarrow\zeta_v\,,$$

is injective; otherwise, it is called reducible. An effective action is irreducible, but an irreducible action may have a discrete nontrivial kernel and thus not be effective.

Let G be a Lie group. A moment map for a smooth action of G on a symplectic manifold (X,ω) is a smooth map

$$\mu\colon X\longrightarrow T_{\mathbb{1}}^*G \qquad \text{s.t.} \qquad \begin{aligned} -\mathrm{d}\big(\{\mu(\cdot)\}(v)\big)&=\iota_{\zeta_v}\omega\equiv\omega(\zeta_v,\cdot) && \forall\, v\in T_{\mathbb{1}}G,\\ \mu\big(\psi_u(x)\big)&=\mathrm{Ad}^*_{u^{-1}}\big(\mu(x)\big) && \forall\, x\in X,\, u\in G, \end{aligned} \tag{1.4}$$

where $\mathrm{Ad}^*_{u^{-1}}: T_{\mathbb{1}}^*G\longrightarrow T_{\mathbb{1}}^*G$ is the dual of the adjoint action $\mathrm{Ad}_{u^{-1}}$ of G on $T_{\mathbb{1}}G$; see [30, Sections 3.46]. On the left-hand side of the first equation in (1.4), $\{\mu(\cdot)\}(v)$ denotes the smooth function on X given by

$$\{\mu(\cdot)\}(v)\colon X\longrightarrow\mathbb{R}, \qquad \big\{\{\mu(\cdot)\}(v)\big\}(x)=\{\mu(x)\}(v).$$

On the right-hand side of this equation, $\omega(\zeta_v,\cdot)$ denotes the 1-form on X given by

$$\omega(\zeta_v,\cdot)\colon TX\longrightarrow\mathbb{R},\qquad \big\{\omega(\zeta_v,\cdot)\big\}(w)=\omega\big(\zeta_v(x),w\big)\quad\forall\,x\in X,\,w\in T_xX.$$

If G is abelian, the second equation in (1.4) is equivalent to μ being G-invariant. A smooth G-action on (X,ω) that admits a moment map is called Hamiltonian. In such a case, μ is determined up to an additive constant fixed by the Ad^*-action of G on $T^*_{\mathbb{1}}G$. If G is connected, Exercise 3.11(a) implies that a smooth action of G on X that admits a smooth map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ satisfying the first condition in (1.4) is in fact an action on (X,ω). By Exercise 3.11(c), such a map is G-invariant if G is connected, abelian, and either $G\approx\mathbb{R}$ or either G or X is compact. We will call a tuple (X,ω,ψ,μ) a Hamiltonian G-manifold if (X,ω) is a symplectic manifold, ψ is a smooth G-action on (X,ω), and μ is a moment map for this action.

A closed manifold is a compact manifold without boundary. We call a closed subset Z of a smooth manifold X a closed submanifold if every topological component, i.e. a maximal connected subset, of Z is open in Z and is a smooth manifold without boundary smoothly embedded into X. In other words, every topological component of Z has an open neighborhood in X disjoint from the rest of Z and is a submanifold of X in the usual sense, but the dimensions of these submanifolds may not be the same. In such a case, the components of Z are also its path components; as usual, we denote the set of these components by $\pi_0(Z)$. We will call a Hamiltonian G-manifold (X,ω,ψ,μ) closed (resp. connected) if the manifold X is closed (resp. connected).

1.1 Convexity Theorem

The convex hull of a subset S of a vector space V is the subset

$$\mathrm{CH}(S)\equiv\bigg\{\sum_{i=1}^{m}r_is_i\colon m\in\mathbb{Z}^+,\ s_1,\ldots,s_m\in S,\ r_1,\ldots,r_m\in\mathbb{R}^{\ge0},\ \sum_{i=1}^{m}r_i=1\bigg\}\subset V.$$

If $S\subset V$ is a finite subset of V, then $\mathrm{CH}(S)$ is called a polytope. In such a case, we denote by $\mathrm{Ver}(S)$ the set of vertices of $\mathrm{CH}(S)$, i.e. the minimal subset of S so that $\mathrm{CH}(\mathrm{Ver}(S))=\mathrm{CH}(S)$. Since this subset of the polytope $P\equiv\mathrm{CH}(S)$ is determined by P itself, we will also denote it by $\mathrm{Ver}(P)$. The dimension of a polytope P is the dimension of the minimal affine subspace of V containing P. A (closed) face of a polytope P is the intersection P with the hyperplane $L^{-1}(c)$ for some nonzero linear functional $L\colon V\longrightarrow\mathbb{R}$ and $c\in\mathbb{R}$ such that

$$L(v)\ge c\qquad\forall\,v\in P.$$

Such a face is the convex hull of $\mathrm{Ver}(P)\cap L^{-1}(c)$ and thus is a polytope in itself. The interior P° of a polytope P is the complement of the proper faces of P in P. An open face of P is the interior of a face of P. An edge (resp. facet) of a polytope P is a face of P of dimension 1 (resp. codimension 1). We denote by $\mathrm{Edg}(P)$ the set of edges of a polytope P and by $\mathrm{Edg}_v(P)\subset\mathrm{Edg}(P)$ for each $v\in\mathrm{Ver}(P)$ the subset of the edges containing v. For $e\in\mathrm{Edg}(P)$, we call $v_e\in V$ an edge vector for e if

$$P\cap\big\{v+tv_e\colon t\in\mathbb{R}\big\}=e\subset V$$

for a vertex $v\in\mathrm{Ver}(e)$. A full tuple of edge vectors for a polytope P is an element $(v_e)_{e\in\mathrm{Edg}(P)}$ of $V^{\mathrm{Edg}(P)}$ so that v_e is an edge vector for each $e\in\mathrm{Edg}_v(P)$.

Following [1], we call a smooth $\mathbb{R}^k$-action ψ on a smooth manifold X almost periodic if there exists a smooth action ψ' of a torus $\mathbb{T}$, i.e. a compact connected abelian Lie group, on X and a group homomorphism

$$\rho\colon \mathbb{R}^k \longrightarrow \mathbb{T} \qquad \text{s.t.} \quad \psi = \psi'\circ\rho\colon \mathbb{R}^k \longrightarrow \mathrm{Diff}(X)\,. \tag{1.5}$$

If the image of ρ is dense in $\mathbb{T}$ (which can be achieved by replacing $\mathbb{T}$ with the closure of $\rho(\mathbb{R}^k)$), then $X^\psi = X^{\psi'}$. If in addition ψ preserves a symplectic form ω on X, then so does ψ'. This implies the existence of an $\mathbb{R}^k$-invariant Riemannian metric on X and thus of an $\mathbb{R}^k$-invariant ω-compatible almost complex structure on X in Theorem 1 below; see Exercises 3.2 and 3.12.

Theorem 1. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a closed connected Hamiltonian $\mathbb{R}^k$-manifold, and the $\mathbb{R}^k$-action ψ is almost periodic.*

(A_k) The subset $\mu^{-1}(\alpha)\subset X$ is connected for every $\alpha\in T_0^*\mathbb{R}^k$.

(B_k) The image $\mu(X)\subset T_0^*\mathbb{R}^k$ of X is a convex subset.

(C_k) The ψ-fixed locus X^ψ is a closed symplectic submanifold of (X,ω), $\mu|_Y$ is constant for each $Y\in\pi_0(X^\psi)$, and $\mu(X)$ is the convex hull of the finite subset $\mu(X^\psi)\subset T_0^*\mathbb{R}^k$ with at most $(\dim X)/2$ edges at each vertex.

(D_k) The map $\mu\colon X\longrightarrow\mu(X)$ is open.

(E_k) The components of a full tuple of edge vectors for the polytope $\mu(X)$ at any given vertex of $\mu(X)$ span $T_0^*\mathbb{R}^k$ if and only if the action ψ is irreducible.

(F_k) If the action ψ is irreducible, then the subset $\mathrm{Crit}(\mu)$ of points $x\in X$ so that $\mathrm{d}_x\mu$ is not surjective is a finite union of (not necessarily disjoint) closed symplectic proper submanifolds of (X,ω), and the image of each such submanifold under μ is contained in a hyperplane of $T_0^*\mathbb{R}^k$.

The first claim in (C_k) is straightforward and holds for any smooth Lie group action on a symplectic manifold (X,ω) preserving a Riemannian metric on X; see Remark 3.4 and the proof of Proposition 3.14(1). The second claim in (C_k) follows from the first and the observation that

$$X^\psi = \big\{x\in X\colon \mathrm{d}_x\mu=0\big\};$$

this identity is a consequence of (1.4) and Proposition 3.8(1). The first claim in (F_k) is a straightforward consequence of the equivariant splitting (3.5) of $TX|_Y$ for each $Y\in\pi_0(X^\psi)$.

The interesting parts of Theorem 1 are (A_k), (B_k), (D_k), and the remaining claims in (C_k) and (F_k). The fundamental reason behind these statements is the local form of the moment map μ provided by the first part of Corollary 3.28. This part of Corollary 3.28 is in a sense a Hamiltonian version of the Darboux Theorem. It ensures that (A_k), (B_k), and (D_k) hold locally and establishes (E_k), the last claim in (F_k), and Theorem 2(1) on page 7. We use the global, Morse-Bott theory statement of Proposition 4.8 and the first, local statement of Proposition 4.5 to obtain

($\mathrm{A}_k^\star$) $\mu^{-1}(\alpha)\subset X$ is connected for every regular value $\alpha\in T_0^*\mathbb{R}^k$ of μ, via an induction of the dimension of the torus $\mathbb{T}$, as in [1, 21], and to deduce (D_k) from its local version. The remaining part of (A_k) follows from ($\mathrm{A}_k^\star$), (D_k), and (F_k) via Exercise 4.10. Claim (B_k) of Theorem 1 is obvious for $k=1$, follows readily from (A_k) for $k\ge 2$, and leads to the last claim in (C_k).

By the last claim in (C_k), $\mu(X)\subset T_0^*\mathbb{R}^k$ is a polytope, called a moment polytope for the Hamiltonian action ψ on (X,ω). It is well-defined up to translation. Since a torus $\mathbb{T}$ is the quotient of a finite-dimensional vector space by a lattice, Theorem 1 immediately implies its statement with $\mathbb{R}^k$ replaced by $\mathbb{T}$. In such a case, the moment polytope $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ has additional properties; see Theorems 2 and 3 on pages 7 and 8, respectively. Figure 1 on page 14 shows moment polytopes for the two torus actions on $(\mathbb{C}P^2,\omega_{\mathrm{FS};2})$ of Exercise 2.7.

Remark 1.1. The three parts of the statement of [1, Theorem 1] are (A_k), (B_k), and (C_k) in Theorem 1, while (F_k) is stated at the beginning of the proof of ($\mathrm{A}_k^\star$) and is justified at the end of Section 2 in [1]. The proof in [1] contains (at least) two gaps, both at the top of page 6.

(G1) It is claimed that ($\mathrm{A}_k^\star$) implies (A_k) *by continuity*. This is indeed the case for $k=1$; see Lemma 4.9. However, this may not be the case for $k\ge 2$, even if the regular values are dense in the image (as is the case at the top of page 6). As an example, pinch the points $+1,-1$ of the unit circle $S^1\subset\mathbb{R}^2$ to the origin (so that the preimage of $0\in\mathbb{R}^2$ consists of two points, while the preimages of all other points contain at most one point). Replacing the circle with a 2-torus, we can ensure that the set of regular values is dense in the image. Statement (D_k) is the key property of μ needed for the *by continuity* claim in [1], but it does not even appear in [1], and neither does the local description of μ of Corollary 3.28 needed for this statement. Both play prominent roles in other approaches to the convexity theorem; see Section 1.4.

(G2) It is implicitly assumed that if $(c_1,\ldots,c_k)\in\mathbb{R}^k$ is a regular value of an $\mathbb{R}^k$-valued smooth function $(f_1,\ldots,f_k)$ on a smooth manifold X, then $(c_1,\ldots,c_{k-1})\in\mathbb{R}^{k-1}$ is a regular value of $(f_1,\ldots,f_{k-1})$. This need not be the case, including in the setting on page 6 in [1].

Remark 1.2. The statements (B_k) and (C_k) in Theorem 1, with $\mathbb{R}^k$ replaced by a torus $\mathbb{T}$, form [2, Theorem IV.4.3] and [21, Theorem 5.5.1]; [8, Theorem 27.1] includes (A_k) as well. The arguments in [2, 8, 21] generally follow [1], with [2] stating (A_k) as part of the proof and replicating the two gaps of Remark 1.1 in the middle of page 115 almost *verbatim*. In [21], only ($\mathrm{A}_k^\star$) is stated as part of the argument, making the issue (G1) in Remark 1.1 extraneous, while the gap (G2) is resolved. In order to deduce (B_k) from ($\mathrm{A}_k^\star$), it is claimed in the last full sentence on page 239 in [21] that any two points in X (M in [21]) with the same value of a Hamiltonian ($A^{\mathrm{tr}}\mu$ in [21]) can be approximated by two points in the preimage of a regular value of the Hamiltonian (the same regular value for both points). However, this is precisely what is needed to deduce (A_k) from ($\mathrm{A}_k^\star$), as indicated by the proof of Lemma 4.9. This property is implied by the Hamiltonian being an open map onto its image, i.e. (D_k), as suggested by Exercise 4.10, but neither the openness of the Hamiltonian nor its local description as in the first part of Corollary 3.28 is ever brought up in [21]. Thus, the attempt in [21] to bypass (G1) while establishing (B_k) contains fundamentally the same gap. In [8], the proof of (A_k) is relegated to Homework 21, which deals only with ($\mathrm{A}_k^\star$), while resolving (G2) as in [21]; neither the openness of the Hamiltonian nor its local description is mentioned in [8] either. While (F_k) is also stated at the beginning of the proof of ($\mathrm{A}_k^\star$) in [21] with a note that it is established later in the proof, (F_k) is never addressed in [21]. The equivariant splitting (3.5) of $TX|_Y$ for each $Y\in\pi_0(X^\psi)$ needed to establish even the first claim in (F_k) does not appear anywhere in [21].

1.2 Kähler case

For a Lie group G and $v\in T_{\mathbb{1}}G$, let $\mathrm{e}^v\in G$ be the exponential of v; see [30, §3.30]. Define

$$(T_{\mathbb{1}}G)_{\mathbb{Z}}=\big\{v\in T_{\mathbb{1}}G\colon \mathrm{e}^v=\{\mathbb{1}\}\big\}\quad\text{and}\quad (T_{\mathbb{1}}^*G)_{\mathbb{Z}}=\big\{\alpha\in T_{\mathbb{1}}^*G\colon \alpha(v)\in\mathbb{Z}\ \forall\,v\in(T_{\mathbb{1}}G)_{\mathbb{Z}}\big\}.$$

If $G'\subset G$ is a Lie subgroup, then

$$(T_{\mathbb{1}}G')_{\mathbb{Z}}=(T_{\mathbb{1}}G)_{\mathbb{Z}}\cap T_{\mathbb{1}}G'\subset T_{\mathbb{1}}G$$

and the image of $(T_{\mathbb{1}}^*G)_{\mathbb{Z}}$ under the restriction homomorphism $T_{\mathbb{1}}^*G)_{\mathbb{Z}}\longrightarrow T_{\mathbb{1}}^*G$ is contained in $(T_{\mathbb{1}}^*G')_{\mathbb{Z}}$. If $\mathbb{T}$ is a torus, then $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ and $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}^*\mathbb{T}$ are lattices, i.e. the homomorphisms

$$(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}\longrightarrow T_{\mathbb{1}}\mathbb{T},\quad v\otimes c\longrightarrow cv,\qquad\text{and}\qquad (T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}\longrightarrow T_{\mathbb{1}}^*\mathbb{T},\quad \alpha\otimes c\longrightarrow c\alpha,$$

are isomorphisms of real vector spaces, and the map

$$T_{\mathbb{1}}\mathbb{T}/(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\longrightarrow\mathbb{T},\qquad [v]\longrightarrow\mathrm{e}^v,$$

is an isomorphism of Lie groups. We call $\alpha\in T_{\mathbb{1}}^*\mathbb{T}$ integral if $\alpha\in(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and a line segment in $T_{\mathbb{1}}^*\mathbb{T}$ rational if it is parallel to an integral element of $T_{\mathbb{1}}^*\mathbb{T}$.

Exercise 1.3. Let $\mathbb{T}$ be a torus.

(a) Suppose that $V\subset T_{\mathbb{1}}\mathbb{T}$ is a linear subspace. Show that

$$\mathrm{e}^V\equiv\big\{\mathrm{e}^v\colon v\in V\big\}\subset\mathbb{T}$$

is a subtorus if and only if V is the $\mathbb{R}$-span of a finite subset of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$.

(b) Suppose $\mathbb{T}'\subset\mathbb{T}$ is a subtorus. Show that there exists a subtorus $\mathbb{T}'^c\subset\mathbb{T}$ so that the Lie group homomorphism

$$\mathbb{T}'\times\mathbb{T}'^c\longrightarrow\mathbb{T},\qquad (u',u'^c)\longrightarrow u'u'^c,$$

is an isomorphism. Conclude that the restriction homomorphism $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}\longrightarrow(T_{\mathbb{1}}^*\mathbb{T}')_{\mathbb{Z}}$ is surjective.

The complexification of a torus $\mathbb{T}$ is the complex Lie group

$$\mathbb{T}_{\mathbb{C}}\equiv\big(T_{\mathbb{1}}\mathbb{T}\otimes_{\mathbb{R}}\mathbb{C}\big)/(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\otimes_{\mathbb{R}}\{1\}$$

with $T_{\mathbb{1}}\mathbb{T}_{\mathbb{C}}=T_{\mathbb{1}}\mathbb{T}\otimes_{\mathbb{R}}\mathbb{C}$. An almost complex structure on a smooth manifold X is an endomorphism J of the real vector bundle $TX\longrightarrow X$ covering the identity on X so that $J^2=-\mathrm{Id}_{TX}$. A complexification of a smooth $\mathbb{T}$-action ψ on a symplectic manifold (X,ω) is a smooth $\mathbb{T}_{\mathbb{C}}$-action on X,

$$\begin{aligned}&\psi_{\mathbb{C}}\colon\mathbb{T}_{\mathbb{C}}\longrightarrow\mathrm{Diff}(X),\quad u\longrightarrow\psi_{\mathbb{C};u}\,,\qquad\text{s.t.}\\ &\mathrm{d}_{\mathbb{1}}\psi_{\mathbb{C}}(v+\mathfrak{i}v')=\mathrm{d}_{\mathbb{1}}\psi(v)+J\mathrm{d}_{\mathbb{1}}\psi(v')\in\Gamma(X;TX)\ \ \forall\,v,v'\in T_{\mathbb{1}}\mathbb{T},\end{aligned}\tag{1.6}$$

for some almost complex structure J on X compatible with ω and preserved by ψ, i.e.

$$\omega(w,Jw)>0 \quad \forall\, w\in TX,\ w\neq 0, \quad \omega(Jw,Jw')=\omega(w,w') \quad \forall\, w,w'\in TX, \quad \text{and}$$
$$\{\mathrm{d}_x\psi_u\}^{-1}\circ J\circ \mathrm{d}_x\psi_u = J\colon T_xX \longrightarrow T_xX \qquad \forall\, u\in\mathbb{T},\ x\in X.$$

If X is compact, the conditions (1.6) with $v=0$ determine $\mathbb{R}$-actions in the imaginary directions. However, these actions may not commute with ψ or each other and thus not give rise to a $\mathbb{T}_{\mathbb{C}}$-action.

Theorem 2. *Suppose $\mathbb{T}$ is a torus and (X,ω,ψ,μ) is a closed connected Hamiltonian $\mathbb{T}$-manifold.*

(1) All edges of the polytope $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}$ are rational.

(2) If $\psi_{\mathbb{C}}$ is a complexification of ψ, $x\in X$, and $\overline{\mathcal{O}_x}\subset X$ is the closure of the $\mathbb{T}_{\mathbb{C}}$-orbit $\mathcal{O}_x\equiv\mathbb{T}_{\mathbb{C}}x$ of x, then

- *(2a)* $\mathrm{Ver}(\mu(\{Y\in\pi_0(X^\psi)\colon Y\cap\overline{\mathcal{O}_x}\neq\emptyset\}))=\mu(\{Y\in\pi_0(X^\psi)\colon Y\cap\overline{\mathcal{O}_x}\neq\emptyset\})$;
- *(2b)* $\mu(\overline{\mathcal{O}_x})=\mathrm{CH}(\mu(\{Y\in\pi_0(X^\psi)\colon Y\cap\overline{\mathcal{O}_x}\neq\emptyset\}))$;
- *(2c) for every open face σ of the polytope $\mu(\overline{\mathcal{O}_x})$, $\mu^{-1}(\sigma)\cap\overline{\mathcal{O}_x}$ is a single $\mathbb{T}_{\mathbb{C}}$-orbit;*
- *(2d) the map $\overline{\mathcal{O}_x}/\mathbb{T}\longrightarrow\mu(\overline{\mathcal{O}_x})$, $[x']\longrightarrow\mu(x')$, is a well-defined homeomorphism.*

If $\mathbb{T}\approx S^1$ or J is an integrable almost complex structure on X compatible with ω and preserved by a smooth $\mathbb{T}$-action ψ on (X,ω), then (1.6) determines a complexification $\psi_{\mathbb{C}}$ of ψ; see Exercise 5.1. In the latter case, Theorem 2(2) reduces to [1, Theorem 2], but the argument in [1] applies to the general case of Theorem 2(2). The crucial implication of (1.6) is that the action of the imaginary components (which correspond to the radial direction in $\mathbb{C}^*$) is given by the gradient flow of projections of μ to one-dimensional subspaces of $T^*_{\mathbb{1}}\mathbb{T}$; see (3.18).

Remark 1.4. As stated, [1, (3.6)] is wrong. For example, it fails if $C\subset N^u-N$ consists of a single point. However, [1, (3.6)] is used in the proof of Theorem 2(2) in [1] only to obtain the second statement in [1, (3.7)]. The latter is correct, as it follows from Proposition 4.7(6), which corrects [1, (3.6)].

1.3 Symplectic toric manifolds

Let $\mathbb{T}$ be a torus. A polytope $P\subset T^*_{\mathbb{1}}\mathbb{T}$ is Delzant if there exists a full tuple $(\alpha_e)_{e\in\mathrm{Edg}(P)}$ of (integral) edge vectors for P such that for each vertex η of P the components α_e with $e\in\mathrm{Edg}_\eta(P)$ form a $\mathbb{Z}$-basis for $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$; this property of P is typically called smoothness in the literature. In particular, all edges of a Delzant polytope are rational. Furthermore, the number of edges containing any given vertex is the same as the dimension of $\mathbb{T}$; this property of P is typically called simplicity. A symplectic toric $\mathbb{T}$-manifold is a closed connected Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) so that

$$\dim X = 2\dim\mathbb{T} \tag{1.7}$$

and the action ψ on X is effective. By Delzant's Theorem, Theorem 3 below, the map

$$(X,\omega,\psi,\mu)\longrightarrow\mu(X)$$

induces a bijection between the equivalence classes of symplectic toric $\mathbb{T}$-manifolds and Delzant polytopes in $T^*_{\mathbb{1}}\mathbb{T}$.

Theorem 3 ([11, Théorème 2.1, Section 3.2]). *Let $\mathbb{T}$ be a torus.*

(0) If (X,ω,ψ,μ) is a symplectic toric $\mathbb{T}$-manifold, then the moment polytope $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}$ is Delzant.

(1) For every Delzant polytope $P\subset T^*_{\mathbb{1}}\mathbb{T}$, there exists a symplectic toric $\mathbb{T}$-manifold (X,ω,ψ,μ) with $\mu(X)=P$.

(2) If (X,ω,ψ,μ) and (X',ω',ψ',μ') are symplectic toric $\mathbb{T}$-manifolds with $\mu(X)=\mu(X')$, then there exists a $\mathbb{T}$-equivariant diffeomorphism

$$\Phi\colon X \longrightarrow X' \qquad s.t. \quad \omega=\Phi^*\omega', \quad \mu=\mu'\circ\Phi.$$

Claim 0 of this theorem is a consequence of the following description of the moment map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}\mathbb{T}$.

(0^+) If (X,ω,ψ,μ) is a symplectic toric $\mathbb{T}$-manifold, then there exist a subtorus $\mathbb{T}_F\subset\mathbb{T}$ for every face F of the polytope $\mu(X)$ and a full tuple $(\alpha_e)_{e\in\mathrm{Edg}(P)}$ of integral edge vectors for $\mu(X)$ so that

- (0^+a) $\mathbb{T}_F=\{\mathrm{e}^v\colon v\in T_{\mathbb{1}}\mathbb{T},\ \alpha_e(v)=0\,\forall\, e\in\mathrm{Edg}_\eta(F)\}$ for every face F of P and $\eta\in\mathrm{Ver}(F)$;
- (0^+b) $\mathbb{T}_x(\psi)=\mathbb{T}_F$ for every face F of P and every $x\in\mu^{-1}(F^\circ)$;
- (0^+c) the restriction $\mu\colon\mu^{-1}(F^\circ)\longrightarrow F^\circ$ is a principal $\mathbb{T}/\mathbb{T}_F$-bundle with ω-isotropic fibers, i.e. $\omega|_{T\mu^{-1}(\eta)}=0$ for every $\eta\in F^\circ$.

Similarly to Theorems 1(E_k) and 2(1), this statement is a consequence of the equivariant splitting (3.5) of $TX|_Y$ for each $Y\in\pi_0(X^\psi)$ and Corollary 3.28; we establish it at the end of Section 5.2. By (1.7) and (0^+c), the dimension of $\mu(X)$ is the same as the dimension of $\mathbb{T}$ and $\mathbb{T}_{\mu(X)}=\{\mathbb{1}\}$. Along with ($0^+$a), the latter implies that for each vertex η of $\mu(X)$ the components α_e with $\eta\in e$ span $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ over $\mathbb{Z}$. Combining this with the last claim of Theorem 1(C_k) and (1.7) again, we conclude that these components form a $\mathbb{Z}$-basis for $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$.

Theorem 3(1) is readily obtained by applying the Hamiltonian symplectic cut of [19, Proposition 2.4], which is detailed in Section 6.2, to the Hamiltonian $\mathbb{T}$-manifold $(T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ of Exercise 2.11 with respect to the collection $\mathscr{H}$ of half-spaces of $T^*_{\mathbb{1}}\mathbb{T}$ cutting out the polytope P. We establish Theorem 3(2) by describing a reverse of the Hamiltonian symplectic cut in Section 6.3; this implements the argument sketched in the proof of [22, Theorem 7.5.10].

1.4 Alternative approaches to Convexity Theorem

Other approaches to Theorem 1(C_k) on page 4, with $\mathbb{R}^k$ replaced by a torus $\mathbb{T}$, have appeared in particular in [15, 16, 5, 6]. In contrast to [1, 2, 21, 8], they clearly emphasize the significance of the local form of the moment map μ as in Corollary 3.28 and make it possible to relax the compactness condition on X to the properness of μ. These approaches differ in how they pass from the local versions of (A_k), (B_k), and (D_k) implied by this local form to the global versions appearing in Theorem 1. The global, Morse-Bott theory statement of Proposition 4.8 and the first, local statement of Proposition 4.5 used in [1, 2, 21, 8] to inductively confirm ($\mathrm{A}_k^\star$) on page 4 is used in [15, §5] instead to deduce (B_k) from its local version and to establish (C_k); the remaining statements of

Theorem 1 do not appear in [15]. An enlightening summary of the reasoning in [15] appears in [27].

The most succinct approach to passing from the local properties provided by the first part of Corollary 3.28 to the global statements of Theorem 1 is arguably presented in [6]. It is motivated by the following point set topology result from the 1920s.

Proposition 1.5 (Tietze-Nakajima Theorem, [29, Satz 1], [6, Theorem 1])**.** *A closed connected locally convex subset of* $\mathbb{R}^k$ *is convex.*

Remark 1.6. Nakajima's paper [26] typically credited for Proposition 1.5 contains several statements in the same spirit, most concerning subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$, but not the actual statement of this proposition.

Let $f\colon X \longrightarrow V$ be a continuous map between topological spaces. Such a map is fiber connected if $f^{-1}(v)\subset X$ is connected for every $v\in V$. If in addition V is a vector space, f is convex if for any $x_0, x_1\in X$ there exists a path

$$\gamma\colon \big([0,1],0,1\big) \longrightarrow \big(X,x_0,x_1\big)$$

from x_0 to x_1 in X such that the map $f\circ\gamma$ is fiber connected and $f(\gamma([0,1]))$ is contained in the line segment from $f(x_0)$ to $f(x_1)$ in V. The conditions on γ mean that the path

$$f\circ\gamma\colon \big([0,1],0,1\big) \longrightarrow \big(V,f(x_0),f(x_1)\big)$$

traces the line segment from $f(x_0)$ to $f(x_1)$ in V without reversing the direction at any point in time.

Proposition 1.7 ([6, Theorem 15])**.** *Suppose X is a connected Hausdorff topological space, V is a finite-dimensional vector space, and $f\colon X\longrightarrow V$ is a proper continuous map. If for every $x\in X$ there exists an open neighborhood $U\subset X$ of x such that $f|_U$ is convex and $f\colon U\longrightarrow \Phi(U)$ is open, then f is a convex map and $f\colon X\longrightarrow f(X)$ is an open map. In particular, f is fiber connected and $f(X)\subset V$ is convex.*

Taking $V=\mathbb{R}^k$, $X\subset\mathbb{R}^k$ to be a closed connected locally convex subset, and $f\colon X\longrightarrow V$ to be the inclusion in this proposition, we recover Proposition 1.5; the closedness of X implies the properness of f. By Exercises 1.8-1.10 below, Corollary 3.28 implies that the moment map μ of Theorem 1 satisfies the local condition on f in Proposition 1.7.

Exercise 1.8. Suppose $f_i\colon X_i\longrightarrow V_i$ for $i=1,2$ are convex maps. Show that the map

$$f_1\times f_2\colon X_1\times X_2 \longrightarrow V_1\times V_2$$

is also convex.

Exercise 1.9. Suppose $k,m\in\mathbb{Z}^{\ge0}$, $f\colon\mathbb{R}^k\longrightarrow\mathbb{R}^m$ is a smooth function so that the differential $\mathrm{d}_0 f$ is surjective, and $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{R}^k$. Show that f is open and convex on some neighborhood $\mathcal{U}'$ of 0 in $\mathcal{U}$.

Exercise 1.10. Suppose $m\in\mathbb{Z}^{\ge0}$, $S\subset\mathbb{R}^m$ is a finite subset,

$$f\colon \mathbb{C}^S \longrightarrow \mathbb{R}^m, \qquad f\big((w_\alpha)_{\alpha\in S}\big) = \sum_{\alpha\in S}|w_\alpha|^2\alpha,$$

and $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{C}^S$. Show that the map

$$f\colon \mathbb{C}^S \longrightarrow \mathcal{C}_0(S) \equiv \Big\{\sum_{\alpha\in S} t_\alpha \alpha : t_\alpha \in \mathbb{R}^{\ge 0}\ \forall\, \alpha\in S\Big\}$$

is open and f is convex on some neighborhood $\mathcal{U}'$ of 0 in $\mathcal{U}$.

Proposition 1.7 is a variation on the purely topological local-to-global theorems of [16, 17, 4, 5], which do not involve geometric input as in Propositions 4.8 and 4.5 used in [1, 15, 2, 21, 8]. The conditions on the continuous function f in [17, Theorem 3.10] and [5, Theorem 2.28], for example, are arguably more ad hoc, explicitly involving an assignment of a cone in the target vector space V of f to each point in the domain topological space X and thus fitting more closely with the output of Corollary 3.28; see Proposition 1.11 below.

Let V be a vector space. For $S\subset V$ and $v\in S$, define

$$\mathbb{R}^+(S-v) = \big\{r(v'-v)\colon v'\in S,\, r\in\mathbb{R}^+\big\}, \qquad L_v(S) = \overline{\mathbb{R}^+(S-v)} \subset V.$$

A subset $S \subset V$ is locally polyhedral if for every $v \in S$ there exists a neighborhood $U \subset V$ of v such that

$$S\cap U = \big\{v+v'\colon v'\in L_v(S)\big\}\cap U.$$

A closed convex subset $S\subset V$ is a (closed) cone with vertex at $v\in V$ if $v+t(v'-v)\in S$ whenever $v'\in S$ and $t\in\mathbb{R}^+$. Such a subset is locally polyhedral. If $S\neq V$ is a cone and V is a finite-dimensional vector space, then S is contained in a (closed) half-space, i.e.

$$S \subset \big\{w\in V\colon L(w)\ge c\big\}$$

for some nonzero linear functional $L\colon V\longrightarrow\mathbb{R}$ and $c\in\mathbb{R}$.

Let $f : X \longrightarrow V$ be a continuous map between topological spaces. Such a map is locally fiber connected if for every $x\in X$ and open neighborhood $U\subset X$ of x there exists an open neighborhood $U'\subset U$ such that $f|_{U'}$ is fiber connected. If in addition V is a finite-dimensional vector space, a tuple $(\mathcal{C}_x)_{x\in X}$ of closed convex cones in V based at 0 is called local convexity data if for every $y\in X$ and an open neighborhood $U\subset X$ of y there exists an open neighborhood $U_y\subset U$ such that $f|_{U_y}$ is fiber connected, $f(x)-f(y)\in\mathcal{C}_y$ for every $x\in U_y$, and

$$U_y \longrightarrow \mathcal{C}_y, \qquad x \longrightarrow f(x)-f(y),$$

is an open map.

Proposition 1.11 ([16, Theorem 3.4],[17, Theorem 3.10]). *Suppose X is a connected Hausdorff topological space, V is a finite-dimensional vector space, and $f : X \longrightarrow V$ is a proper locally fiber connected map. If f admits local convexity data $(\mathcal{C}_x)_{x\in X}$, then f is a fiber connected map, $f(X)\subset V$ is a closed convex locally polyhedral subset, $f\colon X\longrightarrow f(X)$ is an open map, and $\mathcal{C}_x = L_{f(x)}(f(X))$ for every $x\in X$.*

For a topological space X, $A\subset X$, and $x\in A$, the connected component of A containing x is the maximal connected subset $A_x \subset A$ containing x. For a continuous map $f : X \longrightarrow V$ between reasonable topological spaces, the Reeb quotient space,

$$X_f \equiv X/\sim, \qquad x\sim x' \ \text{ if } \ f(x)=f(x')\in V,\ \big(f^{-1}\big(f(x)\big)\big)_x = \big(f^{-1}\big(f(x)\big)\big)_{x'} \subset X,$$

is Hausdorff; see [12, Theorem 4.5]. A continuous map $f\colon X\longrightarrow V$ is fiber connected if and only if the induced map

$$X_f \longrightarrow V, \qquad [x]\longrightarrow f(x),$$

is injective. This perspective on fiber connectivity appearing in [10] provides the motivation for the proof of Proposition 1.11 in [17].

2 Preliminaries

2.1 Notation and terminology

in preparation

For $k\in\mathbb{Z}^{\ge 0}$, let $[k]=\{1,2,\ldots,k\}$. If $Y\subset X$ is a smooth submanifold of a smooth manifold, let

$$\mathcal{N}_X Y \equiv TX|_Y\big/TY \longrightarrow Y$$

denote the normal bundle of Y in X. If $H\colon X\longrightarrow\mathbb{R}$ is a smooth function, we denote by

$$\mathrm{Crit}(H)\equiv\big\{x\in X\colon \mathrm{d}_x H=0\big\}$$

its set of critical points. The gradient of H with respect to a Riemannian metric g on X is the vector field $\nabla^g H$ on M defined by

$$g\big(\nabla^g H|_x, w\big)=\mathrm{d}_x H(w) \qquad \forall\ x\in X,\ w\in T_xX. \tag{2.1}$$

Let X be a smooth manifold and G be a Lie group. For a map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ and $v\in T_{\mathbb{1}}G$, define

$$\mu_v\colon X\longrightarrow\mathbb{R}, \qquad \mu_v(x)=\{\mu(x)\}(v). \tag{2.2}$$

A basis $v_1,\ldots,v_k$ for $T_{\mathbb{1}}G$ determines identifications

$$\mathbb{R}^k\longrightarrow T_{\mathbb{1}}G,\ \ (r_1,\ldots,r_k)\longrightarrow\sum_{i=1}^k r_i v_i, \quad\text{and}\quad T^*_{\mathbb{1}}G\longrightarrow\mathbb{R},\ \ \alpha\longrightarrow\big(\alpha(v_1),\ldots,\alpha(v_k)\big). \tag{2.3}$$

The latter isomorphism identifies smooth (G-invariant) maps $\mu: X\longrightarrow T^*_{\mathbb{1}}G$ with smooth (G-invariant) maps $H\colon X\longrightarrow\mathbb{R}^k$ by

$$\mu \quad\longleftrightarrow\quad H\equiv\big(\mu_{v_1},\ldots,\mu_{v_k}\big). \tag{2.4}$$

If G is a connected abelian Lie group and ω is a symplectic form on X, a moment map $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ for a smooth action ψ of G on (X,ω) corresponds via (2.4) to a smooth G-invariant map

$$H\equiv(H_1,\ldots,H_k)\colon X\longrightarrow\mathbb{R}^k \quad\text{s.t.}\quad -\mathrm{d}H_i=\iota_{\zeta_{v_i}}\omega,\quad\text{where } \zeta_{v_i}=\mathrm{d}_{\mathbb{1}}\psi(v_i)\in\Gamma(X;TX). \tag{2.5}$$

We will call such a smooth function a Hamiltonian for ψ with respect to basis $v_1,\ldots,v_k$ for $T_{\mathbb{1}}G$.

We denote by $e_1,\ldots,e_k\in\mathbb{R}$ the standard orthonormal basis and by $\mathbb{R}_i\subset\mathbb{R}^k$ the $\mathbb{R}$-span of e_i. A smooth action ψ of $\mathbb{R}^k$ on a smooth manifold X (resp. symplectic manifold (X,ω)) is equivalent to k commuting smooth $\mathbb{R}$-actions $\psi_i\equiv\psi|_{\mathbb{R}_i}$ on X (resp. (X,ω)). In such a case, we will call

$$\zeta_i\equiv \mathrm{d}_{\mathbb{1}}\psi(e_i)=\frac{\mathrm{d}}{\mathrm{d}t}\psi_{i;t}\bigg|_{t=0}\in\Gamma(X;TX),\qquad i\in[k], \tag{2.6}$$

the generating vector fields of ψ and a Hamiltonian H as in (2.5) with respect to the basis $e_1,\ldots,e_k$ for $T_0\mathbb{R}^k$ simply a Hamiltonian for ψ. If $k=1$ or X is compact, such a Hamiltonian corresponds to k smooth functions $H_i\colon X\longrightarrow\mathbb{R}$ satisfying the condition in (2.5) with $\zeta_{v_i}=\zeta_i$; see Exercise 3.11(c).

Suppose $\mathbb{T}$ is a torus and $v_1,\ldots,v_k\in T_{\mathbb{1}}\mathbb{T}$ is a $\mathbb{Z}$-basis for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ and thus an $\mathbb{R}$-basis for $T_{\mathbb{1}}\mathbb{T}$. The isomorphisms in (2.3) then identify $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T^*_{\mathbb{1}}\mathbb{T}$ with $\mathbb{Z}^k\subset\mathbb{R}^k$. The first isomorphism in (2.3) also induces a Lie group identification of $\mathbb{T}$ with the standard torus $\mathbb{T}^k$,

$$\phi_{v_1\ldots v_k}\colon \mathbb{T}^k\equiv(\mathbb{R}/\mathbb{Z})^k=\mathbb{R}^k/\mathbb{Z}^k\longrightarrow\mathbb{T},\quad \phi_{v_1\ldots v_k}\big([r_1,\ldots,r_k]\big)=\prod_{i=1}^k\mathrm{e}^{r_iv_i}\,. \tag{2.7}$$

Exercise 2.1. Suppose $\mathbb{T}$ is a torus and $v_1,\ldots,v_k\in T_{\mathbb{1}}\mathbb{T}$ is $\mathbb{Z}$-basis for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$. Let $\alpha_1,\ldots,\alpha_k\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ be the dual basis. For $u\in\mathbb{T}$, let $m_u\colon\mathbb{T}\longrightarrow\mathbb{T}$ be the multiplication by u. Show that the diffeomorphism

$$\Phi_{v_1\ldots v_k}\colon\mathbb{R}^k\times\mathbb{T}^k\longrightarrow T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T},\quad \Phi_{v_1\ldots v_k}\big((x_1,\ldots,x_k),[r]\big)=\bigg(\sum_{i=1}^k x_i\alpha_i,\phi_{v_1\ldots v_k}\big([r]\big)\bigg), \tag{2.8}$$

is equivariant with respect to the identification $\phi_{v_1\ldots v_k}$ in (2.7) and satisfies

$$\begin{aligned} &\mathrm{d}\pi_2\big(\big\{\mathrm{d}\Phi_{v_1\ldots v_k}\big\}(\partial_{x_i})\big)=0 \qquad\text{and}\\ &\big\{\pi_1\big(\Phi_{v_1\ldots v_k}\big((x_1,\ldots,x_k),[r]\big)\big)\big\}\Big(\mathrm{d}_{\phi_{v_1\ldots v_k}([r])}m_{\phi_{v_1\ldots v_k}([-r])}\big(\mathrm{d}\pi_2\big(\big\{\mathrm{d}\Phi_{v_1\ldots v_k}\big\}(\partial_{r_i})\big)\big)\Big)=x_i \end{aligned}\quad\forall\, i\in[k], \tag{2.9}$$

where $\pi_1,\pi_2\colon T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T}\longrightarrow T^*_{\mathbb{1}}\mathbb{T},\mathbb{T}$ are the projections, ∂_{x_i} is the i-th coordinate vector field on $\mathbb{R}^k$, and ∂_{r_i} is the coordinate vector field on $\mathbb{T}^k$ induced by the i-th coordinate vector field on $\mathbb{R}^k$.

We take the standard $\mathbb{Z}$-basis for $(T_{\mathbb{1}}\mathbb{T}^k)_{\mathbb{Z}}$ to be

$$2\pi\mathrm{i}e_1,\ldots,2\pi\mathrm{i}e_k\in T_{\mathbb{1}}\mathbb{T}^k\subset T_{(1,\ldots,1)}\mathbb{C}^k.$$

A smooth action ψ of $\mathbb{T}^k$ on a smooth manifold X (resp. symplectic manifold (X,ω)) is equivalent to k commuting smooth S^1-actions $\psi_i\equiv\psi|_{S^1_i}$ on X (resp. (X,ω)), where $S^1_i\subset\mathbb{T}^k=(S^1)^k$ is the i-th component subgroup S^1. Similarly to the affine case, we then call

$$\zeta_i\equiv\mathrm{d}_{\mathbb{1}}\psi(2\pi\mathrm{i}e_i)=\frac{\mathrm{d}}{\mathrm{d}t}\psi_{i;\mathrm{e}^{2\pi\mathrm{i}t}}\bigg|_{t=0}\in\Gamma(X;TX),\qquad i\in[k],$$

the generating vector fields of ψ and a Hamiltonian H as in (2.5) with respect to the basis $2\pi\mathrm{i}e_1,\ldots,2\pi\mathrm{i}e_k$ for $T_{\mathbb{1}}\mathbb{T}^k$ simply a Hamiltonian for ψ. By Exercise 3.11(c), such a Hamiltonian corresponds to k smooth functions $H_i\colon X\longrightarrow\mathbb{R}$ satisfying the condition in (2.5) with $\zeta_{v_i}=\zeta_i$.

Exercise 2.2. Suppose G, G' are Lie groups, (X,ω,ψ,μ) is a Hamiltonian G-manifold, and $\rho\colon G'\longrightarrow G$ is a Lie group homomorphism. Show that $(X,\omega,\psi\circ\rho,\rho^*\circ\mu)$, where

$$\rho^*\equiv(\mathrm{d}_{\mathbb{1}}\rho)^*: T^*_{\mathbb{1}}G\longrightarrow T^*_{\mathbb{1}}G'$$

is the homomorphism induced by ρ, is a Hamiltonian G'-manifold.

Exercise 2.3. Suppose G is a compact Lie group, $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$ is a Hamiltonian G-manifold, $\widetilde{Y}\subset\widetilde{X}$ is a smooth submanifold preserved by $\widetilde{\psi}$, (X,ω) is a symplectic manifold, ψ is a smooth G-action on X, $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ is a map, and $p\colon\widetilde{Y}\longrightarrow X$ is a G-equivariant surjective submersion so that

$$p^*\omega=\widetilde{\omega}|_{T\widetilde{Y}}\qquad\text{and}\qquad\mu\circ p=\widetilde{\mu}|_{\widetilde{Y}}.$$

Show that (X,ω,ψ,μ) is also a Hamiltonian G-manifold.

Exercise 2.4. Suppose ψ is a smooth $\mathbb{R}^k$-action on a symplectic manifold (X,ω) with Hamiltonian $H\colon X\longrightarrow\mathbb{R}^k$ and A is a real $k{\times}m$-matrix (determining a linear map from $\mathbb{R}^m$ to $\mathbb{R}^k$). Show that $\psi\circ A$ is a smooth $\mathbb{R}^m$-action on (X,ω) with Hamiltonian $A^{\mathrm{tr}}\circ H\colon X\longrightarrow\mathbb{R}^m$, where A^{tr} is the transpose of A.

2.2 Paradigmatic examples

in preparation

Exercise 2.5. Let $k,n\in\mathbb{Z}^+$ with $k\le n$ and

$$\omega_{\mathbb{C}^n}\equiv\sum_{i=1}^n\mathrm{d}x_i\wedge\mathrm{d}y_i \tag{2.10}$$

be the standard symplectic form on $\omega_{\mathbb{C}^n}$. Show that the action of S^1 on $\mathbb{C}^n$ given by

$$\mathrm{e}^{2\pi\mathrm{i}t}\cdot(z_1,\ldots,z_n)=\big(z_1,\ldots,z_{k-1},\mathrm{e}^{2\pi\mathrm{i}t}z_k,z_{k+1},\ldots,z_n\big)$$

is Hamiltonian with respect to $\omega_{\mathbb{C}^n}$ with a Hamiltonian

$$H\colon\mathbb{C}^n\longrightarrow\mathbb{R},\qquad H(z_1,\ldots,z_n)=\pi|z_k|^2.$$

Exercise 2.6. Suppose $(V,\mathfrak{i})$ is a finite-dimensional complex vector space and Ω is a nondegenerate 2-form on V compatible with $\mathfrak{i}$, i.e.

$$\Omega(w,\mathfrak{i}w)>0\ \ \forall\,w\in V-\{0\},\quad\Omega(\mathfrak{i}w,\mathfrak{i}w')=\Omega(w,w')\ \ \forall\,w,w'\in V.$$

Via the canonical identification $T_wV\approx V$ for each $w\in V$, $\mathfrak{i}$ and Ω determine an almost complex structure J on V and a symplectic form ω compatible with J. Let $\psi\colon\mathbb{T}\longrightarrow\mathrm{GL}_{\mathbb{C}}V$ be a complex representation of a torus $\mathbb{T}$ on V. Show that

(a) there exist a subset $S(Y)\subset(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and a splitting

$$V=\bigoplus_{\alpha\in S}V_\alpha\qquad\text{s.t.}\quad\psi_{\mathrm{e}^v}\big((w_\alpha)_{\alpha\in S}\big)=\big(\mathrm{e}^{2\pi\mathrm{i}\,\alpha(v)}w_\alpha\big)_{\alpha\in S}\ \ \forall\,v\in T_{\mathbb{1}}\mathbb{T},\ (w_\alpha)_{\alpha\in S}\in\bigoplus_{\alpha\in S}V_\alpha\,;$$

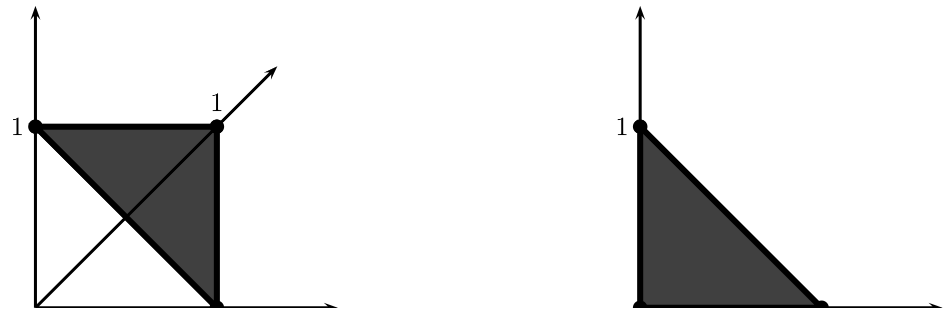

Figure 1: The images of $\mathbb{C}P^2$ under Hamiltonians for the torus actions of Exercise 2.7.

(b) the action ψ is Hamiltonian with respect to ω with a moment map

$$\mu\colon V \longrightarrow T_{\mathbb{1}}^*\mathbb{T}, \qquad \mu\big((w_\alpha)_{\alpha\in S}\big) = \pi\sum_{\alpha\in S}|w_\alpha|^2\alpha \ \ \forall\ (w_\alpha)_{\alpha\in S}\in\bigoplus_{\alpha\in S}V_\alpha\,,$$

where $|\cdot|$ is the norm on V with respect to the metric $g(\cdot,\cdot)\equiv\Omega(\cdot,\mathfrak{i}\cdot)$.

Exercise 2.7. Let $n\in\mathbb{Z}^+$ and $q\colon \mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}$ be the usual quotient projection.

(a) Suppose $U\subset\mathbb{C}P^{n-1}$ is an open subset and $s\colon U\longrightarrow\mathbb{C}^n-\{0\}$ is a holomorphic section of q, i.e. $q\circ s=\mathrm{id}_U$. Show that the 2-form

$$\omega_{\mathrm{FS};n-1}\big|_U \equiv \frac{\mathfrak{i}}{2\pi}\partial\bar\partial\ln|s|^2, \tag{2.11}$$

where $|\cdot|$ is the standard (round) norm on $\mathbb{C}^n$, is independent of the choice of s.

(b) By (a), (2.11) determines a global 2-form $\omega_{\mathrm{FS};n-1}$ on $\mathbb{C}P^{n-1}$, called the Fubini-Study symplectic form. Show that this form is indeed symplectic,

$$q^*\omega_{\mathrm{FS};n-1}\big|_{TS^{2n-1}} = \frac{1}{\pi}\omega_{\mathbb{C}^n}\big|_{TS^{2n-1}}, \qquad\text{and}\qquad \int_{\mathbb{C}P^1}\omega_{\mathrm{FS};1}=1.$$

Hint. The restriction of q to the interior of the upper hemisphere $S^2_+\subset S^2=S^3\cap(\mathbb{C}\times\mathbb{R})$ is a diffeomorphism onto the complement of a point in $\mathbb{C}P^1$.

(c) Show that the actions of $\mathbb{T}^n\equiv(S^1)^n$ and $\mathbb{T}^{n-1}\equiv(S^1)^{n-1}$ on $\mathbb{C}P^{n-1}$ given by

$$\begin{aligned}\big(\mathrm{e}^{2\pi\mathfrak{i}t_1},\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_n}\big)\cdot[z_1,\ldots,z_n] &= \big[\mathrm{e}^{2\pi\mathfrak{i}t_1}z_1,\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_n}z_n\big],\\ \big(\mathrm{e}^{2\pi\mathfrak{i}t_1},\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_{n-1}}\big)\cdot[z_1,\ldots,z_n] &= \big[\mathrm{e}^{2\pi\mathfrak{i}t_1}z_1,\ldots,\mathrm{e}^{2\pi\mathfrak{i}t_{n-1}}z_{n-1},z_n\big]\end{aligned}$$

are Hamiltonian with respect to the symplectic form $\omega_{\mathrm{FS};n-1}$. Determine the moment polytopes for these actions, in particular showing that in the $n=3$ they are as depicted in Figure 1.

Exercise 2.8. Let $k\in\mathbb{Z}^+$, $\mathbb{T}^k\equiv\mathbb{R}^k/\mathbb{Z}^k$ is the standard k-torus, and

$$x\equiv(x_1,\ldots,x_k),\, y\equiv(y_1,\ldots,y_k)\colon \mathbb{R}^k\times\mathbb{R}^k\longrightarrow\mathbb{R}^k$$

be the projections to the two components. Show that

(a) $\omega_k \equiv \sum_{i=1}^{k} \mathrm{d}x_i \wedge \mathrm{d}y_i$ is a well-defined symplectic form on $\mathbb{R}^k \times \mathbb{T}^k$;

(b) the action ψ_k of $\mathbb{T}^k$ on $\mathbb{R}^k \times \mathbb{T}^k$ given by

$$\psi_{\mathbb{T};[r]}\big(x,[y]\big) = \big(x,[y+r]\big),$$

is well-defined, free, and smooth, preserves ω_k, and has

$$H_k\colon \mathbb{R}^k \times \mathbb{T}^k \longrightarrow \mathbb{R}^k, \qquad H_k\big(x,[y]\big) = x,$$

as a Hamiltonian with respect to ω_k.

An automorphism $\phi\colon X \longrightarrow X$ of a set X is an involution if $\phi^2 = \mathrm{id}_X$. If (X,ω) is a symplectic manifold, a smooth involution ϕ on X is called anti-symplectic if $\phi^*\omega = -\omega$. A Lagrangian submanifold of a symplectic manifold (X,ω) is a submanifold $Y \subset X$ such that

$$\dim Y = \frac{1}{2}\dim X \qquad \text{and} \qquad \omega|_{TY} = 0.$$

Exercise 2.9. Suppose (X,ω) and (X',ω') are symplectic manifolds of the same dimension and $f\colon X \longrightarrow X'$ is a smooth map. Show that f is a symplectomorphism with respect to ω and ω' if and only if the graph of f,

$$\mathrm{Gr}(f) \equiv \big\{\big(x,f(x)\big)\colon x \in X\big\} \subset X \times X',$$

is a Lagrangian submanifold of $X \times X'$ with respect to the symplectic form $\pi_1^*\omega - \pi_2^*\omega'$, where

$$\pi_1,\pi_2\colon X \times X' \longrightarrow X, X'$$

are the component projections.

Exercise 2.10. Let Y be a smooth manifold and λ_{T^*Y} be the 1-form on (the total space of) its cotangent bundle $\pi\colon T^*Y \longrightarrow Y$ given by

$$\lambda_{T^*Y}\big|_\theta(w) = \{\pi^*\theta\}(w) \equiv \theta\big(\{\mathrm{d}_\theta\pi\}(w)\big) \qquad \forall\ \theta \in T^*Y,\ w \in T_\theta(T^*Y).$$

Show that

(a) $\{\{\mathrm{d}f\}^*\}^*\lambda_{T^*Y} = \lambda_{T^*Y'}$ for every diffeomorphism $f\colon Y \longrightarrow Y'$ between smooth manifolds (thus $\{\mathrm{d}f\}^*\colon T^*Y' \longrightarrow T^*Y$ is well-defined);

(b) $\omega_{TY^*} \equiv -\mathrm{d}\lambda_{T^*Y}$ is a symplectic form on (the total space of) T^*Y and $\omega_{T^*\mathbb{R}^n} = \omega_{\mathbb{C}^n}$ under the natural identification of $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$;

(c) for every $y \in Y$, there is a canonical decomposition $T_y(T^*Y) = T_yY \oplus T_y^*Y$ and

$$\omega_{T^*Y}\big|_y(v,w) = \begin{cases} 0, & \text{if } v,w \in T_yY \text{ or } v,w \in T_y^*Y; \\ w(v), & v \in T_yY \text{ and } w \in T_y^*Y. \end{cases}$$

Suppose in addition α is 1-form on X. Show that

(d) $\alpha^*\lambda_{T^*Y}=\alpha$ and the map

$$\phi_\alpha\colon T^*Y \longrightarrow T^*Y, \quad \phi_\alpha(\theta)=\alpha_{\pi(\theta)}-\theta,$$

is a smooth involution satisfying $\phi_\alpha^*\lambda_{T^*Y}=\pi^*\alpha-\lambda_{T^*Y}$;

(e) the involution ϕ_α above is anti-symplectic with respect to ω_{T^*Y} if and only if $\mathrm{d}\alpha=0$.

Exercise 2.11. Let $\mathbb{T}$, $v_1,\ldots,v_k$, $\Phi_{v_1\ldots v_k}$, and m_u for $u\in\mathbb{T}$ be as in Exercise 2.1, $(\mathbb{R}^k\times\mathbb{T}^k,\omega_k,\psi_k,H_k)$ be as in Exercise 2.8, and $\omega_{T*\mathbb{T}}$ be as in Exercise 2.10. Define

$$\Phi_{\mathbb{T}}\colon T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}, \quad \Phi_{\mathbb{T}}(\alpha,u)=\big\{\mathrm{d}_u m_{u^{-1}}\big\}^*\alpha, \quad \text{and} \quad \omega_{\mathbb{T}}=\big\{\Phi_{v_1\ldots v_k}^{-1}\big\}^*\omega_k\,.$$

Denote by $\psi_{\mathbb{T}}$ the action of $\mathbb{T}$ on $T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}$ by the multiplication on the second component and by $\mu_{\mathbb{T}}\colon T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}$ the projection to the first component. Show that $\omega_{\mathbb{T}}=-\Phi_{\mathbb{T}}^*\omega_{T^*\mathbb{T}}$ and $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ is a Hamiltonian $\mathbb{T}$-manifold which does not depend on the choice of a $\mathbb{Z}$-basis $v_1,\ldots,v_k\in T_{\mathbb{1}}\mathbb{T}$ for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ and is identified with $(\mathbb{R}^k\times\mathbb{T}^k,\omega_k,\psi_k,H_k)$ via $\Phi_{v_1\ldots v_k}$.

Exercise 2.12. (a) Suppose ϕ is an involution on a neighborhood of $0\in\mathbb{R}^n$ with $\phi(0)=0$. Let $\mathrm{Jac}_0(\phi)\colon\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be its Jacobian at 0 so that

$$\phi(x)=\big\{\mathrm{Jac}_0(\phi)\big\}x+Q(x)$$

for some quadratic term $Q\colon\mathbb{R}^n\longrightarrow\mathbb{R}^n$ ($Q(0)=0$, $\mathrm{Jac}_0(Q)=0$) and all x in a neighborhood of $0\in\mathbb{R}^n$. Show that there exist neighborhoods U and W of $0\in\mathbb{R}^n$ so that

$$h\colon U\longrightarrow W, \qquad h(x)=x+\frac{1}{2}\big\{\mathrm{Jac}_0(\phi)\big\}Q(x),$$

is a well-defined diffeomorphism satisfying $h\circ\phi=\{\mathrm{Jac}_0(\phi)\}h$.

(b) Let X be a smooth manifold and $\phi: X\longrightarrow X$ be a smooth involution. Show that every connected component of the fixed locus of ϕ,

$$X^\phi\equiv\big\{x\in X\colon\phi(x)=x\big\},$$

is a smooth submanifold of X.

(c) Suppose in addition ω is a nondegenerate 2-form on X such that $\phi^*\omega=-\omega$. Show that $X^\phi\subset X$ is a Lagrangian submanifold of (X,ω).

Exercise 2.13. Suppose (X,ω) is a symplectic manifold, $Y\subset X$ is a Lagrangian submanifold, J is an ω-compatible almost complex structure on X, and ω_{T^*Y} is the canonical symplectic form on T^*Y as in Exercise 2.10. Show that

(a) $J(TY)\subset TX|_Y$ is a subbundle complementary to TY;

(b) the map $\Phi_{Y;J}\colon J(TY)\longrightarrow T^*Y$, $\Phi_{Y;J}(w)=\omega(\cdot,w)$, is an isomorphism of vector bundles over Y;

(c) $\Phi_{Y;J}^*\omega_{T^*Y}|_{T_y(J(TY))}=\omega|_{T_yX}$ under the canonical identification

$$T_y\big(J(TY)\big)=T_yY\oplus J(T_yY)=T_yX.$$

3 Group Actions on Manifolds

3.1 Basic properties of group actions

This section collects basic facts about smooth actions of Lie groups, especially abelian ones, on smooth manifolds.

Exercise 3.1. Let ψ be a smooth action of a Lie group G on a smooth manifold X as in (1.1). Show that

$$\zeta_{\mathrm{Ad}_{u^{-1}}(v)} = \psi_u^*\zeta_v \equiv \mathrm{d}\psi_u^{-1}(\zeta_v\circ\psi_u), \quad \zeta_{[v,v']} = -\big[\zeta_v,\zeta_{v'}\big] \in \Gamma(X;TX) \qquad \forall\, u\in G,\, v,v'\in T_{\mathbb{1}}G,$$

where $[v,v']$ is the Lie bracket on $T_{\mathbb{1}}G$; see [30, Sections 3.8]. Furthermore, the maps

$$\mathrm{d}\psi\colon G\longrightarrow \mathrm{Diff}(TX), \quad u\longrightarrow \mathrm{d}\psi_u, \quad \text{and} \quad \mathrm{d}\psi^*\colon G\longrightarrow \mathrm{Diff}(T^*X), \quad u\longrightarrow \mathrm{d}\psi_u^*\equiv\big\{\mathrm{d}\psi_u^{-1}\big\}^*,$$

are smooth actions of G on TX and T^*X, respectively, lifting the G-action ψ on X and linear on the fibers of the vector bundles $TX,T^*X\longrightarrow X$. The 1-form λ_{T^*X} of Exercise 2.10 is preserved by the action $\mathrm{d}\psi^*$.

Exercise 3.2. Let ψ be a smooth action of a compact Lie group on a smooth manifold X as in (1.1). Show that there exists a ψ-invariant Riemannian metric on X, i.e. a Riemannian metric g on X such that

$$g\big(\mathrm{d}_x\psi_u(w),\mathrm{d}_x\psi_u(w')\big) = g(w,w') \qquad \forall\, u\in G,\, x\in X,\, w,w'\in T_xX.$$

Suppose $Y\subset X$ is a closed submanifold of a smooth manifold. A tubular neighborhood identification for Y in X is a diffeomorphism $\Phi\colon\mathcal{U}\longrightarrow U$ from an open neighborhood of Y in a subbundle $TY^c\subset TX|_Y$ complementary to TY to an open neighborhood of $Y\subset X$ such that

$$\Phi(y)=y,\ \mathrm{d}_y\Phi=\mathrm{id}\colon T_y\mathcal{U}=T_y(TY^c)=T_yY\oplus TY^c\big|_y\longrightarrow T_yY\oplus TY^c\big|_y=T_yX=T_yU \quad \forall\, y\in Y. \tag{3.1}$$

Proposition 3.3. *Let ψ be a smooth action of a compact Lie group G on a smooth manifold X as in (1.1).*

(1) The fixed locus $X^\psi\subset X$ of ψ is a closed submanifold with

$$T(X^\psi) = (TX)^{\mathrm{d}\psi}. \tag{3.2}$$

(2) If $Y\subset X$ is a closed submanifold preserved by ψ and $TY^c\subset TX|_Y$ is a subbundle complementary to TY and preserved by ψ, then there exists a tubular neighborhood identification $\Phi\colon\mathcal{U}\longrightarrow U$ for Y in X with $\mathcal{U}\subset TY^c$ which is G-equivariant with respect to the actions ψ on X and $\mathrm{d}\psi$ on TX.

Proof. Let g be a Riemannian metric preserved by the group action ψ, as provided by Exercise 3.2. Its Levi-Civita connection ∇ is also preserved by G. If $w\in TX$, $\gamma_w\colon(a,b)\longrightarrow X$ with $a<0<b$ is the geodesic with respect to ∇ of g with $\gamma_w'(0)=w$, and $u\in G$, then

$$\psi_u\circ\gamma_w\colon(a,b)\longrightarrow X$$

is the geodesic with respect to ∇ with $(\psi_u\circ\gamma_w)'(0)=\{\mathrm{d}_{\gamma_w(0)}\psi_u\}(w)$, i.e. $\psi_u\circ\gamma_w=\gamma_{\{\mathrm{d}_{\gamma_w(0)}\psi_u\}(w)}$. Thus, the exponential map

$$\exp\colon\mathcal{W}\longrightarrow X,\qquad \exp(w)=\gamma_w(1)\ \ \forall\,w\!\in\!\mathcal{W}\!\subset\!TX,$$

with respect to ∇ satisfies

$$\{\mathrm{d}\psi_u\}(\mathcal{W})=\mathcal{W}\quad\text{and}\quad\exp\big(\{\mathrm{d}\psi_u\}(w)\big)=\psi_u\big(\exp(w)\big)\qquad\forall\;u\!\in\!G,\,w\!\in\!\mathcal{W},\tag{3.3}$$

i.e. it is G-equivariant with respect to the actions ψ on X and $\mathrm{d}\psi$ on TX.

(2) Since

$$\{\mathrm{d}_x\exp\}(w)=w\in T_xX\qquad\forall\,x\!\in\!X,\,w\!\in\!W,\tag{3.4}$$

for each $y\!\in\!Y$ the restriction of exp to a neighborhood of y in $TY^c\!\cap\!\mathcal{W}$ is a diffeomorphism onto an open neighborhood of y in X by the Inverse Function Theorem. Since $Y\subset X$ is closed, it follows that there exists a neighborhood $\mathcal{U}'$ of Y in $TY^c\!\cap\!\mathcal{W}$ so that

$$\exp\colon\mathcal{U}'\longrightarrow\Psi(\mathcal{U}')$$

is a diffeomorphism onto an open subset of X. This map satisfies both conditions in (3.1) by the definition of the exponential map. Since G is compact,

$$\mathcal{U}\equiv\bigcap_{u\in G}\mathrm{d}\psi_u(\mathcal{U}')\subset\mathcal{U}'\subset TY^c\!\cap\!\mathcal{W}$$

is a neighborhood of $Y\!\subset\!TY^c$ preserved by the G-action. By (3.3), the restriction

$$\Phi\!\equiv\!\exp|_{\mathcal{U}}\colon\mathcal{U}\longrightarrow U\!\equiv\!\exp(\mathcal{U})\subset X$$

is a G-equivariant diffeomorphism from an open neighborhood of Y in TY^c to an open neighborhood of Y in X with the required properties. This establishes (2).

(1) It is immediate that $X^\psi\subset X$ is a closed subset. For each $y\in X^\psi$, let $\Phi_y:\mathcal{U}_y\longrightarrow U_y$ be a G-equivariant tubular neighborhood identification as in (2) with $Y=\{y\}$ and $\mathcal{U}_y\subset T_yX$. By the G-equivariance of Φ_y,

$$\Phi_y\colon(T_yX)^{\mathrm{d}\psi}\!\cap\!\mathcal{U}_y\longrightarrow X^\psi\!\cap\!U_y$$

is a homeomorphism for every $y\!\in\!X^\psi$. Thus, each topological component of X^ψ is a submanifold of X; see [30, 1.33(b)]. By the G-equivariance of Φ_y, (3.2) holds as well. □

Remark 3.4. The conclusion of Proposition 3.3(1) also holds if $\pi_0(G)$ is finite and X admits a G-invariant metric (but G is not necessarily compact). The first paragraph in the proof of Proposition 3.3 still applies. For (1) in this proof, $\mathcal{U}_y\!\subset\!T_yX$ can be taken to be any neighborhood of $0\!\in\!T_yX$ on which the map exp is injective.

Corollary 3.5. *Let ψ be an irreducible almost periodic action of $\mathbb{R}^k$ on a smooth manifold X as in (1.1). The subspace $\mathrm{Crit}(\psi)$ of points of X with stabilizers containing a one-dimensional linear subspace of $\mathbb{R}^k$ is a countable union of (not necessarily disjoint) closed proper submanifolds of X.*

Proof. Let $\rho\colon \mathbb{R}^k \longrightarrow \mathbb{T}$ and ψ' be as in (1.5). For each one-dimensional linear subspace $L\subset\mathbb{R}^k$, the closure $\mathbb{T}_L\subset\mathbb{T}$ of $\rho(L)$ in $\mathbb{T}$ is a torus. By (1.5), the fixed locus $X^{\mathbb{T}_L}\subset X$ of the smooth action $\psi'|_{\mathbb{T}_L}$ on X is the same as the fixed locus X^L of the smooth action $\psi|_L$. By Proposition 3.3(1), X^L is thus a closed submanifold of X (possibly empty). Since the action ψ is irreducible, $\psi|_L$ is a nontrivial action and $X^L\neq X$. Thus, $\mathrm{Crit}(\psi)$ is the union of the closed submanifolds $X^{\mathbb{T}'}\neq X$ taken over the subcollection

$$\mathcal{A}\equiv\big\{\mathbb{T}_L\colon L\in\mathbb{RP}^{k-1}\big\}$$

of subtori of $\mathbb{T}$. Since the subtori of $\mathbb{T}$ are generated by finite sets of vectors in $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$, the collection $\mathcal{A}$ is (at most) countable. $\square$

For a torus $\mathbb{T}$ and $\alpha\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$, let

$$\mathbb{T}_\alpha=\big\{\mathrm{e}^v\colon v\in T_{\mathbb{1}}\mathbb{T},\ \alpha(v)\in\mathbb{Z}\big\}.$$

If $\alpha\neq0$, $\mathbb{T}_\alpha\subset\mathbb{T}$ is a codimension 1 closed subgroup. If α is primitive, i.e. $\alpha\neq k\alpha'$ for any $\alpha'\in(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and $k\in\mathbb{Z}$ with $k\geq2$, then $\mathbb{T}_\alpha\subset\mathbb{T}$ is a codimension 1 subtorus. For a subset $S\subset(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and a closed subgroup $G\subset\mathbb{T}$, let

$$\mathbb{T}_S=\bigcap_{\alpha\in S}\mathbb{T}_\alpha \qquad\text{and}\qquad S_G=\big\{\alpha\in S\colon G\subset\mathbb{T}_\alpha\big\}.$$

Thus, $\mathbb{T}_S\subset\mathbb{T}$ is a closed subgroup of codimension at most $|S|$, $S_G\subset S$ is the maximal subset so that $G\subset\mathbb{T}_{S_G}$, and $S_{\mathbb{T}_S}=S$.

Proposition 3.6. *Let ψ be a smooth action of a torus $\mathbb{T}$ on a smooth manifold X as in (1.1) and $Y\subset X^\psi$ be a topological component of the ψ-fixed locus.*

(1) There exist a subset $S(Y)\subset(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}-\{0\}$ and a splitting

$$TX|_Y=TY\oplus\bigoplus_{\alpha\in S(Y)}\mathcal{N}^\alpha_XY\longrightarrow Y \tag{3.5}$$

of $TX|_Y$ into a direct sum of vector bundles preserved by $\mathrm{d}\psi$ so that the bundles $\mathcal{N}^\alpha_XY$ are nonzero and complex with

$$\mathrm{d}\psi_{\mathrm{e}^v}(w)=\mathrm{e}^{2\pi\mathrm{i}\,\alpha(v)}w \qquad \forall\ v\in T_{\mathbb{1}}\mathbb{T},\ w\in\mathcal{N}^\alpha_XY,\ \alpha\in S(Y). \tag{3.6}$$

In particular, $2|S(Y)|\leq\mathrm{codim}_XY$. If X is connected and the action ψ is irreducible (resp. effective), then the $\mathbb{R}$-span (resp. $\mathbb{Z}$-span) of $S(Y)$ is $T^*_{\mathbb{1}}\mathbb{T}$ (resp. $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$). If $TX|_Y$ is a complex vector bundle and $\mathrm{d}\psi$ preserves its complex structure J, then the complex structure on each subbundle $\mathcal{N}^\alpha_XY\subset TX|_Y$ can be taken to be the restriction of J.

(2) If $G\subset\mathbb{T}$ is a closed subgroup, $Z\subset X^G$ is a topological component of the G-action $\psi|_G$ on X, and $Y\subset Z$, then

$$TZ|_Y=TY\oplus\bigoplus_{\alpha\in S(Y)_G}\mathcal{N}^\alpha_XY\subset TX|_Y\longrightarrow Y$$

and Z is a topological component of the fixed locus $X^{\mathbb{T}_{S(Y)_G}}$ of the $\mathbb{T}_{S(Y)_G}$-action $\psi|_{\mathbb{T}_{S(Y)_G}}$ on X.

Proof. (1) For each $y \in Y$, $\mathrm{d}_y\psi$ is a real representation of $\mathbb{T}$ on T_yX. Every such representation splits as a direct sum of a trivial real representation and of one-dimensional complex representations with the action on each factor given by (3.6) for some $\alpha \in (T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ nonzero. By (3.2), the trivial representation summand is T_yY. Since $\mathrm{d}_y\psi$ depends smoothly on y, the weights α are independent of $y \in Y$ and the corresponding component representations vary smoothly with $y \in Y$. Thus, the latter form vector subbundles $\mathcal{N}_X^{\alpha}Y \subset TX|_Y$ as in (3.5) with complex structures. If $TX|_Y$ is a complex vector bundle and $\mathrm{d}\psi$ preserves its complex structure, then $\mathrm{d}_y\psi$ is a complex representation of $\mathbb{T}$ on T_yY and the same reasoning applies.

If $S(Y)$ does not span $T^*_{\mathbb{1}}\mathbb{T}$ over $\mathbb{R}$ (resp. $(T^*_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ over $\mathbb{Z}$), there exists $v \in T_{\mathbb{1}}\mathbb{T} - (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ such that $\alpha(v) = 0$ (resp. $\alpha(v) \in \mathbb{Z}$) for all $\alpha \in S(Y)$. Let G be the closure of the subgroup $\{\mathrm{e}^{tv} : t \in \mathbb{R}\}$ (resp. the subgroup generated by v) in $\mathbb{T}$. This subgroup acts trivially on $TX|_Y$. By Proposition 3.3(1), this implies that the connected component of the G-fixed locus X^G containing Y is a connected component of X, i.e. G acts trivially on X (and so the action ψ is not effective) if X is connected. If $\alpha(v) = 0$ for all $\alpha \in S(Y)$ and X is connected, then

$$\zeta_v \equiv \mathrm{d}_{\mathbb{1}}\psi(v) = 0 \in \Gamma(X; TX),$$

i.e. the action ψ is reducible.

(2) By Proposition 3.3(1) applied to $\psi|_G$, ψ, and $\psi|_{\mathbb{T}_{S(Y)_G}}$,

$$TZ|_Y = \big\{w \in TX|_Y : \mathrm{d}\psi_u(w) = w \ \forall\, u \in G\big\} = TY \oplus \bigoplus_{\alpha \in S(Y)_G} \mathcal{N}_X^{\alpha}Y = TX^{\mathbb{T}_{S(Y)_G}}\big|_Y .$$

This establishes both claims. $\square$

Corollary 3.7. *Let ψ be an irreducible almost periodic action of $\mathbb{R}^k$ on a smooth manifold X as in (1.1). For each $L \in \mathbb{RP}^{k-1}$, let $X^L \subset X$ be the fixed locus of the action $\psi|_L$. If X is compact, then the set*

$$\widetilde{\pi}_0^*\big(\mathrm{Crit}(\psi)\big) \equiv \big\{Z \in \pi_0(X^L) : L \in \mathbb{RP}^{k-1},\ Z \cap X^{\psi} \neq \emptyset\big\}$$

is finite.

Proof. Let $\rho: \mathbb{R}^k \longrightarrow \mathbb{T}$ and ψ' be as in (1.5). We can assume that the image of ρ is dense in $\mathbb{T}$ and so $X^{\psi} = X^{\psi'}$. For each $L \in \mathbb{RP}^{k-1}$, let $\mathbb{T}_L \subset \mathbb{T}$ be as in the proof of Corollary 3.5. For each subtorus $\mathbb{T}' \subset \mathbb{T}$, let $X^{\mathbb{T}'} \subset X$ be the fixed locus of the action $\psi'|_{\mathbb{T}'}$. In particular, $X^L = X^{\mathbb{T}_L}$. By Proposition 3.6(2), every element Z of $\widetilde{\pi}_0^*(\mathrm{Crit}(\psi))$ intersecting a topological component Y of $X^{\psi} = X^{\psi'}$ is thus the unique topological component of $X^{\mathbb{T}_S} \subset X$ for some $S \subset S(Y)$ intersecting Y. The number of subsets of $S \subset S(Y)$ is finite for each $Y \in \pi_0(X^{\psi})$. Since X is compact, $\pi_0(X^{\psi})$ is finite as well. $\square$

Proposition 3.8. *Let X be a smooth manifold.*

(1) The flow of a complete vector field ζ on X determines a smooth $\mathbb{R}$-action ψ on X by

$$\psi_0 = \mathrm{id}_X, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = \zeta\big(\psi_t(x)\big) \quad \forall\, t \in \mathbb{R},\ x \in X.$$

Conversely, a smooth $\mathbb{R}$-action ψ on X is the flow of the vector field ζ on X given by

$$\zeta(x)=\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x)\Big|_{t=0}\quad\forall\,x\in X. \tag{3.7}$$

In particular, $X^\psi=\{x\in X:\zeta(x)=0\}$.

(2) If ψ is a smooth $\mathbb{R}$-action on X with associated vector field ζ and $x\in X^\psi$, the linear $\mathbb{R}$-action $\mathrm{d}_x\psi$ on T_xX is the flow of the vector field

$$\nabla\zeta|_x\colon T_xX\longrightarrow T_xX,\qquad w\longrightarrow\nabla_w\zeta, \tag{3.8}$$

on T_xX, where ∇ is any connection in the vector bundle $TX\longrightarrow X$. If in addition J is a ψ-invariant endomorphism of this vector bundle, i.e.

$$\big\{\mathrm{d}_{x'}\psi_t\big\}^{-1}\circ J\circ\mathrm{d}_{x'}\psi_t=J\colon T_{x'}X\longrightarrow T_{x'}X\qquad\forall\,t\in\mathbb{R},\,x'\in X, \tag{3.9}$$

then

$$\nabla_{Jw}\zeta=J\nabla_w\zeta\qquad\forall\,w\in T_xX\,. \tag{3.10}$$

Proof. (1) If $\psi_t\colon X\longrightarrow X$ is the time t flow of $\zeta\in\Gamma(X;TX)$ for each $t\in\mathbb{R}$, then

$$\psi_{s+t}=\psi_s\circ\psi_t\colon X\longrightarrow X;$$

see [30, Theorem 1.48]. Thus, the map (1.1) is a group homomorphism and ψ is a smooth $\mathbb{R}$-action on X. Conversely, if ψ is a smooth $\mathbb{R}$-action on X and $\zeta\in\Gamma(X;TX)$ is given by (3.7), then

$$\psi_0=\mathrm{id}_X,\qquad\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x)=\frac{\mathrm{d}}{\mathrm{d}s}\psi_{s+t}(x)\bigg|_{s=0}=\frac{\mathrm{d}}{\mathrm{d}s}\psi_s\big(\psi_t(x)\big)\bigg|_{s=0}\equiv\zeta\big(\psi_t(x)\big);$$

the second equality above holds because (1.1) is a group homomorphism. Thus, ψ_t is the time t flow of ζ. The last claim in (1) follows immediately.

(2) Let $x\in X^\psi$ and $\gamma\colon(-\delta,\delta)\longrightarrow X$ be a smooth curve such that $\gamma(0)=x$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}_x\psi_t\big(\gamma'(0)\big)\bigg|_{t=0}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\psi_t\big(\gamma(s)\big)\big|_{s,t=0}=\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t\big(\gamma(s)\big)\bigg|_{s,t=0}=\frac{\mathrm{D}}{\mathrm{d}s}\zeta\big(\gamma(s)\big)\bigg|_{s=0}=\nabla_{\gamma'(0)}\zeta\,,$$

where $\mathrm{D}/\mathrm{d}s$ denotes the covariant derivative with respect to any torsion-free connection in TX; the penultimate equality above holds by the second claim in (1) for the $\mathbb{R}$-action ψ on X. By second claim of (1) for the $\mathbb{R}$-action $\mathrm{d}_x\psi$ on T_xX, $\mathrm{d}_x\psi_t$ is thus the time t flow of the vector field $\nabla\zeta|_x$ on T_xX given by (3.8) (which is independent of the choice of ∇ because $\zeta(x)=0$).

If in addition J is a ψ-invariant endomorphism of the vector bundle $TX\longrightarrow X$, then

$$\mathrm{d}_{x'}\psi_t(Jw)=J\mathrm{d}_{x'}\psi_t(w)\qquad\forall\,w\in T_{x'}X,\,x'\in X\,.$$

Setting $x'=x$ above, differentiating the resulting equation at $t=0$, and using the previous statement, we obtain (3.10). $\square$

Exercise 3.9. Suppose $k\in\mathbb{Z}^{\geq 0}$, ψ is a smooth $\mathbb{R}^k$-action on a smooth manifold X, and $x\in X$. Let $\mathbb{R}^k_x\subset\mathbb{R}^k$ be the largest linear subspace fixing x and $\mathbb{R}^c_x\subset\mathbb{R}^k$ be a complementary linear subspace. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_{tv}(x)\bigg|_{t=0}\neq 0\in T_xX \qquad \forall\ v\in\mathbb{R}^c_x-\{0\}.$$

Exercise 3.10. Let ψ be a nontrivial smooth action of $\mathbb{R}^k$ (resp. k-torus $\mathbb{T}^k\equiv(\mathbb{R}/\mathbb{Z})^k$) on a smooth manifold X as in (1.1). Show that there exist an irreducible smooth action ψ' of $\mathbb{R}^m$ (resp. $\mathbb{T}^m$) on X for some $m\in\mathbb{Z}^+$ and a full-rank real (resp. integer) $m\times k$-matrix A so that $\psi=\psi'\circ A$, i.e.

$$\psi_v=\psi'_{Av}\in\mathrm{Diff}(X) \qquad \forall\, v\in\mathbb{R}^k \quad \big(\text{resp. } v\in\mathbb{T}^k\big).$$

Furthermore, if the action ψ is almost periodic in the first case, then so is the action ψ'. *Hint.* Let G be a compact Lie group. For every $v\in T_{\mathbb{1}}G$, the closure of the one-parameter subgroup $\{\mathrm{e}^{tv}: t\in\mathbb{R}\}$ is a torus.

3.2 Group actions on symplectic manifolds

This section provides an analogue of Proposition 3.3 in the symplectic setting. In particular, Proposition 3.14(2) is an equivariant version of the Symplectic Tubular Neighborhood Theorem.

Exercise 3.11. Let (X,ω) be a symplectic manifold and ψ be a smooth action of a Lie group G on X as in (1.1).

(a) Suppose G is connected. Show that ψ preserves ω if and only if $\mathrm{d}(\iota_{\zeta_v}\omega)=0$ for all $v\in T_{\mathbb{1}}G$.

(b) Suppose ψ preserves ω. Show that

$$\mathrm{d}\big(\omega(\zeta_v,\zeta_{v'})\big)=-\iota_{[\zeta_v,\zeta_{v'}]}\omega \qquad \forall\ v,v'\in T_{\mathbb{1}}G.$$

(c) Suppose in addition G is connected and abelian. Let $\mu\colon X\longrightarrow T^*_{\mathbb{1}}G$ be a smooth map satisfying the first condition in (1.4). Show that μ is G-invariant if $G\approx\mathbb{R}$, or G is compact, or X is compact.

Hint: use Proposition 3.8(1) for the restriction of ψ to the one-parameter subgroup $\{\mathrm{e}^{tv}: t\in\mathbb{R}\}$ of G.

Exercise 3.12. Let ψ be a smooth action of a compact Lie group on a symplectic manifold (X,ω) as in (1.1). Show that there exists a ψ-invariant ω-compatible almost complex structure on X, i.e. an ω-compatible almost complex structure J on X such that

$$\big\{\mathrm{d}_x\psi_u\big\}^{-1}\circ J\circ\mathrm{d}_x\psi_u=J\colon T_xX\longrightarrow T_xX \qquad \forall\ u\in G,\ x\in X.$$

Hint: a Riemannian metric g and a nondegenerate 2-form ω on X determine an ω-compatible almost complex structure $J_{g,\omega}$ on X; see the proof of Proposition 2.3 in [32].

Exercise 3.13. Suppose ψ is a smooth action of a Lie group on a symplectic manifold (X,ω) as in (1.1), J is a ψ-invariant ω-compatible almost complex structure on X, and $Y\subset X$ is a Lagrangian submanifold. Show that the isomorphism

$$\Phi_{Y;J}\colon J(TY)\longrightarrow T^*Y, \qquad \Phi_{Y;J}(w)=\omega(\cdot,w),$$

of real vector bundles over Y is G-equivariant with respect to the actions $\mathrm{d}\psi$ and $\mathrm{d}\psi^*$ of Exercise 3.1.

Proposition 3.14. *Let ψ be a smooth action of a compact Lie group G on a symplectic manifold (X,ω) as in (1.1).*

(1) The fixed locus $X^{\psi}\subset X$ of ψ is a closed symplectic submanifold with $T(X^{\psi})=(TX)^{\mathrm{d}\psi}$.

(2) Suppose $Y\subset X$ is a closed submanifold preserved by ψ, $TY^c\subset TX|_Y$ is a subbundle complementary to TY and preserved by $\mathrm{d}\psi$, and $\widetilde{\omega}$ is a G-invariant closed 2-form on a neighborhood of Y in TY^c preserved by $\mathrm{d}\psi$. If

$$\widetilde{\omega}\big|_{T_y(TY^c)}=\omega\big|_{T_yX} \qquad \forall\, y\in Y\,, \tag{3.11}$$

there exists a G-equivariant tubular neighborhood identification $\Phi\colon\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\subset TY^c$ and $\Phi^*\omega=\widetilde{\omega}|_{\mathcal{U}}$.

Proof. (1) Let J be a G-invariant ω-compatible almost complex structure on X, as provided by Exercise 3.12. By Proposition 3.3(1), $X^{\psi}\subset X$ is a closed submanifold with $T(X^{\psi})=(TX)^{\mathrm{d}\psi}$. Since J is G-invariant, $J(TX^{\psi})\subset TX^{\psi}$ by (3.2). Since J is ω-compatible, $\omega(v,Jv)>0$ for all $v\in TX$ nonzero. Thus, $\omega|_{TX^{\psi}}$ is nondegenerate.

(2) Let $\Phi\colon\mathcal{U}\longrightarrow U$ be a G-equivariant tubular neighborhood identification for Y in X with $\mathcal{U}\subset TY^c$, as provided by Proposition 3.3(2). In particular, $\Phi^*\omega$ is a symplectic form on $\mathcal{U}$. By (3.11) and (3.1),

$$(\Phi^*\omega)\big|_{T_y(TY^c)}=\widetilde{\omega}\big|_{T_y(TY^c)} \qquad \forall\, y\in Y\,. \tag{3.12}$$

Since Φ is G-equivariant, the 2-form $\Phi^*\omega$ is G-invariant. Since the subset of $\mathcal{U}$ on which $\widetilde{\omega}$ is nondegenerate contains Y by (3.11) and is open and preserved by G, we can assume that the 2-form $\widetilde{\omega}$ is nondegenerate on $\mathcal{U}$ (by replacing $\mathcal{U}$ by its subset on which $\widetilde{\omega}$ is nondegenerate) .

Let $m_\tau\colon TY^c\longrightarrow TY^c$ be the scalar multiplication by τ as in (A.1) and $\zeta_{TY^c}\in\Gamma(TY^c;T(TY^c))$ be the canonical vertical vector field as in (A.5). Define a 1-form α on $\mathcal{U}$ by

$$\alpha=\int_0^1 m_\tau^*\big(\iota_{\tau^{-1}\zeta_{TY^c}}(\Phi^*\omega-\widetilde{\omega})\big)\mathrm{d}\tau.$$

By Exercise A.1 and (3.12),

$$\mathrm{d}\alpha=\Phi^*\omega-\widetilde{\omega},\quad \mathrm{d}\alpha\big|_{T(TY^c)|_Y}=0,\quad \alpha\big|_{T(TY^c)|_Y}=0,\quad \nabla\alpha\big|_{T(TY^c)|_Y}=0, \tag{3.13}$$

where ∇ is any connection in $T^*(TY^c)$.

By the second statement in (3.13) and the compactness of $[0,1]$,

$$\omega_t\equiv\widetilde{\omega}+t\,\mathrm{d}\alpha$$

is a symplectic form on a neighborhood $\mathcal{U}'$ of $Y\subset\mathcal{U}$ for every $t\in[0,1]$. For each $t\in[0,1]$, define

$$\xi_t\in\Gamma\big(\mathcal{U}';T\mathcal{U}'\big) \qquad \text{by} \quad \iota_{\xi_t}\omega_t=-\alpha. \tag{3.14}$$

By the third statement in (3.13), $\xi_t|_Y=0$. Since $[0,1]$ is compact, it follows that there exists a neighborhood $\mathcal{U}''$ of Y in $\mathcal{U}'$ so that the flow of ξ_t,

$$\psi_t\colon\mathcal{U}''\longrightarrow\mathcal{U}',\qquad \psi_0(w)=w\ \ \forall\,w\in\mathcal{U}'',\quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t=\xi_t\circ\psi_t,$$

is well-defined for every $t\in[0,1]$. By the first, third, and fourth statements in (3.13),

$$\widetilde{\omega}=\psi_t^*\omega_t,\quad \psi_t(w)=w\ \ \forall\,w\in\mathcal{U}'',\quad\text{and}\quad \mathrm{d}\psi_t|_{T(TY^c)|_Y}=\mathrm{id}_{T(TY^c)|_Y}\qquad \forall\,t\in[0,1]; \tag{3.15}$$

see Exercise 3.15 below for the first identity.

Since G is compact, the set

$$\mathcal{U}'''\equiv\bigcap_{u\in G}\mathrm{d}\psi_u(\mathcal{U}'')\subset\mathcal{U}'\subset TY^c$$

is a neighborhood of $Y\subset TY^c$ preserved by the G-action so that $\psi_1\!:\mathcal{U}'''\longrightarrow\mathcal{U}$ is a well-defined diffeomorphism onto an open subset of $\mathcal{U}$. Since the 2-forms $\Phi^*\omega$ and $\widetilde{\omega}$ are G-invariant, so are the 1-form α and the vector fields ξ_t. Thus, the smooth map ψ_1 is G-equivariant, as is the diffeomorphism

$$\Phi\circ\psi_1\!:\mathcal{U}'''\longrightarrow\Phi\big(\psi_1(\mathcal{U}''')\big)\subset U\subset X.$$

By the second and third statements in (3.15), this diffeomorphism is a tubular neighborhood identification for Y in X (because Φ is). By the first statement in (3.15), $\{\Phi\circ\psi_1\}^*\omega=\widetilde{\omega}|_{\mathcal{U}'''}$. $\square$

Exercise 3.15. Suppose X is a smooth manifold, $(\omega_t)_{t\in[0,1]}$ is a smooth family of symplectic forms on X, $(\xi_t)_{t\in[0,1]}$ is the smooth family of vector fields on X defined by

$$\mathrm{d}\big(\iota_{\xi_t}\omega_t\big)=-\frac{\mathrm{d}}{\mathrm{d}t}\omega_t,$$

and $\psi_t\!:\mathcal{U}\longrightarrow X$ is a flow of $(\xi_t)_{t\in[0,1]}$ on an open subset of X, i.e.

$$\psi_0(x)=x,\quad \frac{\mathrm{d}}{\mathrm{d}t}\big(\psi_t(x)\big)=\xi_t\big(\psi_t(x)\big)\qquad\forall\ x\in\mathcal{U},\ t\in[0,1].$$

Show that $\psi_t^*\omega_t=\omega_0|_{\mathcal{U}}$ for all $t\in[0,1]$. *Hint:* differentiate both sides and use Cartan's formula.

Example 3.16 (Symplectic Tubular Neighborhood Theorem)**.** Suppose ψ is a smooth action of a compact Lie group G on a symplectic manifold (X,ω) and Y is a symplectic submanifold of (X,ω) preserved by ψ. The restriction $\omega_Y^{\perp}$ of ω to the ω-symplectic complement

$$TY^{\omega}\equiv\big\{w\in TX|_Y:\omega(w,w')=0\ \forall\,w'\in TY\big\}\longrightarrow Y$$

of TY in $TX|_Y$ is then a G-invariant nondegenerate fiberwise 2-form. Let $\zeta_{TY^{\omega}}\in\Gamma(TY^{\omega};T(TY^{\omega}))$ be the canonical vertical vector field as in (A.5). A G-invariant connection ∇ in the real vector bundle $\pi\!:TY^{\omega}\longrightarrow Y$ extends $\omega_Y^{\perp}$ to a G-invariant 2-form $\omega_{\nabla}^{\perp}$ on (the total space of) TY^{ω}; see Exercise A.5. By (A.19), the G-invariant closed 2-form

$$\widetilde{\omega}_{\nabla}\equiv\pi^*\omega+\frac{1}{2}\mathrm{d}\big(\iota_{\zeta_{TY^{\omega}}}\omega_{\nabla}^{\perp}\big)$$

satisfies (3.11) with $\widetilde{\omega}=\widetilde{\omega}_{\nabla}$. If in addition $Y\subset X$ is a closed subspace, by Proposition 3.14(2) there then exists a G-equivariant tubular neighborhood identification $\Phi\!:\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\subset TY^{\omega}$ and $\Phi^*\omega=\widetilde{\omega}_{\nabla}|_{\mathcal{U}}$. If $Y\equiv\{x\}$ is a one-point set, $TY^{\omega}=T_xX$ and $\widetilde{\omega}_{\nabla}=\pi^*\omega_x$. This yields Corollary 3.17 below.

Corollary 3.17 (Darboux Theorem)**.** *Suppose ψ is a smooth action of a compact Lie group G on a symplectic manifold (X,ω) and $x\in X^{\psi}$. There exist a G-invariant tubular neighborhood $\mathcal{U}$ of 0 in T_xX and a G-equivariant diffeomorphism $\Phi\colon \mathcal{U}\longrightarrow U$ onto a neighborhood U of x in X such that*

$$\Phi(0)=x, \qquad \mathrm{d}_0\Phi=\mathrm{id}\colon T_0(T_xX)=T_xX\longrightarrow T_xX, \qquad \textit{and} \qquad \Phi^*\omega=\pi^*\omega_x|_{\mathcal{U}}\,,$$

where $\pi\colon T_xX\longrightarrow\{x\}$ is the projection.

Corollary 3.18 (Lagrangian Tubular Neighborhood Theorem)**.** *Suppose ψ is a closed action of a compact Lie group G on a symplectic manifold (X,ω), $Y\subset X$ is a compact Lagrangian submanifold preserved by ψ, and ω_{T^*Y} is the canonical symplectic form on T^*Y as in Exercise 2.10. There exists a G-equivariant diffeomorphism $\Phi\colon\mathcal{U}\longrightarrow U$ from an open neighborhood of Y in T^*Y onto an open neighborhood of Y in X so that*

$$\Phi(y)=y \quad \forall\, y\in Y \qquad \textit{and} \qquad \Phi^*\omega=\omega_{T^*Y}\big|_{\mathcal{U}}\,. \tag{3.16}$$

Proof. By Exercise 3.12, there exists a ψ-invariant ω-compatible almost complex structure J on X. By Exercise 3.13, the isomorphism

$$\Phi_{Y;J}\colon J(TY)\longrightarrow T^*Y, \qquad \Phi_{Y;J}(w)=\omega(\cdot,w),$$

of real vector bundles over Y is G-equivariant with respect to the actions $\mathrm{d}\psi$ and $\mathrm{d}\psi^*$ of Exercise 3.1. Along with the latter exercise, this implies that the closed 2-form $\Phi_{Y;J}^*\omega_{T^*X}$ on (the total space of) $J(TY)$ is G-invariant. By Exercise 2.13(c), this form satisfies (3.11). By Proposition 3.14(2), there thus exists a G-equivariant tubular neighborhood identification $\Phi\colon\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\subset J(TY)$ and $\Phi^*\omega=\Phi_{Y;J}^*\omega_{T^*Y}|_{\mathcal{U}}$. The map

$$\Phi\circ\Phi_{Y;J}^{-1}\colon \Phi_{Y;J}(\mathcal{U})\longrightarrow U$$

is then a G-equivariant diffeomorphism satisfying (3.16) with Φ replaced by $\Phi\circ\Phi_{Y;J}^{-1}$. □

Exercise 3.19 (Moser's Stability)**.** Let X be a closed manifold. Suppose $p\in\mathbb{Z}^{\ge0}$ and $(\omega_t)_{t\in[0,1]}$ is a smooth family of cohomologous closed smooth p-forms on X, i.e. $[\omega_t]=[\omega_0]\in H^p_{\mathrm{deR}}(X)$ for every $t\in[0,1]$.

(a) Show that there exists a smooth family $(\eta_t)_{t\in[0,1]}$ of $(p-1)$-forms on X such that

$$\omega_t-\omega_0=\mathrm{d}\eta_t \qquad \forall\ t\in[0,1]\,.$$

(b) With η_t as in (a), suppose also that $(\zeta_t)_{t\in[0,1]}$ is a smooth family of vector fields on X satisfying

$$\iota_{\zeta_t}\omega_t=-\frac{\mathrm{d}}{\mathrm{d}t}\eta_t \qquad \forall\ t\in[0,1]$$

and $\psi_t\colon X\longrightarrow X$ for $t\in[0,1]$ is its flow. Show that $\psi_t^*\omega_t=\omega_0$ for all $t\in[0,1]$.

(c) Suppose ω_t is a symplectic form on X for every $t\in[0,1]$. Show that there exists a diffeomorphism $\psi\colon X\longrightarrow X$ such that $\psi^*\omega_1=\omega_0$.

(d) Suppose X is connected and oriented and Ω_0,Ω_1 are volume forms on X. Show that there exists a diffeomorphism $\psi\colon X\longrightarrow X$ such that $\psi^*\Omega_1=\Omega_0$ if and only if $\displaystyle\int_X\Omega_0=\int_X\Omega_1$.

The assumptions that X is compact and the symplectic forms ω_t are cohomologous necessary for the conclusion of Exercise 3.19(c). For example, $\mathbb{C}^n$ with $n\ge 2$ admits a symplectic structure ω so that $(\mathbb{C}^n,\omega)$ is not symplectomorphic to $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$; see [14, 0.4.A$_2'$]. A smooth family $(\omega_t)_{t\in\mathbb{R}}$ of symplectic forms on a closed 8-dimensional smooth manifold $\widetilde{Y}$ is constructed in [20] so that all forms ω_k with $k\in\mathbb{Z}$ are cohomologous and the symplectic manifolds $(\widetilde{Y},\omega_k)$ and $(\widetilde{Y},\omega_\ell)$ with $k,\ell\in\mathbb{Z}$ are symplectomorphic if and only if $|k|=|\ell|$; see Theorem 2.1 in [20]. In the case of Exercise 3.19(d), the equality of the integrals implies that

$$\Omega_t \equiv (1-t)\Omega_0+t\Omega_1, \qquad t\in[0,1],$$

is a smooth family of cohomologous volume forms.

Remark 3.20. The analogue of Proposition 3.14(2) in [21] is Lemma 3.2.1, which unnecessarily requires Y (Q in [21]) to be compact. As a consequence, the Symplectic and Lagrangian Tubular Neighborhood Theorems, i.e. Example 3.16 and Corollary 3.18 above, are restricted to compact submanifolds in [21]; see Theorems 3.4.10 and 3.4.13 in [21]. Even if one is interested only in compact symplectic manifolds, the Symplectic Tubular Neighborhood Theorem without the compactness restriction is needed for the proof of Proposition 6.13. The latter is a key step in the proof of Delzant's Theorem, Theorem 3, following the modern efficient approach sketched in [19, 22]; see page 62.

3.3 Hamiltonian group actions

We next obtain structural results for Hamiltonian group actions and their moment maps, in particular Proposition 4.5 and Corollary 3.28.

Exercise 3.21. Suppose G is a Lie group, (X,ω,ψ,μ) is a Hamiltonian G-manifold, and $x\in X$. For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be as in (1.2). Show that

$$\begin{aligned}\ker \mathrm{d}_x\mu &= \big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}^{\omega} \equiv \big\{w\in T_xX\colon \omega\big(w,\zeta_v(x)\big)=0\ \forall\, v\in T_{\mathbb{1}}G\big\},\\ \operatorname{Im}\mathrm{d}_x\mu &= \operatorname{Ann}\big(\big\{v\in T_{\mathbb{1}}G\colon \zeta_v(x)=0\big\}\big) \equiv \big\{\alpha\in T_{\mathbb{1}}^*G\colon \alpha(v)=0\ \forall\, v\in T_{\mathbb{1}}G\,\text{s.t.}\,\zeta_v(x)=0\big\}.\end{aligned}\tag{3.17}$$

Conclude that

(a) the G-orbit $Gx\subset X$ of x is open if and only if $\mathrm{d}_x\mu$ is injective;

(b) the stabilizer $\mathrm{Stab}_x(\psi)\subset G$ of x is discrete if and only if $\mathrm{d}_x\mu$ is surjective.

Exercise 3.22. Suppose $\mathbb{T}$ is a torus and (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold so that (1.7) holds, the action ψ is free, and the fibers of μ are connected.

(a) Show that $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ is an open subset, $\mu\colon X\longrightarrow\mu(X)$ is a principal $\mathbb{T}$-bundle, and the fibers of μ are Lagrangian submanifolds of (X,ω), i.e.

$$\dim\mu^{-1}(\alpha)=n \quad\text{and}\quad \omega\big|_{T\mu^{-1}(\alpha)}=0 \qquad \forall\ \alpha\in\mu(X).$$

(b) Let η be a 1-form on $\mu(X)$. Show that the vector field ζ_η on X defined by $\iota_{\zeta_\eta}\omega=\mu^*\eta$ is μ-vertical, i.e.

$$\mathrm{d}\mu\big(\zeta_\eta\big)=0\in\Gamma\big(\mu(X);T\mu(X)\big).$$

(c) Let $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, \mu_{\mathbb{T}})$ be the Hamiltonian $\mathbb{T}$-manifold of Exercise 2.11, with $k=n$. Suppose $s\colon \mu(X)\longrightarrow X$ is a (smooth) Lagrangian section of μ, i.e. $\mu\circ s=\mathrm{id}_{\mu(X)}$ and $s^*\omega=0$. Show that the map

$$\Phi\colon \mu(X)\times\mathbb{T} \longrightarrow X, \qquad \Phi(\alpha,u) = \psi_u\big(s(\alpha)\big),$$

is a $\mathbb{T}$-equivariant diffeomorphism such that $\Phi^*\omega = \omega_{\mathbb{T}}|_{\mu(X)\times\mathbb{T}}$ and $\mu\circ\Phi=\mu_{\mathbb{T}}|_{\mu(X)\times\mathbb{T}}$. *Hint:* choose a $\mathbb{Z}$-basis $v_1,\ldots,v_n\in T_{\mathbb{1}}\mathbb{T}$ for the lattice $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\subset T_{\mathbb{1}}\mathbb{T}$ and replace μ by the corresponding Hamiltonian $H\colon X\longrightarrow\mathbb{R}^n$ and $(\mathbb{T}_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}, \omega_{\mathbb{T}}, \psi_{\mathbb{T}}, H_{\mathbb{T}})$ by the Hamiltonian $\mathbb{T}$-manifold $(\mathbb{R}^n\times\mathbb{T}^n, \omega_n, \psi_n, H_n)$ as in Exercise 2.8.

Exercise 3.23. Suppose G is a positive-dimensional Lie group, (X,ω) is a compact positive-dimensional symplectic manifold, and ψ is a smooth G-action on (X,ω).

(a) Suppose ψ is a Hamiltonian action. Show that the ψ-fixed locus X^ψ contains at least 2 points.

(b) Give an example of a compact positive-dimensional symplectic manifold (X,ω) and an action ψ on (X,ω) so that $X^\psi=\emptyset$.

Exercise 3.24. Suppose $\mathbb{T}$ is a torus, (X,ω,ψ,μ) is a compact Hamiltonian $\mathbb{T}$-manifold, $x\in X$, and $Z\in\pi_0(X^{\mathbb{T}_\psi(x)})$ is the topological component of the $\psi|_{\mathbb{T}_\psi(x)}$-fixed locus containing x. For each $Y\in\pi_0(X^\psi)$, let $S(Y)\subset T_{\mathbb{1}}^*\mathbb{T}$ be as in Proposition 3.6(1). Show that

(a) $Z\cap X^\psi\neq\emptyset$;

(b) if $Y\in\pi_0(X^\psi)$ and $Y\subset Z$, then $\mathbb{T}_x(\psi)=\mathbb{T}_S$ for some $S\subset S(Y)$.

Hint: use Exercise 3.23(a) and Proposition 3.6(2).

Exercise 3.25. Suppose G is a Lie group, (X,ω) is a symplectic manifold, ψ is a smooth G-action on (X,ω), J is an ω-compatible almost complex structure on X, and $x\in X$. For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(X;TX)$ be as in (1.2).

(a) Let $\mu\colon X\longrightarrow T_{\mathbb{1}}^*G$ be a smooth map satisfying the first condition in (1.4) and $g(\cdot,\cdot)\equiv\omega(\cdot,J\cdot)$ be the Riemannian metric on X determined by ω and J. Show that

$$\nabla^g\mu_v = -J\zeta_v \in \Gamma(X;TX) \qquad \forall\, v\in T_{\mathbb{1}}G. \tag{3.18}$$

(b) Let $\mu : X \longrightarrow T_{\mathbb{1}}^*G$ be a G-invariant smooth map satisfying the first condition in (1.4). Show that

$$\big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}\cap\big\{J\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\} = \{0\}\subset T_xX.$$

Suppose in addition that $\alpha\equiv\mu(x)\in T_{\mathbb{1}}^*G$ is a regular value of μ and thus $\mu^{-1}(\alpha)\subset X$ is a smooth submanifold. Show that

$$T_xX = T_x\big(\mu^{-1}(\alpha)\big)\oplus\big\{J\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}. \tag{3.19}$$

(c) Give an example of a positive-dimensional symplectic manifold (X,ω) and an action ψ on (X,ω) so that

$$\big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\} = \big\{J\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\} = T_xX.$$

Exercise 3.26. Suppose (X,ω) is a symplectic manifold, ψ is a smooth action of $\mathbb{R}^k$ on (X,ω), $X'\subset X$ is an ω-symplectic manifold preserved ϕ so that the inclusion $i\colon X'\longrightarrow X$ is a homotopy equivalence, and $\mu'\colon X'\longrightarrow T_0^*\mathbb{R}^k$ is a moment map for the restriction of the action ψ to X'. Show that μ' extends to a moment map $\mu\colon X\longrightarrow T_0^*\mathbb{R}^k$ for ψ. *Hint:* first show this for $k=1$.

Proposition 3.27. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a Hamiltonian $\mathbb{R}^k$-manifold, ψ' is a smooth action of a torus $\mathbb{T}$ on X, $\rho\colon\mathbb{R}^k\longrightarrow\mathbb{T}$ is a homomorphism with dense image so that $\psi=\psi'\circ\rho$, $Y\subset X$ is a topological component of $X^\psi=X^{\psi'}$, and J is a ψ-invariant (or equivalently ψ'-invariant) ω-compatible almost complex structure on X. Let $S(Y)\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and $\mathcal{N}_X^\alpha Y\subset TX|_Y$ for each $\alpha\in S(Y)$ be as in Proposition 3.6(1) with ψ replaced by ψ' so that the complex structure on $\mathcal{N}_X^\alpha Y$ is induced by J. For every $y\in Y$, there exists a $\mathbb{T}$-equivariant tubular neighborhood identification $\Phi_y\colon\mathcal{U}_y\longrightarrow U_y$ for y in X such that*

$$\Phi_y^*\omega=\omega_y|_{\mathcal{U}_y}\quad\text{and}\quad \mu\big(\Phi_y\big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)\big)=\mu(Y)+\pi\sum_{\alpha\in S(Y)}|w_\alpha|^2\rho^*\alpha$$
$$\forall\ \big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)\in\mathcal{U}_y\subset T_yX=T_yY\oplus\bigoplus_{\alpha\in S(Y)}\mathcal{N}_X^\alpha Y\big|_y\,, \tag{3.20}$$

where $|\cdot|$ is the norm on TX with respect to the metric $g(\cdot,\cdot)\equiv\omega(\cdot,J\cdot)$. If in addition X is closed and connected, then

$$\mu(X)\subset\mathcal{C}_{\mu(Y)}\big(\rho^*S(Y)\big)\equiv\Big\{\mu(Y)+\sum_{\alpha\in S(Y)}t_\alpha\rho^*\alpha: t_\alpha\in\mathbb{R}^{\ge0}\ \forall\,\alpha\in S(Y)\Big\}. \tag{3.21}$$

Proof. By Corollary 3.17 with ψ replaced by ψ', there exists a $\mathbb{T}$-equivariant (or equivalently $\mathbb{R}^k$-equivariant) tubular neighborhood identification $\Phi_y\colon\mathcal{U}_y\longrightarrow U_y$ for y in X satisfying the first condition in (3.20). By Proposition 3.6(1), the complex vector space (T_yX,J_y) splits as

$$T_yX|_Y=T_yY\oplus\bigoplus_{\alpha\in S(Y)}\mathcal{N}_X^\alpha Y\big|_y$$

with the $\mathbb{T}$-action $\mathrm{d}_{y'}\psi'$ given by (3.6) with ψ replaced by ψ'. By Example 2.6, a moment map for this action with respect to ω_y is

$$\mu'\colon T_yX\longrightarrow T_{\mathbb{1}}^*\mathbb{T},\quad \mu'\big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)\big)=\pi\sum_{\alpha\in S(Y)}|w_\alpha|^2\alpha\ \ \forall\ \big(w_0,(w_\alpha)_{\alpha\in S(Y)}\big)\in T_yY\oplus\bigoplus_{\alpha\in S(Y)}\mathcal{N}_X^\alpha Y\big|_y\,.$$

Since a moment map is unique up to an additive constant on each connected component of the domain, it follows that

$$\Phi_y^*\mu=\mu(y)+\rho^*\circ\mu'\big|_{\mathcal{U}_y}\colon\mathcal{U}_y\longrightarrow T_0^*\mathbb{R}^k\,.$$

This establishes the second condition in (3.20).

Suppose in addition that X is closed and connected and $\eta_0\in T_0^*\mathbb{R}^k-\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$. Thus, $\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$ is contained in a (closed) half-space in $T_0^*\mathbb{R}^k$ and there exists $v\in T_0\mathbb{R}^k$ so that

$$\eta_0(v)<\inf\big\{\eta(v)\colon\eta\in\mathcal{C}_{\mu(Y)}\big(\rho^*S(Y)\big)\big\}=\big\{\mu(Y)\big\}(v)\equiv\mu_v(Y). \tag{3.22}$$

By the second equality in (3.20), this implies that $\{\rho^*\alpha\}(v)\ge 0$ for all $\alpha\in S(Y)$. Thus,

$$\mu_v(Y)=\inf\big\{\mu_v(x)\colon x\in U_y\big\}.$$

Combining this with Proposition 4.5(2), we conclude that

$$\mu_v(Y)=\inf\big\{\mu_v(x)\equiv\{\mu(x)\}(v)\colon x\in X\big\}\qquad\forall\,y\in Y.$$

Along with (3.22), this implies that $\eta_0\notin\mu(X)$ and establishes (3.21). $\square$

Suppose ψ is a smooth action of $\mathbb{R}^k$ on a smooth manifold X. For $y\in X$, let $\mathbb{R}^k_y\subset\mathbb{R}^k$ be the largest linear subspace preserving y and $\mathbb{R}^c_y\subset\mathbb{R}^k$ be a complementary linear subspace. The decomposition $\mathbb{R}^k=\mathbb{R}^k_y\oplus\mathbb{R}^c_y$ induces decompositions

$$T_0\mathbb{R}^k=T_0\mathbb{R}^k_y\times T_0\mathbb{R}^c_y\qquad\text{and}\qquad \mu\equiv\big(\mu_y,\mu^c_y\big)\colon X\longrightarrow T_0^*\mathbb{R}^k_y\times T_0^*\mathbb{R}^c_y=T_0^*\mathbb{R}^k\,,\tag{3.23}$$

for any map $\mu: X\longrightarrow T_0^*\mathbb{R}^k$. If μ is a moment map for the action ψ on X with respect to a symplectic form ω, then μ_y and μ^c_y are moment maps for the $\mathbb{R}^k_y$-action $\psi_y\equiv\psi|_{\mathbb{R}^k_y}$ and $\mathbb{R}^c_y$-action $\psi^c_y\equiv\psi|_{\mathbb{R}^k_c}$, respectively, on X with respect to ω. If the action ψ is almost periodic, then so are the actions ψ_y and ψ^c_y. By Proposition 3.14(1), $(X^{\psi_y},\omega|_{X^{\psi_y}},\psi^c_y,\mu^c_y)$ is then a Hamiltonian $\mathbb{R}^c_y$-manifold. By Exercises 3.21(b) and 3.9 with (X,ω,ψ,μ) replaced by $(X^{\psi_y},\omega|_{X^{\psi_y}},\psi^c_y,\mu^c_y)$, the differential

$$\mathrm{d}_y\mu^c_y\colon T_yX^{\psi_y}\longrightarrow T_0^*\mathbb{R}^c_y\tag{3.24}$$

is surjective in this case.

Corollary 3.28. *Suppose $k\in\mathbb{Z}^+$, (X,ω,ψ,μ) is a Hamiltonian $\mathbb{R}^k$-manifold, the $\mathbb{R}^k$-action ψ is almost periodic, $y\in X$, and $\mathbb{R}^k_y,\mathbb{R}^c_y\subset\mathbb{R}^k$, μ_y, μ^c_y, and ψ_y are as above. There exist a finite subset $S(y)\subset T_0^*\mathbb{R}^c_y$, neighborhoods $\mathcal{U}_{y;1}$ of y in X^{ψ_y}, $\mathcal{U}_{y;2}$ of 0 in $\mathbb{C}^{S(y)}$, and U_y of y in X, and a diffeomorphism $\Phi_y\colon\mathcal{U}_{y;1}\times\mathcal{U}_{y;2}\longrightarrow U_y$ such that*

$$\big|S(y)\big|\le(\dim X)/2-\dim\mathbb{R}^c_y\,,\tag{3.25}$$

$$\begin{gathered}\Phi_y(y',0)=y',\quad \mu^c_y\big(\Phi_y(y',w)\big)=\mu^c_y(y'),\\ \mu_y\big(\Phi_y\big(y',(w_\alpha)_{\alpha\in S(y)}\big)\big)=\mu_y(y)+\pi\sum_{\alpha\in S(y)}|w_\alpha|^2\alpha\qquad\forall\,y'\in\mathcal{U}_{y;1},\ w\equiv(w_\alpha)_{\alpha\in S(y)}\in\mathcal{U}_{y;2}\subset\mathbb{C}^{S(y)}\,.\end{gathered}$$

In particular, the map μ is locally convex. If in addition X is closed and connected, then there exist a cone $\mathcal{C}_y(\psi)\subset T_0^*\mathbb{R}^k$ with vertex at $\mu(y)$ and a neighborhood $U_y\subset X$ of y so that $\mu(X)\subset\mathcal{C}_y(\psi)$ and the restriction $\mu\colon U_y\longrightarrow\mathcal{C}_y(\psi)$ is an open map.

Proof. Let $Y\in\pi_0(X^{\psi_y})$ be the connected component of ψ_y-fixed locus containing y and ψ^c_y be as above. With $S(Y)$ and ρ as in Proposition 3.27 with (ψ,μ) replaced by (ψ_y,μ_y), let

$$S(y)=\rho^*S(Y)\subset T_{\mathbb{1}}^*\mathbb{R}^k_y.$$

By Proposition 3.14(1), $Y\subset X$ is a symplectic submanifold with $\mu_y(Y)=\mu_y(y)$. Since the actions ψ_y and ψ^c_y commute, X^{ψ_y} is preserved by ψ^c_y. Since the differential (3.24) is surjective, the second equations in (3.17) and in (3.18) applied to $(Y,\omega|_Y,\psi^c_y|_Y,\mu^c_y|_Y)$ imply that

$$\dim Y\ge 2\dim\mathbb{R}^c_y\quad\Longrightarrow\quad 2\big|S(y)\big|\le\dim T_yX-\dim T_yY\le\dim X-2\dim\mathbb{R}^c_y\,.$$

This establishes (3.25). The remainder of the first claim of the corollary follows from the first statement of Proposition 3.27 with (ψ,μ) replaced by (ψ_y,μ_y) and Exercise 3.29 below with $k=\dim Y$, $\ell=\operatorname{codim} Y$, $m=\mathbb{R}^c_y$, and $f=\mu^c_y$. Along with Exercises 1.8-1.10, this claim implies the convexity claim.

Suppose in addition that X is closed and connected. By the second statement of Proposition 3.27,

$$\mu(X)\subset\mathcal{C}_y(\psi)\equiv\mathcal{C}_{\mu(y)}\big(S(y)\big)\times T^*_0\mathbb{R}^c_y\subset T^*_0\mathbb{R}^k\,.$$

By the first claim of the corollary and Exercises 1.8-1.10, there exists a neighborhood $U_y\subset X$ of y so that the restriction $\mu\colon U_y\longrightarrow\mathcal{C}_y(\psi)$ is an open map. $\square$

Exercise 3.29. Suppose $k,\ell,m\in\mathbb{Z}^{\ge0}$ and $f:\mathbb{R}^k\times\mathbb{R}^\ell\longrightarrow\mathbb{R}^m$ is a smooth function so that the restriction of the differential $\mathrm{d}_{(0,0)}f$ to $\mathbb{R}^k\times\{0\}$ is surjective. Show that there exist neighborhoods $\mathcal{U}_1$ of $0\in\mathbb{R}^k$ and $\mathcal{U}_2$ of $0\in\mathbb{R}^\ell$ and a smooth map

$$\phi\colon\mathcal{U}_1\times\mathcal{U}_2\longrightarrow\mathbb{R}^k\qquad\text{s.t.}\quad \phi(x,0)=x,\ \ f\big(\phi(x,w),w\big)=f(x,0)\ \ \forall\,x\in\mathcal{U}_1,\ w\in\mathcal{U}_2,$$

and for each $w\in\mathcal{U}_2$ the map $\mathcal{U}_1\longrightarrow\mathbb{R}^k$, $x\longrightarrow\phi(x,w)$, is a diffeomorphism onto an open subset of $\mathbb{R}^k$. *Hint:* assume that the restriction of $\mathrm{d}_{(0,0)}f$ to $\mathbb{R}^m\times\{0\}\times\{0\}\subset\mathbb{R}^k\times\{0\}$ is surjective; show that there exist neighborhoods $\mathcal{U}_1$ of $0\in\mathbb{R}^k$ and $\mathcal{U}_2$ of $0\in\mathbb{R}^\ell$ so that for each $w\in\mathcal{U}_2$ the map

$$\Phi_w\colon\mathcal{U}_1\longrightarrow\mathbb{R}^k,\quad \Phi_w(x_1,x_2)=\big(f\big((x_1,x_2),w\big),x_2\big)\ \ \forall\,(x_1,x_2)\in\mathcal{U}_1\subset\mathbb{R}^m\times\mathbb{R}^{k-m},$$

is a diffeomorphism onto an open subset of $\mathbb{R}^k$.

Proposition 3.30. *Suppose $\mathbb{T}$ is a torus and (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold so that (1.7) holds, the action ψ is free, the fibers of μ are connected, and $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}$ is contractible. Let $(T^*_{\mathbb{1}}\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ be the Hamiltonian $\mathbb{T}$-manifold of Exercise 2.11. Then $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}$ is an open subset and there exists a $\mathbb{T}$-equivariant diffeomorphism*

$$\Phi\colon\mu(X)\times\mathbb{T}\longrightarrow X\qquad\textit{s.t.}\qquad \Phi^*\omega=\omega_{\mathbb{T}}\big|_{\mu(X)\times\mathbb{T}},\quad \mu\circ\Phi=\mu_{\mathbb{T}}\big|_{\mu(X)\times\mathbb{T}}.$$

Proof. The subset $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}$ is open by Exercise 3.21(b). By Exercise 3.22(c), it remains to show that μ admits a Lagrangian section $\mu(X)\longrightarrow X$. Since $\mu(X)\subset T^*_{\mathbb{1}}\mathbb{T}$ is contractible, μ admits a section $s\colon\mu(X)\longrightarrow X$ and $s^*\omega=\mathrm{d}\eta$ for some 1-form η on $\mu(X)$. Let $\zeta_\eta\in\Gamma(X;TX)$ be the μ-vertical vector field of Exercise 3.22(b). Since the fibers of μ are compact and the vector field ζ_η is vertical, the flow of $-\zeta_\eta$,

$$\psi_t\colon X\longrightarrow X,\qquad \psi_0=\mathrm{id}_X,\quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_t=-\zeta_\eta\circ\psi_t\,,$$

is defined for all $t\in\mathbb{R}$, $\psi_t\circ s\colon\mu(X)\longrightarrow X$ is a section of μ for every $t\in\mathbb{R}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega=\psi_t^*\big(\mathcal{L}_{-\zeta_\eta}\omega\big)=\psi_t^*\big(\mathrm{d}(\iota_{-\zeta_\eta}\omega)+\iota_{-\zeta_\eta}(\mathrm{d}\omega)\big)=\psi_t^*\big(\mathrm{d}(\mu^*(-\eta))+0\big)=-\mathrm{d}(\mu^*\eta)=-\mu^*s^*\omega,$$

where $\mathcal{L}$ is the Lie derivative; the second equality above holds by Cartan's formula. Thus,

$$s^*\psi_1^*\omega=s^*\big(\omega-\mu^*s^*\omega\big)=0,$$

i.e. $\psi_1\circ s\colon\mu(X)\longrightarrow X$ is a Lagrangian section of μ. $\square$

4 Morse-Bott Theory

4.1 Definitions and notation

Let X be a smooth manifold and $H: X \longrightarrow \mathbb{R}$ be a smooth function. If $x \in \mathrm{Crit}(H)$, then the gradient $\nabla^g H$ of H with respect to any Riemannian metric g vanishes at x and the Hessian

$$\nabla^2 H\big|_x \equiv \nabla\big(\nabla^g H\big)\big|_x : T_x X \longrightarrow T_x X, \qquad \nabla^2 H\big|_x(w) = \nabla_w\big(\nabla^g H\big), \tag{4.1}$$

of H at x does not depend on the choice of a connection ∇ in TX (it does depend on the metric g though). If in addition ξ, ξ' are vector fields on X and ∇ is the Levi-Civita connection of g, then

$$\begin{aligned} g\big(\nabla^2 H\big|_x(\xi(x)), \xi'(x)\big) &\equiv g\big(\nabla_{\xi(x)}\nabla^g H, \xi'(x)\big) = \big\{\xi(x)\big\}\big(g(\nabla^g H, \xi')\big) - g\big(\nabla^g H|_x, \nabla_{\xi(x)}\xi'\big) \\ &= \big\{\xi(x)\big\}\big(\mathrm{d}H(\xi')\big) - \mathrm{d}_x H\big(\nabla_{\xi(x)}\xi'\big) = \big\{\xi(x)\big\}\big(\xi'(H)\big) - \big\{\nabla_{\xi(x)}\xi'\big\}(H) \\ &= \big\{\xi'(x)\big\}\big(\xi(H)\big) + \big\{[\xi, \xi'](x)\big\}(H) - \big\{\nabla_{\xi(x)}\xi'\big\}(H) \\ &= \big\{\xi'(x)\big\}\big(\xi(H)\big) - \big\{\nabla_{\xi'(x)}\xi\big\}(H) = g\big(\nabla^2 H\big|_x(\xi'(x)), \xi(x)\big). \end{aligned}$$

Thus, the linear automorphism (4.1) is symmetric with respect to the metric g and therefore diagonalizable. We denote by

$$\mathrm{E}_x^0(H), \mathrm{E}_x^-(H), \mathrm{E}_x^+(H) \subset T_x X \qquad \text{and} \qquad n_x^0(H), n_x^-(H), n_x^+(H) \in \mathbb{Z}^{\ge 0}$$

the nullspace of $\nabla^2 H|_x$, the negative eigenspace of $\nabla^2 H|_x$, the positive eigenspace of $\nabla^2 H|_x$, and their respective dimensions. In particular,

$$T_x X = \mathrm{E}_x^0(H) \oplus \mathrm{E}_x^-(H) \oplus \mathrm{E}_x^+(H) \qquad \text{and} \qquad \dim X = n_x^0(H) + n_x^-(H) + n_x^+(H).$$

Exercise 4.1. Let X be a smooth manifold, $H: X \longrightarrow \mathbb{R}$ be a smooth function, and $x \in \mathrm{Crit}(H)$. Show that

(a) the negative and positive eigenspaces $\mathrm{E}_x^-(H), \mathrm{E}_x^+(H) \subset T_x X$ of $\nabla^2 H|_x$ depend on the choice of a Riemannian metric g on X, but

(b) their dimensions $n_x^-(H), n_x^+(H)$ and the nullspace $\mathrm{E}_x^0(H) \subset T_x X$ of $\nabla^2 H|_x$ do not.

Definition 4.2. Let X be a smooth manifold. A smooth function $H: X \longrightarrow \mathbb{R}$ is Morse-Bott if $\mathrm{Crit}(H) \subset X$ is a closed submanifold of X with $T_x Y = E_x^0(H)$ for all $Y \in \pi_0(\mathrm{Crit}(H))$ and $x \in Y$.

If $H: X \longrightarrow \mathbb{R}$ is a Morse-Bott function and $Y \in \pi_0(\mathrm{Crit}(H))$, $H|_Y$ is constant. Furthermore, the numbers $n_x^-(H), n_x^+(H)$ do not depend on $x \in Y$; we denote them by $n_Y^-(H), n_Y^+(H)$, respectively. The subspaces $\mathrm{E}_x^-(H), \mathrm{E}_x^+(H)$ of $T_x X$ form subbundles $\mathrm{E}_Y^-(H), \mathrm{E}_Y^+(H)$ of $TX|_Y$ so that

$$TX|_Y = TY \oplus \mathrm{E}_Y^-(H) \oplus \mathrm{E}_Y^+(H).$$

Exercise 4.3. Suppose X is a smooth manifold, $H: X \longrightarrow \mathbb{R}$ is a Morse-Bott function, and $Y \in \pi_0(\mathrm{Crit}(H))$. Show that H reaches a local minimum (resp. maximum) on Y if and only if $n_Y^-(H) = 0$ (resp. $n_Y^+(H) = 0$).

Exercise 4.4. Suppose $H: X \longrightarrow \mathbb{R}$ and $Y \subset \mathrm{Crit}(H)$ are in Exercise 4.3 and $Z \subset X$ is a smooth submanifold transverse to the closed submanifold $Y \subset X$. Show that $Y \cap Z$ is a closed submanifold of Z, is an open subset of $\mathrm{Crit}(H|_Z)$, and

$$T_x(Y \cap Z) = T_x Y \cap T_x Z = E_x^0\big(H|_Z\big), \quad n_x^\pm\big(H|_Z\big) = n_x^\pm(H) \qquad \forall\, x \in Y \cap Z.$$

Proposition 4.5. *Let ψ be an almost periodic $\mathbb{R}$-action on a symplectic manifold (X,ω). If $H\colon X\longrightarrow\mathbb{R}$ is a Hamiltonian for ψ, then $\mathrm{Crit}(H)\subset X$ is a closed symplectic submanifold and H is a Morse-Bott function with $n_x^{\pm}(H)\in 2\mathbb{Z}^{\ge 0}$ for every $x\in\mathrm{Crit}(H)$. If in addition X is compact and connected, then*

(2) H has a unique local minimum and a unique local maximum;

(3) $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$.

Proof. Let $\rho:\mathbb{R}\longrightarrow\mathbb{T}$ and ψ' be as in (1.5) and $\zeta\in\Gamma(X;TX)$ be the generating vector field for the ψ-action as in (3.7). We can assume that the image of ρ is dense in $\mathbb{T}$ and so $X^{\psi}=X^{\psi'}$. By Exercise 3.12, there exist a ψ-invariant ω-compatible almost complex structure J on X. Let $g(\cdot,\cdot)\equiv\omega(\cdot,J\cdot)$ be the Riemannian metric on X determined by ω and J. By (2.1) and (2.5), the gradient of H with respect to g is then given by

$$\nabla^g H=-J\zeta\in\Gamma(X;TX). \tag{4.2}$$

By (2.5) and Proposition 3.8(1),

$$\mathrm{Crit}(H)\equiv\big\{x\in X\colon \mathrm{d}_xH=0\big\}=\big\{x\in X\colon \zeta(x)=0\big\}=X^{\psi}=X^{\psi'}\,. \tag{4.3}$$

Along with (4.1), (4.2), and (3.10), this implies that the Hessian ∇^2H of H satisfies

$$\nabla^2H\big|_x(w)=-J\nabla_w\zeta,\quad \nabla^2H\big|_x(Jw)=J\nabla^2H\big|_x(w)\qquad\forall\ w\in T_xX,\ x\in\mathrm{Crit}(H)\,. \tag{4.4}$$

By (4.3), Propositions 3.14(1) and 3.8(2), and the first equation in (4.4), $\mathrm{Crit}(H)\subset X$ is thus a closed symplectic submanifold of (X,ω) with

$$T_xY=(T_xX)^{\mathrm{d}\psi'}=(T_xX)^{\mathrm{d}\psi}=\big\{w\in T_xX\colon \nabla_w\zeta=0\big\}=E_x^0(H)\quad\forall\ Y\in\pi_0\big(\mathrm{Crit}(H)\big),\ x\in Y.$$

By the second equation in (4.4), the subspaces $\mathrm{E}_x^{\pm}(H)\subset T_xX$ are preserved by J for every $x\in\mathrm{Crit}(H)$ and thus $n_x^{\pm}(H)\in 2\mathbb{Z}^{\ge 0}$ for every $x\in\mathrm{Crit}(H)$. The remaining claims of the proposition now follow immediately from Proposition 4.8. $\square$

The second proof of Proposition 4.8 is based on standard properties of gradient flows of Morse-Bott functions. As these properties are also used in the proof of Theorem 2(2) in Section 5.1, we collect them in Proposition 4.7 below and justify at the end of this section.

Exercise 4.6. Suppose (X,g) is a compact Riemannian manifold and $H:X\longrightarrow\mathbb{R}$ is a smooth function. Since X is compact, the negative gradient flow of H,

$$\psi_{H;t}\colon X\longrightarrow X,\qquad \psi_{H;0}=\mathrm{id}_X,\quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_{H;t}=-\nabla^gH\big|_{\psi_{H;t}},$$

is defined for all $t\in\mathbb{R}$. Show that the limits

$$x_H^{\pm}\equiv\lim_{t\longrightarrow\pm\infty}\psi_{H;t}(x)$$

exist for every $x\in X$.

Let (X,g) be a compact Riemannian manifold and $H\colon X\longrightarrow\mathbb{R}$ be a Morse-Bott function. For $Y\in\pi_0(\mathrm{Crit}(H))$, we denote by

$$\pi^{\pm}_{H;Y}\colon \mathrm{E}^{\pm}_Y(H)\longrightarrow Y$$

the bundle projections and by $S(\mathrm{E}^{\pm}_Y(H))\subset\mathrm{E}^{\pm}_Y(H)$ the sphere bundle of $\mathrm{E}^{\pm}_Y(H)$. Let

$$X^{\pm}_Y(H)\equiv\big\{x\in X\colon x^{\pm}_H\in Y\big\}\supset Y$$

be the H-stable and unstable manifolds of Y. For $A\subset X^{\pm}_Y(H)$, let

$$A^{\pm}_H=\big\{x^{\pm}_H\colon x\in A\big\}\subset Y.$$

Proposition 4.7. *Suppose (X,g) is a compact Riemannian manifold, $H\colon X\longrightarrow\mathbb{R}$ is a Morse-Bott function, and $Y\in\pi_0(\mathrm{Crit}(H))$.*

(1) The subspaces $X^{\pm}_Y(H)\subset X$ are smooth submanifolds with

$$T\big(X^{\pm}_Y(H)\big)\big|_Y=TY\oplus\mathrm{E}^{\pm}_Y(H)\subset TX|_Y\,. \tag{4.5}$$

(2) There exist diffeomorphisms $\Phi^{\pm}_{H;Y}\colon\mathrm{E}^{\pm}_Y(H)\longrightarrow X^{\pm}_Y(H)$ such that

$$\begin{gathered}\big(\Phi^{\pm}_{H;Y}(w)\big)^{\pm}_H=\pi^{\pm}_{H;Y}(w)\quad\forall\,w\in\mathrm{E}^{\pm}_Y(H),\qquad\Phi^{\pm}_{H;Y}(y)=y\quad\forall\,y\in Y,\\ \mathrm{d}_y\Phi^{\pm}_{H;Y}(w)=w\quad\forall\;y\in Y,\,w\in\mathrm{E}^{\pm}_Y(H)\subset T_y\big(\mathrm{E}^{\pm}_Y(H)\big).\end{gathered} \tag{4.6}$$

(3) For every $c\in\mathbb{R}$, the submanifolds $X^{\pm}_Y(H)$ and $H^{-1}(c)-\mathrm{Crit}(H)$ of X are transverse.

(4) For every $\epsilon\in\mathbb{R}^+$ such that $H(Y)$ is the only critical value of H in $[H(Y)-\epsilon,H(Y)+\epsilon]$, there exist diffeomorphisms

$$\Phi^{\pm}_{H;Y;\epsilon}\colon S\big(\mathrm{E}^{\pm}_Y(H)\big)\longrightarrow X^{\pm}_Y(H)\cap H^{-1}\big(H(Y)\pm\epsilon\big)\subset X$$

satisfying the first property in (4.6) with $(\Phi^{\pm}_{H;Y},\mathrm{E}^{\pm}_Y(H))$ replaced by $(\Phi^{\pm}_{H;Y;\epsilon},S(\mathrm{E}^{\pm}_Y(H)))$.

(5) The intersection of the closure $\overline{X^{\pm}_Y(H)}\subset X$ of $X^{\pm}_Y(H)$ with the level set $H^{-1}(H(Y))$ is Y.

(6) If $A\subset X^{\pm}_Y(H)$ and $\overline{A}\subset X$ is the closure of A, $\overline{A}\cap H^{-1}(H(Y))\subset\overline{A^{\pm}_H}$. If in addition A is preserved by the gradient flow of H, i.e. $\psi_{H;t}(A)=A$ for all $t\in\mathbb{R}$, then $\overline{A}\cap H^{-1}(H(Y))=\overline{A^{\pm}_H}$.

Proof. By the Tubular Neighborhood Theorem, there are neighborhoods $\mathcal{U}\subset\mathrm{E}^-_Y(H)\oplus\mathrm{E}^+_Y(H)$ and $U\subset X$ of Y and a diffeomorphism $\Phi\colon\mathcal{U}\longrightarrow U$ such that

$$\Phi(y)=y\quad\forall\,y\in Y,\qquad\mathrm{d}_y\Phi(w)=w\quad\forall\;y\in Y,\,w\in\mathrm{E}^-_Y(H)\oplus\mathrm{E}^+_Y(H)\subset T_y\big(\mathrm{E}^-_Y(H)\oplus\mathrm{E}^+_Y(H)\big).$$

By the statement and proof of [3, Theorem A.9], there are then neighborhoods $\mathcal{U}'\subset\mathcal{U}$ and $U'\subset U$ of Y and smooth embeddings

$$\Phi^{\pm}_{H;Y}\colon\mathcal{U}^{\pm}_{H;Y}\equiv\mathcal{U}'\cap\mathrm{E}^{+}_Y(H)\longrightarrow U$$

so that $\Phi_{H;Y}^{\pm}(\mathcal{U}_{H;Y}^{\pm})=U'\cap X_Y^{\pm}(H)$, the first identity in (4.6) holds whenever $w\in\mathcal{U}_{H;Y}^{\pm}$, and the other two identities in (4.6) hold as stated. For all $\delta\in\mathbb{R}^+$ sufficiently small, $\mathcal{U}'$ contains the closed disk bundle of $\mathrm{E}_Y^-(H)\oplus\mathrm{E}_Y^+(H)$ of radius δ and

$$\pm\frac{\mathrm{d}}{\mathrm{d}t}H\big(\Phi_{H;Y}^{\pm}(tw)\big)\Big|_{t=1}>0 \qquad \forall\, w\in\mathrm{E}_Y^{\pm}(H),\, |w|=\delta. \tag{4.7}$$

The smooth maps

$$\Psi_{H;Y}^{\pm}\colon \mathrm{E}_Y^{\pm}(H)-Y\longrightarrow X, \qquad \Psi_{H;Y}^{\pm}(w)=\psi_{H;\mp\ln(|w|/\delta)}\big(\Phi_{H;Y}^{\pm}(\delta w/|w|)\big),$$

are then smooth embeddings onto $X_Y^{\pm}(H)-Y$ that agree with $\Phi_{H;Y}^{\pm}$ on the sphere bundle $S_\delta(\mathrm{E}_Y^{\pm}(H))$ in $\mathrm{E}_Y^{\pm}(H)$ of radius δ, satisfy the first identity in (4.6) whenever $w\neq0$, and satisfy (4.7) with $\Phi_{H;Y}^{\pm}$ replaced by $\Psi_{H;Y}^{\pm}$. We can thus paste $\Phi_{H;Y}^{\pm}$ and $\Psi_{H;Y}^{\pm}$ together on a neighborhood of $S_\delta(\mathrm{E}_Y^{\pm}(H))$ in $\mathrm{E}_Y^{\pm}(H)$ to obtain smooth embeddings $\Phi_{H;Y}^{\pm}$ of $\mathrm{E}_Y^{\pm}(H)$ into X with image $X_Y^{\pm}(H)$ which satisfy (4.6). This establishes (1) and (2).

Let $c\in\mathbb{R}$ and $x\in X_Y^{\pm}(H)\cap(H^{-1}(c)-\mathrm{Crit}(H))$. Thus, $\mathrm{d}_xH\neq0$, $H^{-1}(c)-\mathrm{Crit}(H)\subset X$ is a smooth submanifold with

$$T_x\big(H^{-1}(c)-\mathrm{Crit}(H)\big)=\ker\mathrm{d}_xH\,,$$

and $\psi_{H;t}(x)$ is a curve in $X_Y^{\pm}(H)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}H\big(\psi_{H;t}(x)\big)\Big|_{t=0}=\mathrm{d}_xH(-\nabla^gH)=-g(\nabla^gH,\nabla^gH)\neq0.$$

This gives (3).

Let $\epsilon,\delta\in\mathbb{R}^+$ be as in (4) and above, respectively, with

$$\Psi_{H;Y}^{\pm}\big(S_\delta(\mathrm{E}_Y^{\pm}(H))\big)\subset H^{-1}\big((H(Y)-\epsilon,H(Y)+\epsilon)\big).$$

Since the norm of ∇^gH is bounded below on $H^{-1}((H(Y)-\epsilon,H(Y)-\epsilon'))$ and $H^{-1}((H(Y)+\epsilon',H(Y)+\epsilon))$ for every $\epsilon'\in(0,\epsilon)$, (4.7) and the smoothness of the negative gradient flow $\psi_{H;t}$ imply that there is a smooth function

$$\rho\colon S_\delta\big(\mathrm{E}_Y^{\pm}(H)\big)\longrightarrow\mathbb{R} \qquad \text{s.t.} \quad H\big(\psi_{H;\rho(w)}\big(\Phi_{H;Y}^{\pm}(w)\big)\big)=H(Y)\pm\epsilon.$$

Along with (4.7) again, the map

$$\Phi_{H;Y;\epsilon}^{\pm}\colon S\big(\mathrm{E}_Y^{\pm}(H)\big)\longrightarrow X_Y^{\pm}(H)\cap H^{-1}\big(H(Y)\pm\epsilon\big), \quad \Phi_{H;Y;\epsilon}^{\pm}=\psi_{H;\rho(w)}\big(\Phi_{H;Y}^{\pm}(w)\big),$$

is then a diffeomorphism satisfying the first property in (4.6) with $(\Phi_{H;Y}^{\pm},\mathrm{E}_Y^{\pm}(H))$ replaced by $(\Phi_{H;Y;\epsilon}^{\pm},S(\mathrm{E}_Y^{\pm}(H)))$.

Suppose $x'\in H^{-1}(H(Y))-Y$. Choose disjoint open neighborhoods $U,U'\subset X$ of Y and x', respectively. By (4.7), there exists $\epsilon\in\mathbb{R}^+$ so that

$$\big|H(x)-H(Y)\big|\geq\epsilon \qquad \forall\, x\in X_Y^{\pm}(H)-U\,.$$

By shrinking U' if necessary, we can assume that

$$\big|H(x)-H(Y)\big| < \epsilon \qquad \forall\, x\in U'.$$

It then follows that U' is disjoint from $X_Y^{\pm}(H)$ and thus from $\overline{X_Y^{\pm}(H)}$. This establishes (5).

Let $A\subset X_Y^{\pm}(H)$. By ((5)), $\overline{A}\cap H^{-1}(H(Y))=\overline{A}\cap Y$. Suppose $y\in\overline{A}\cap Y$ and $U\subset X$ is a neighborhood of y. By (2), there exists a neighborhood U' of y in U so that $x_H^{\pm}\in U'$ for every $x\in U'$. Since $y\in\overline{A}$, U' contains some $x\in A$ and thus $x_H^{\pm}\in A_H^{\pm}$. We conclude that $y\in\overline{A_H^{\pm}}$.

Suppose A is preserved by the gradient flow of H and $y\in\overline{A_H^{\pm}}$. Let $U\subset X$ be a neighborhood of y and $x\in A$ a point such that $x_H^{\pm}\in U$. Thus, $\psi_{H;\pm}(x)\in A\cap U$ for all $t\in\mathbb{R}$ sufficiently large. Therefore, $y\in\overline{A}$. □

4.2 Fiber connectedness

The next proposition is the main point-set topology input in the proof of ($\mathrm{A}_k^{\star}$) on page 4 in [1] and [21].

Proposition 4.8 ([1, Lemma 2.1], [21, Lemma 5.5.5])**.** *Suppose M is a compact connected smooth manifold and $H\colon X\longrightarrow\mathbb{R}$ is a Morse-Bott function. If $n_x^{\pm}(H)\neq 1$ for every $x\in\mathrm{Crit}(H)$, then*

(1) H has a unique local minimum and a unique local maximum;

(2) $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$.

We give two proofs of this proposition, which are essentially two different formulations of the same reasoning. The first one is in the style of classical Morse theory, as in [24]. It is based on describing the changes in the homotopy type of $H^{-1}((-\infty,c])$ as c passes through critical values as adding "handles" of various kinds; see (4.8) below. The second proof is in the style of the modern take on Morse theory originating in [31]. It is based on partitioning X into stable or unstable manifolds of the negative gradient flow; see (4.9) below. In both cases, we first show that there are unique connected critical submanifolds $Y_-,Y_+\subset X$ with $n_{Y_-}^-(Y_-)=0$ and $n_{Y_+}^+(Y_+)=0$. The function H reaches its global minimum along Y_- and maximum along Y_+; there are no other local minima or maxima. We then show that $H^{-1}(c)$ is connected whenever $c\in(\min H,\max H)$ is a regular value of H. The claim for arbitrary $c\in\mathbb{R}$ then follows from Lemma 4.9 below.

Proof 1 of Proposition 4.8 ([1]). For $Y\in\pi_0(\mathrm{Crit}(H))$, we denote by $D(\mathrm{E}_Y^-(H))\subset\mathrm{E}_Y^-(H)$ the disk bundle of $\mathrm{E}_Y^-(H)$. For $c\in\mathbb{R}$, let

$$X_c(H)=H^{-1}\big((-\infty,c]\big)\subset X.$$

If c is a regular value of H, i.e. $c\notin H(\mathrm{Crit}(H))$, then $X_c(H)$ is a smooth manifold with boundary $\partial X_c(H)=H^{-1}(c)$. If $c_-,c_+\in\mathbb{R}$ are regular values of H with $c_-<c_+$, then

$$X_{c_+}(H)\sim X_{c_-}(H)\cup\bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ H(Y)\in(c_-,c_+)}}D\big(\mathrm{E}_Y^-(H)\big),\tag{4.8}$$

with $\sim$ denoting homotopy equivalence; see [7, Section 1]. The boundaries of the disk bundles on the right-hand side above are attached to $\partial X_{c_-}(H)$; the right-hand side is then a deformation retract of the left-hand side.

If $Y\in\pi_0(\mathrm{Crit}(H))$ and $n_Y^-(H)=0$ (i.e. H has a local minimum along Y), then adding $D(\mathrm{E}_Y^-(H))$ as in (4.8) adds a topological component to $X_{c_-}(H)$. If $Y\in\pi_0(\mathrm{Crit}(H))$ and $n_Y^-(H)\geq 2$, then attaching $D(\mathrm{E}_Y^-(H))$ as in (4.8) has no impact on the topological components of $X_{c_-}(H)$ vs $X_{c_+}(H)$. Since $n_Y^-(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$ and X is connected, it follows that there is a unique $Y_-\in\pi_0(\mathrm{Crit}(H))$ with $n_{Y_-}^-(H)=0$; thus, $H(Y_-)=\min H$. Since $n_Y^-(-H)=n_Y^+(H)$, the same reasoning shows that there is a unique $Y_+\in\pi_0(\mathrm{Crit}(H))$ with $n_{Y_+}^+(H)=0$; thus, $H(Y_+)=\max H$. Furthermore, $X_c(H)$ is connected for every $c\in\mathbb{R}$.

We can assume that H is not a constant function. Let $c\in(\min H,\max H)$ be a regular value. By (4.8), $X_c(H)$ is a homotopy equivalent to a CW complex with cells of dimension at most the maximum of the numbers

$$\dim D\big(\mathrm{E}_Y^-(H)\big)=n_Y^0(H)+n_Y^-(H)=\dim X-n_Y^+(H)<\dim X-1=\dim\partial X_c(H)$$

taken over $Y\in\pi_0(\mathrm{Crit}(H))$ with $H(Y)<c$. The inequality above holds because $n_Y^+(H)\neq 0$ for $Y\neq Y_+$ and $n_Y^+(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$. Thus, $H_k(X_c(H);\mathbb{Z}_2)=0$ for $k\geq\dim\partial X_c(H)$ and the boundary homomorphism

$$\partial\colon H_{\dim X}\big(X_c(H);\mathbb{Z}_2\big)\longrightarrow H_{\dim\partial X_c(H)}\big(\partial X_c(H);\mathbb{Z}_2\big)$$

in the homology exact sequence for $(X_c(H),\partial X_c(H))$ is surjective. Since $X_c(H)$ is connected, the domain of this homomorphism is isomorphic to $\mathbb{Z}_2$. It follows that $\partial X_c(H)=H^{-1}(c)$ is connected.

Thus, $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}-H(\mathrm{Crit}(H))$. Since $H(\mathrm{Crit}(H))\subset\mathbb{R}$ is a finite subset, Lemma 4.9 below implies that $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$. $\square$

Proof 2 of Proposition 4.8 (modification of [21, pp233,4]). By definition,

$$X=\bigsqcup_{Y\in\pi_0(\mathrm{Crit}(H))}X_Y^+(H)=\bigsqcup_{Y\in\pi_0(\mathrm{Crit}(H))}X_Y^-(H)\,. \tag{4.9}$$

Since $n_Y^-(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$,

$$\bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ n_Y^-(H)=0}}X_Y^+(H)=X-\bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ n_Y^-(H)\geq 2}}X_Y^+(H).$$

Since X is connected and each submanifold $X_Y^+(H)\subset X$ on the right-hand side above is of codimension $n_Y^-(H)\geq 2$ by Proposition 4.7(1), the union on the left-hand side is connected. Since each submanifold $X_Y^+(H)\subset X$ on the left-hand side is open, it follows that there is a unique $Y_-\in\pi_0(\mathrm{Crit}(H))$ with $n_{Y_-}^-(H)=0$; thus, $H(Y_-)=\min H$. By the same reasoning with the second partition in (4.9), there is a unique $Y_+\in\pi_0(\mathrm{Crit}(H))$ with $n_{Y_+}^+(H)=0$; thus, $H(Y_+)=\max H$.

We can assume that H is not a constant function. Let ϵ be as in Proposition 4.7(4) with $Y=Y_-$ and $c\in(\min H,\max H)$ be a regular value. By (4.9),

$$\begin{aligned} H^{-1}(c)-\bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ n_Y^-(H)\ge 2}} X_Y^+(H) &= H^{-1}(c)\cap X_{Y_-}^+(H) \\ &\approx H^{-1}\big(\min H+\epsilon\big)-\bigcup_{\substack{Y\in\pi_0(\mathrm{Crit}(H))\\ H(Y)\in(\min H,c)}} H^{-1}\big(\min H+\epsilon\big)\cap X_Y^-(H)\,; \end{aligned}\tag{4.10}$$

the diffeomorphism $\approx$ above is obtained via the gradient flow. Since $H^{-1}(\min H+\epsilon)$ is transverse to each $X_Y^-(H)$, the codimension of the last intersection above in $H^{-1}(\min H+\epsilon)$ is the codimension of $X_Y^-(H)$ in X, i.e. $n_Y^+(H)\ge 2$. The last inequality holds because $n_Y^+(H)\neq 0$ for $Y\neq Y_+$ and $n_Y^+(H)\neq 1$ for any $Y\in\pi_0(\mathrm{Crit}(H))$. Since $H^{-1}(\min H+\epsilon)$ is diffeomorphic to the connected manifold $S(\mathrm{E}_{Y_-}^+(H))$ by Proposition 4.7(2), the right-hand side in (4.10) is connected as well. Since the codimension of $H^{-1}(c)\cap X_Y^+(H)$ in $H^{-1}(c)$ is $n_Y^-(H)$ and $n_Y^-(H)>0$ whenever $Y\neq Y_-$, it then follows from (4.10) that $H^{-1}(c)$ is also connected.

By the above $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}-H(\mathrm{Crit}(H))$. Since $H(\mathrm{Crit}(H))\subset\mathbb{R}$ is a finite subset, Lemma 4.9 below implies that $H^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$. $\square$

Lemma 4.9. *Let X be a compact connected manifold (or more generally a topological space which is sequentially compact, connected, locally connected, and normal). Suppose $f: X\longrightarrow\mathbb{R}$ is a continuous function and $P^*\subset\mathbb{R}$ is a dense subset. If $f^{-1}(c)\subset X$ is connected for every $c\in P^*$, then $f^{-1}(c)\subset X$ is connected for every $c\in\mathbb{R}$.*

Proof. Suppose $c\in\mathbb{R}-P^*$ and $f^{-1}(c)=A\cup B$ for some disjoint nonempty subsets A,B that are closed in $f^{-1}(c)$ and thus in X. Let $U_A,U_B\subset X$ be disjoint open subsets such that $A\subset U_A$ and $B\subset U_B$. Let $W\subset\mathbb{R}$ be a neighborhood of c such that $f^{-1}(W)\subset U_A\cup U_B$ (its existence follows from the first countability of $\mathbb{R}$, sequential compactness of X, and the continuity of f). For each $x\in A\cup B$, choose a connected neighborhood V_x of x in $f^{-1}(W)$. The subsets

$$V_A\equiv\bigcup_{x\in A}V_x \qquad\text{and}\qquad V_B\equiv\bigcup_{x\in B}V_x$$

of X are then open disjoint neighborhoods of A and B, respectively, in X. If $f^{-1}(c_A)\cap V_A\neq\emptyset$ for some $c_A<c$, then

$$f^{-1}(c')\cap V_A\neq\emptyset \qquad \forall\ c'\in(c_A,c).$$

If in addition $f(x)<c$ for some $x\in V_B$, then there exists $c^*\in(c_A,c)$ such that

$$c^*\in P^*,\qquad f^{-1}(c^*)\cap U_A\neq\emptyset,\quad\text{and}\quad f^{-1}(c^*)\cap U_B\neq\emptyset.$$

Since $f^{-1}(c^*)\subset U_A\cap U_B$, this would contradict the assumption that $f^{-1}(c^*)\subset M$ is connected for every $c^*\in P^*$. We can thus assume that $f(x)\le c$ for all $x\in V_A$ and $f(x)\ge c$ for all $x\in V_B$. Then

$$\widetilde{U}_A\equiv f^{-1}\big((-\infty,c)\big)\cup V_A \qquad\text{and}\qquad \widetilde{U}_B\equiv f^{-1}\big((c,\infty)\big)\cup V_B$$

are disjoint nonempty open subsets of X that cover X. However, this contradicts the assumption that X is connected. $\square$

Exercise 4.10. Let X be a compact connected manifold (or more generally a topological space which is sequentially compact, connected, and normal). Suppose P is a first countable topological space, $f\colon X\longrightarrow P$ is a continuous open surjective map, and $P^*\subset P$ is a dense subset. Show that if $f^{-1}(c)\subset X$ is connected for every $c\in P^*$, then $f^{-1}(c)\subset X$ is connected for every $c\in P$.

5 Properties of Moment Polytopes

5.1 Complexified Hamiltonian group actions

This section establishes Theorem 2(2). The key steps in the proof are Lemma 5.2, which describes the behavior of the moment map μ on $\mathcal{O}_x$, and Proposition 5.6, which concerns the images of $\mathcal{O}_x$ and $\overline{\mathcal{O}_x}-\mathcal{O}_x$ under μ.

Exercise 5.1. Suppose ψ is a smooth action of a torus $\mathbb{T}$ on a compact almost complex manifold (M,J), i.e. ψ preserves J. Show that (1.6) determines a complexification $\psi_{\mathbb{C}}$ of ψ if either $\mathbb{T}$ is one-dimensional or J is integrable. *Hint:* J is preserved by ψ if and only if $\mathcal{L}_{\zeta_v}J=0$ for every $v\in T_{\mathbb{1}}\mathbb{T}^k$, where $\mathcal{L}$ is the Lie derivative and $\zeta_v\in\Gamma(X;TX)$ is as in (1.2); J is integrable if and only $\mathcal{L}_{J\xi}J=J(\mathcal{L}_\xi J)$ for every $\xi\in\Gamma(X;TX)$.

Lemma 5.2. *Suppose $\mathbb{T}$ is a torus, (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold, and $\psi_{\mathbb{C}}$ is a complexification of ψ with respect to a $\mathbb{T}$-invariant ω-compatible almost complex structure J as in (1.6). For each $v\in T_{\mathbb{1}}\mathbb{T}$, let $\zeta_v\in\Gamma(X;TX)$ and $\mu_v\in C^\infty(X)$ be as in (1.2) and (2.2), respectively. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_v\big(\psi_{\mathbb{C};[\mathrm{i}tv']}(x)\big)=-g(\zeta_v,\zeta_{v'})\big|_{\psi_{\mathbb{C};[\mathrm{i}tv']}(x)}\qquad\forall\ v,v'\in T_{\mathbb{1}}\mathbb{T},\ x\in X,\tag{5.1}$$

where $g(\cdot,\cdot)\equiv\omega(\cdot,J\cdot)$ is the Riemannian metric determined by ω and J. Furthermore,

$$\mu\big(\psi_{\mathbb{C};[\mathrm{i}v]}(x)\big)\neq\mu(x)\in T_{\mathbb{1}}\mathbb{T}^*\qquad\forall\ v\in T_{\mathbb{1}}\mathbb{T},\ x\in X\ \text{s.t.}\ \zeta_v(x)\neq0.$$

Proof. By (1.4), (1.6), and Proposition 3.8(1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_v\big(\psi_{\mathbb{C};[\mathrm{i}tv']}(x)\big)\equiv\mathrm{d}_{\psi_{\mathbb{C};[\mathrm{i}tv']}(x)}\mu_v\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\mathbb{C};[\mathrm{i}tv']}(x)\bigg)=-\omega\Big(\zeta_v\big(\psi_{\mathbb{C};[\mathrm{i}tv']}(x)\big),\big(J\zeta_{v'}\big(\psi_{\mathbb{C};[\mathrm{i}tv']}(x)\big)\big)\Big).$$

This gives (5.1). Along with Proposition 3.8(1) again, this implies that

$$\mathbb{R}\longrightarrow\mathbb{R},\qquad t\longrightarrow\mu_v\big(\psi_{\mathbb{C};[\mathrm{i}tv]}(x)\big),$$

is a strictly decreasing function unless $\zeta_v(x)=0$. This gives the second claim of the lemma. $\square$

Exercise 5.3. Suppose $\mathbb{T}$, (X,ω,μ,ψ), $\psi_{\mathbb{C}}$, J, g, ζ_v, and μ_v are as in Lemma 5.2. Suppose in addition X is compact. Show that the limit

$$x_\infty(v)\equiv\lim_{t\longrightarrow\infty}\psi_{\mathbb{C};[\mathrm{i}tv]}(x)\in X\tag{5.2}$$

exists for all $v\in T_{\mathbb{1}}\mathbb{T}$ and $x\in X$ and satisfies

$$\begin{gathered}\zeta_v\big(x_\infty(v)\big)=0,\quad \mathrm{d}_{x_\infty(v)}\mu_v=0,\quad \mu_v\big(x_\infty(v)\big)=\inf_{t\in\mathbb{R}}\mu_v\big(x_\infty(v)\big),\\ \big(\psi_{\mathbb{C};u}(x)\big)_\infty(v)=\psi_{\mathbb{C};u}\big(x_\infty(v)\big)\qquad\forall\,u\in\mathbb{T}_{\mathbb{C}}\,.\end{gathered}\tag{5.3}$$

Suppose ψ is a smooth action of a torus $\mathbb{T}$ on a smooth manifold X. For $x\in X$ and $v\in T_{\mathbb{1}}\mathbb{T}$, let $\mathbb{T}_x(\psi)\subset\mathbb{T}$ be as in (1.3) with $G=\mathbb{T}$ and $\mathbb{T}_{x;v}(\psi)\subset\mathbb{T}$ be the closed subgroup spanned by $\mathbb{T}_x(\psi)$ and the closure of $\{\mathrm{e}^{tv}: t\in\mathbb{R}\}$ in $\mathbb{T}$.

Corollary 5.4. *Suppose $\mathbb{T}$, (X,ω,ψ,μ), $\psi_{\mathbb{C}}$, and μ_v are as in Lemma 5.2, with X compact, $x\in X$, $v\in T_{\mathbb{1}}\mathbb{T}$, and $x'_\infty(v)\in X$ is as in (5.2). Let $\mathcal{O}_x\subset\overline{\mathcal{O}_x}$ be the $\mathbb{T}_{\mathbb{C}}$-orbit of x and its closure in X, respectively. Then, there exists a topological component $Z_{x;v}$ of the $\mathbb{T}_{x;v}(\psi)$-fixed locus $X^{\mathbb{T}_{x;v}(\psi)}\subset X$ so that*

$$x'_\infty(v)\in Z_{x;v} \quad \textit{and} \quad \inf_{t\in\mathbb{R}}\mu_v\big(x'_\infty(v)\big)=\mu_v\big(Z_{x;v}\big)=\inf_{\overline{\mathcal{O}_x}}\mu_v=\inf_{\substack{Y\in\pi_0(X^\psi)\\ Y\cap\overline{\mathcal{O}_x}\neq\emptyset}}\mu_v(Y) \qquad \forall\, x'\in\mathcal{O}_x. \tag{5.4}$$

Proof. Let $x'\in\mathcal{O}_x$. By the continuity of the action ψ, $x'_\infty(v)$ is fixed by $\mathbb{T}_{x'}(\psi)=\mathbb{T}_x(\psi)$. By the first equation in (5.3), $x'_\infty(v)$ is also fixed by the closure of $\{\mathrm{e}^{tv}: t\in\mathbb{R}\}$ in $\mathbb{T}$. Thus, $x'_\infty(v)$ lies in $X^{\mathbb{T}_{x;v}(\psi)}$, which is a closed symplectic submanifold of (X,ω) by Proposition 3.14(1). Let $Z_{x;v}\subset X^{\mathbb{T}_{x;v}(\psi)}$ be the component containing $x_\infty(v)$. Since $\mathbb{T}_{\mathbb{C}}$ is connected, the last equation in (5.3) implies that $x'_\infty(v)\in Z_{x;v}$ as well.

Since $\mathrm{d}\mu_v$ vanishes on $X^{\mathbb{T}_{x;v}(\psi)}$, μ_v is constant on $Z_{x;v}$. Along with the third equation in (5.3), this yields the first equality in the second equation in (5.4). Thus,

$$\inf_{\overline{\mathcal{O}_x}}\mu_v=\inf_{\mathcal{O}_x}\mu_v=\mu_v(Z_{x;v}). \tag{5.5}$$

If $v\in T_{\mathbb{1}}\mathbb{T}$ is generic, $\mathbb{T}_{x;v}(\psi)=\mathbb{T}$ and so $Z_{x;v}\subset X^\psi$. Thus,

$$\inf_{\overline{\mathcal{O}_x}}\mu_v\le\inf_{\substack{Y\in\pi_0(X^\psi)\\ Y\cap\overline{\mathcal{O}_x}\neq\emptyset}}\mu_v(Y)\le\mu_v\big(Z_{x;v}\big)=\inf_{\overline{\mathcal{O}_x}}\mu_v \quad\Longrightarrow\quad \inf_{\overline{\mathcal{O}_x}}\mu_v=\inf_{\substack{Y\in\pi_0(X^\psi)\\ Y\cap\overline{\mathcal{O}_x}\neq\emptyset}}\mu_v(Y).$$

By the compactness of $\overline{\mathcal{O}_x}$ and the continuity of μ in both inputs, the last equality holds for all $v\in T_{\mathbb{1}}\mathbb{T}$. Combining this equality with (5.5), we obtain the last equality in the second equation in (5.4). $\square$

Corollary 5.5. *Suppose $\mathbb{T}$, (X,ω,ψ,μ), $\psi_{\mathbb{C}}$, and μ_v are as in Lemma 5.2, $x\in X$, and $\mathcal{O}_x\subset\overline{\mathcal{O}_x}$ are as in Corollary 5.4. Then,*

$$\overline{\mathcal{O}_x}-\mathcal{O}_x=\big\{x'\in\overline{\mathcal{O}_x}:\exists\ v\in T_{\mathbb{1}}\mathbb{T}\ \textit{s.t.}\ \mu_v(x')=\inf_{\overline{\mathcal{O}_x}}\mu_v,\ \mathrm{d}_x\mu_v\neq0\big\}. \tag{5.6}$$

Proof. Let $T^c_{\mathbb{1}}\mathbb{T}\subset\mathbb{T}_{\mathbb{1}}\mathbb{T}$ be a complement of

$$T_{\mathbb{1}}\mathbb{T}_x(\psi)=\big\{v\in T_{\mathbb{1}}\mathbb{T}:\mathrm{d}_x\mu_v=0\big\} \tag{5.7}$$

and $S(T^c_{\mathbb{1}}\mathbb{T})\subset T^c_{\mathbb{1}}\mathbb{T}$ be the unit sphere with respect to some metric. In particular,

$$\mathcal{O}_x=\big\{\psi_{\mathbb{C};v+\mathrm{i}v'}(x): v,v'\in T^c_{\mathbb{1}}\mathbb{T}\big\}.$$

By (1.6) and (3.18), $\psi_{\mathbb{C};\mathrm{i}tv}$ is the negative gradient flow $\psi_{\mu_v;t}$ of μ_v with respect to the metric $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$, with J as in (1.6). Since $\mathrm{d}_{x'}\mu_v\neq0$ for all $x'\in\mathcal{O}_x$, the continuous function

$$\mathbb{T}\times S(T^c_{\mathbb{1}}\mathbb{T})\times\mathbb{R}\longrightarrow\mathbb{R},\qquad h\big(u,v,t\big)=\mu_v\big(\psi_{\mathbb{C};\mathrm{i}tv}(\psi_u(x))\big),$$

is thus decreasing in t. In particular, the left-hand set in (5.6) contains the right-hand set. By (5.4),

$$\lim_{t\longrightarrow\infty} h(u,v,t) = \inf_{\overline{\mathcal{O}_x}} \mu_v \qquad \forall\, u\in\mathbb{T},\, v\in S(T_{\mathbb{1}}^c\mathbb{T}). \tag{5.8}$$

Suppose $y\in\overline{\mathcal{O}_x}-\mathcal{O}_x$ is the limit of a sequence $\psi_{\mathbb{C};\mathrm{i}t_k v_k}(\psi_{u_k}(x))$ with $u_k\in\mathbb{T}$ converging to some u, $v_k\in S(T_{\mathbb{1}}^c\mathbb{T})$ converging to some v, and $t_k\in\mathbb{R}$ converging to ∞. Since the functions $h(u_k,v_k,t)$ are decreasing in t and μ_v is a continuous function on X, (5.8) implies that $\mu_v(y)=\inf_{\overline{\mathcal{O}_x}}\mu_v$. $\square$

For a polytope $P\subset\mathbb{T}_{\mathbb{1}}^*\mathbb{T}$, we denote by $\partial P\subset P$ the union of the proper faces of P. Let $\mathrm{Int}\,P=P-\partial P$. With $\mathbb{T}$, (X,ω,ψ,μ), $\psi_{\mathbb{C}}$, $x\in X$, and $\overline{\mathcal{O}_x}\subset X$ as in Theorem 2(2), let

$$P_x(\psi_{\mathbb{C}}) = \mathrm{CH}\big(\mu\big\{Y\in\pi_0(X^\psi)\colon Y\cap\overline{\mathcal{O}_x}\neq\emptyset\big\}\big).$$

Proposition 5.6. *Suppose* $\mathbb{T}$, (X,ω,ψ,μ), $\psi_{\mathbb{C}}$, $x\in X$, *and* $\mathcal{O}_x\subset\overline{\mathcal{O}_x}$ *are as in Theorem 2(2). Then,*

$$\mu(\mathcal{O}_x) = \mathrm{Int}\,P_x\big(\psi_{\mathbb{C}}\big), \qquad \mu\big(\overline{\mathcal{O}_x}-\mathcal{O}_x\big) = \partial P_x\big(\psi_{\mathbb{C}}\big), \tag{5.9}$$

and the map $\mathcal{O}_x/\mathbb{T}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}$ *induced by* μ *is injective.*

Proof. For each $v\in T_{\mathbb{1}}\mathbb{T}$, the map

$$L_v\colon T_{\mathbb{1}}^*\mathbb{T}\longrightarrow\mathbb{R}, \quad L_v(\alpha)=\alpha(v), \tag{5.10}$$

is a linear functional and $\mu_v\equiv L_v\circ\mu\colon X\longrightarrow\mathbb{R}$. Let $\mathbb{T}_x^c\subset\mathbb{T}$ be a subtorus complementary to the identity component $(\mathbb{T}_x(\psi))_0$ of $\mathbb{T}_x(\psi)$ and $\iota\colon\mathbb{T}_x^c\longrightarrow\mathbb{T}$ be the inclusion so that

$$\mu_x^c\equiv\iota^*\circ\mu\colon X\longrightarrow T_{\mathbb{1}}^*\mathbb{T}_x^c$$

is a moment map for the restriction of the $\mathbb{T}$-action ψ on $\mathbb{T}_x^c$. In particular,

$$\begin{gathered}\mathcal{O}_x\equiv\mathbb{T}_{\mathbb{C}}x=(\mathbb{T}_x^c)_{\mathbb{C}}x, \quad \big\{Y\in\pi_0(X^\psi)\colon Y\cap\overline{\mathcal{O}_x}\neq\emptyset\big\}=\big\{Y\in\pi_0(X^{\mathbb{T}_x^c})\colon Y\cap\overline{\mathcal{O}_x}\neq\emptyset\big\},\\ \mathcal{O}_x/\mathbb{T}=\mathcal{O}_x/\mathbb{T}_x^c, \quad \mu\big(\overline{\mathcal{O}_x}\big)\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mu;x}\equiv\big\{\alpha\in T_{\mathbb{1}}^*\mathbb{T}\colon L_v(\alpha)=L_v\big(\mu(x)\big)\ \forall\,v\in T_{\mathbb{1}}\mathbb{T}_x(\psi)\big\}.\end{gathered}$$

Since $\iota^*\colon(T_{\mathbb{1}}^*\mathbb{T})_{\mu;x}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}_x^c$ is a homeomorphism sending line segments to line segments, it suffices to establish the claims with (ψ,μ) replaced by $(\psi|_{\mathbb{T}_x^c},\mu_x^c)$. We can thus assume that $(\mathbb{T}_x(\psi))_0=\{\mathbb{1}\}$, as is done below.

For $v\in T_{\mathbb{1}}\mathbb{T}$, let $\zeta_v\in\Gamma(X;TX)$ be as in (1.2). Since

$$\big(\mathbb{T}_{x'}(\psi)\big)_0=\big(\mathbb{T}_x(\psi)\big)_0=\{\mathbb{1}\} \qquad \forall\ x'\in\mathcal{O}_x,$$

$\zeta_v(x')\neq0$ for all $v\in T_{\mathbb{1}}\mathbb{T}-\{0\}$ and $x'\in\mathcal{O}_x$. By Lemma 5.2, the map

$$T_{\mathbb{1}}\mathbb{T}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}, \qquad v\longrightarrow\mu\big(\psi_{\mathbb{C};\mathrm{i}v}(x)\big),$$

is thus a diffeomorphism onto an open subset of $T_{\mathbb{1}}^*\mathbb{T}$. Since μ is $\mathbb{T}$-invariant, this open subset is $\mu(\mathcal{O}_x)$. By the last equality in the second equation in (5.4), $\mu(\overline{\mathcal{O}_x})\subset P_x(\psi_{\mathbb{C}})$. Thus, the polytope $P_x(\psi_{\mathbb{C}})\subset T_{\mathbb{1}}^*\mathbb{T}$ is of full dimension, $\mu(\mathcal{O}_x)\subset\mathrm{Int}\,P_x(\psi_{\mathbb{C}})$, and the last claim of the proposition holds.

For $v \in T_{\mathbb{1}}\mathbb{T}-\{0\}$, the level sets of L_v are hyperplanes. Thus, the restriction of $\mu_v \equiv L_v \circ \mu$ to $\overline{\mathcal{O}_x}$ achieves its minimum along the preimage of a proper face of $P_x(\psi_{\mathbb{C}})$ under μ and not at any point of $\mu^{-1}(\mathrm{Int}\, P_x(\psi_{\mathbb{C}}))$; the former preimage contains $Y \cap \overline{\mathcal{O}}_x$ for at least one element $Y \in \pi_0(X^{\psi})$ with $Y \cap \overline{\mathcal{O}}_x \neq \emptyset$. From the compactness of $\overline{\mathcal{O}_x}$ and (5.6), we then conclude that

$$\overline{\mu(\mathcal{O}_x)}-\mu(\mathcal{O}_x) \subset \mu\big(\overline{\mathcal{O}_x}-\mathcal{O}_x\big) = \big\{\mu(x'): x' \in \overline{\mathcal{O}_x}, \exists\, v \in T_{\mathbb{1}}\mathbb{T}-\{0\} \text{ s.t. } \mu_v(x') = \inf_{\overline{\mathcal{O}_x}} \mu_v\big\} \subset \partial P_x(\psi_{\mathbb{C}}).$$

Thus, $\mu(\mathcal{O}_x) \supset \mathrm{Int}\, P_x(\psi_{\mathbb{C}})$, which establishes the first equality in (5.9). The second equality in (5.9) then follows from the compactness of $\overline{\mathcal{O}_x}$. $\square$

Proof of Theorem 2(2). The two equations in (5.9) give (2b), as well as (2c) with $\sigma = \mathrm{Int}\, P_x(\psi_{\mathbb{C}})$. Suppose σ is the interior of a proper face F of $P_x(\psi_{\mathbb{C}})$. Choose $v \in T_{\mathbb{1}}\mathbb{T}$ and $c \in \mathbb{R}$ be so that

$$F = P_x(\psi_{\mathbb{C}}) \cap L_v^{-1}(0),$$

with L_v as in (5.10). Let $Z_{x;v} \subset X$ be as in (5.4). Thus,

$$\mu^{-1}(F) \cap \overline{\mathcal{O}_x} \subset \mathrm{Crit}(\mu_v), \qquad \mu_v|_{\mu^{-1}(F)} = c,$$

and $Z_{x;v}$ is a topological component of $\mathrm{Crit}(\mu_v)$ with $\mu_v|_{Z_{x;v}} = c$. By (1.6) and (3.18), $\psi_{\mathbb{C};\mathrm{i}tv}$ is the negative gradient flow $\psi_{\mu_v;t}$ of μ_v with respect to the metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, with J as in (1.6). By the first statement in (5.4), $\mathcal{O}_x \subset X^+_{Z_{x;v}}(\mu_v)$. The first statement of Proposition 4.5, Proposition 4.7(6), and the last equation in (5.3) then imply that

$$\mu^{-1}(F) \cap \overline{\mathcal{O}_x} = \mu_v^{-1}(c) \cap \overline{\mathcal{O}_x} \subset \overline{\{x'_\infty(v) : x' \in \mathcal{O}_x\}} = \overline{\mathcal{O}_{x_\infty(v)}}.$$

Thus, $\overline{\mathcal{O}_{x_\infty(v)}} = \mu^{-1}(F) \cap \overline{\mathcal{O}_x}$. From (5.9) with x replaced by $x_\infty(v)$, we then conclude that

$$\mu^{-1}(\sigma) \cap \overline{\mathcal{O}_x} = \mu^{-1}(\sigma) \cap \overline{\mathcal{O}_{x_\infty(v)}} = \mathcal{O}_{x_\infty(v)}.$$

This establishes (2c). Since $\mathcal{O}_y = \{y\}$ for any $y \in X^{\psi}$, (2a) follows from (2c).

Since the moment map μ is $\mathbb{T}$-invariant, the map

$$\overline{\mathcal{O}_x}/\mathbb{T} \longrightarrow \mu(\overline{\mathcal{O}_x}), \qquad [x'] \longrightarrow \mu(x'), \tag{5.11}$$

is well-defined. It is surjective by definition. Its domain is compact, while the target is Hausdorff. By (2c), for every open face σ of the polytope $\mu(\overline{\mathcal{O}_x})$ there exists $x_\sigma \in \overline{\mathcal{O}_x}$ so that $\mu^{-1}(\sigma) \cap \overline{\mathcal{O}_x} = \mathcal{O}_{x_\sigma}$. By the last statement of Proposition 5.6 with x replaced by x_σ, the restriction of the map (5.11) to $\mu^{-1}(\sigma) \cap \overline{\mathcal{O}_x}/\mathbb{T}$ is thus injective for every open face σ of $\mu(\overline{\mathcal{O}_x})$. It follows that the entire map (5.11) is injective as well and thus a homeomorphism. $\square$

Exercise 5.7. Show that

(a) the S^1-action on $\mathbb{C}P^2$ given by

$$S^1 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2, \qquad u \cdot [z_0, z_1, z_2] = [z_0, u^2 z_1, u^3 z_2],$$

is effective and Hamiltonian with respect to the symplectic form $\omega_{\mathrm{FS};2}$ of Exercise 2.7;

(b) the closure $\overline{\mathcal{O}_x}$ of the $\mathbb{C}^*$-orbit $\mathcal{O}_x$ is a rational cubic curve for any point $x \equiv [z_0, z_1, z_2]$ in $\mathbb{C}P^2$ with $z_0, z_1, z_2 \neq 0$.

5.2 Proofs of Theorems 1, 2(1), and 3(0^+)

The last statement of Corollary 3.28 establishes (D_k). As already noted in Section 1, (F_k) is a straightforward consequence of the equivariant splitting (3.5) of $TX|_Y$ for each $Y\in\pi_0(X^\psi)$ and is deduced first below. We then establish the main part of the proof of Theorem 1, ($\mathrm{A}_k^\star$) on page 4, and wrap up this section with the remaining statements of Theorem 1, Theorem 2(1), and (0^+) on page 8.

Proof of (F_k). For each $L\in\mathbb{RP}^{k-1}$, let $X^L\subset X$ be the fixed locus of the action $\psi|_L$. By the second equality in (3.17) and Proposition 3.8(1),

$$\mathrm{Crit}(\mu)=\big\{x\in X^L\colon L\in\mathbb{R}P^{k-1}\big\}.$$

By Proposition 3.14(1), $X^L\subset X$ is a compact symplectic submanifold. Every topological component $Z\subset X^L$ is preserved ψ. The restriction of the ψ-action to such a component is Hamiltonian. Thus,

$$Z\cap X^\psi=Z^\psi\neq\emptyset \qquad \forall\; Z\in\pi_0(X^L),\, L\in\mathbb{R}P^{k-1}$$

by Exercise 3.23(a). Along with Corollary 3.7, this implies that $\mathrm{Crit}(\mu)$ is a *finite* union of the topological components Z of the symplectic submanifolds $X^L\subset X$ with $L\in\mathbb{R}P^{k-1}$. By the second equality in (3.17) and Proposition 3.8(1), the smooth map

$$\mu_v\colon X\longrightarrow T_0^*\mathbb{R}^k,\qquad \mu_v(x)=\{\mu(x)\}(v),\tag{5.12}$$

is constant along each topological component $Z\subset X^L$ for every $v\in L$, i.e. for any $v\in L$ there exists $c_v\in\mathbb{R}$ so that

$$\mu(Z)\subset\big\{\alpha\in T_0^*\mathbb{R}^k\colon \alpha(v)\!=\!c_v\big\};$$

the right-hand side above is a *hyperplane* in $T_0^*\mathbb{R}^k$ if $v\neq0$. □

Lemma 5.8. *Suppose $k\in\mathbb{Z}^{\ge0}$, (X,ω) is a symplectic manifold, $\psi_1,\ldots,\psi_{k+1}$ are $\mathbb{R}$-actions on (M,ω) with Hamiltonians*

$$H_1,\ldots,H_{k+1}\colon X\longrightarrow\mathbb{R},$$

respectively, the $\mathbb{R}$-action ψ_{k+1} is almost periodic, and its Hamiltonian H_{k+1} is ψ_i-invariant for every $i\in[k]$. If $c\in\mathbb{R}^k$ is a regular value of the map

$$H\equiv\big(H_1,\ldots,H_k\big)\colon X\longrightarrow\mathbb{R}^k,$$

then the submanifolds $Z\equiv H^{-1}(c)$ and $\mathrm{Crit}(H_{k+1})$ of X are transverse, $Z\cap\mathrm{Crit}(H_{k+1})$ is an open subset of $\mathrm{Crit}(H_{k+1}|_Z)$, and

$$T_x\big(Z\cap\mathrm{Crit}(H_{k+1})\big)=T_xZ\cap T_x\mathrm{Crit}(H_{k+1})=\mathrm{E}_x^0\big(H_{k+1}|_Z\big),\quad n_x^\pm\big(H_{k+1}|_Z\big)=n_x^\pm(H_{k+1})\in2\mathbb{Z}^{\ge0}$$

for all $x\in Z\cap\mathrm{Crit}(H_{k+1})$.

Proof. By the first statement of Proposition 4.5, $\mathrm{Crit}(H_{k+1})\subset X$ is a closed symplectic submanifold and $H_{k+1}\colon X\longrightarrow\mathbb{R}$ is a Morse-Bott function. Since $c\in\mathbb{R}^k$ is a regular value of H, $Z\equiv H^{-1}(c)$ is a submanifold of X. In light of Exercise 4.4 and the first statement of Proposition 4.5, it remains to prove that the submanifolds $\mathrm{Crit}(H_{k+1}),Z\subset X$ are transverse. Let $Y\in\pi_0(\mathrm{Crit}(H_{k+1}))$.

Let $\zeta_1,\ldots,\zeta_k\in\Gamma(X;TX)$ be the vector fields generating the $\mathbb{R}$-actions $\psi_1,\ldots,\psi_k$ and thus satisfying the middle equation in (2.5) with $\zeta_{v_i}=\zeta_i$. Since $H_{k+1}\circ\psi_i=H_{k+1}$ for each $i\in[k]$, ψ_i preserves $\mathrm{Crit}(H_{k+1})$ and thus Y. Therefore,

$$\zeta_i\big|_Y\in\Gamma(Y;TY)\subset\Gamma\big(Y;TX|_Y\big)\qquad\forall\ i\in[k].$$

If $x\in Z$, then $\mathrm{d}_xH_1,\ldots,\mathrm{d}_xH_k$ vanish on T_xZ. Since d_xH is surjective, it follows that

$$\mathrm{d}_xH\equiv\big(\mathrm{d}_xH_1,\ldots,\mathrm{d}_xH_k\big)\colon T_xX/T_xZ\longrightarrow\mathbb{R}^k \tag{5.13}$$

is a well-defined isomorphism. The middle equation in (2.5) with $\zeta_{v_i}=\zeta_i$ then implies that the tangent vectors $\zeta_1(x),\ldots,\zeta_k(x)\in T_xX$ are linearly independent.

Suppose $x\in Y\cap Z$ and $(r_1,\ldots,r_k)\in\mathbb{R}^k-\{0\}$. Since

$$r\cdot\zeta(x)\equiv r_1\zeta_1(x)+\ldots+r_k\zeta_k(x)\in T_xY-\{0\}$$

and $\omega|_{T_xY}$ is a nondegenerate, there exists $w\in T_xY$ so that

$$\sum_{i=1}^k r_i\mathrm{d}_xH_i(w)\equiv-\omega\big(r\cdot\zeta(x),w\big)\neq0.$$

Thus, the restrictions of $\mathrm{d}_xH_1,\ldots,\mathrm{d}_xH_k$ to T_xY are linearly independent. Since (5.13) is a well-defined isomorphism, it follows that $T_xX=T_xY\oplus T_xZ$, i.e. the submanifolds $Y,Z\subset X$ are transverse at x. □

Corollary 5.9. *Suppose $k\in\mathbb{Z}^{\ge0}$, (X,ω) is a symplectic manifold, $\widetilde\psi$ is an almost periodic $\mathbb{R}^{k+1}$-action on (X,ω) with Hamiltonian*

$$\widetilde H\equiv(H,H_{k+1})\colon X\longrightarrow\mathbb{R}^k\times\mathbb{R}=\mathbb{R}^{k+1}.$$

If $c\in\mathbb{R}^k$ is a regular value of H, then the restriction of H_{k+1} to the submanifold $Z\equiv H^{-1}(c)$ of X is a Morse-Bott function with $n_x^{\pm}(H)\in2\mathbb{Z}^{\ge0}$ for every $x\in\mathrm{Crit}(H_{k+1}|_Z)$.

Proof. Let $H=(H_1,\ldots,H_k)$ and $x\in\mathrm{Crit}(H_{k+1}|_Z)$. Suppose $x\in\mathrm{Crit}(H_{k+1}|_Z)$. Since the map (5.13) is a well-defined isomorphism,

$$\mathrm{d}_xH_{k+1}=r_1H_1+\ldots+r_kH_k\colon T_xX\longrightarrow\mathbb{R}$$

for some $r\equiv(r_1,\ldots,r_k)\in\mathbb{R}^k$. The map

$$H_{k+1;r}\equiv H_{k+1}-\big(r_1H_1+\ldots+r_kH_k\big)\colon X\longrightarrow\mathbb{R}$$

is then a Hamiltonian for an almost periodic $\mathbb{R}$-action on (X,ω) so that $x\in Z\cap\mathrm{Crit}(H_{k+1;r})$. This Hamiltonian is preserved by the restriction of the action $\widetilde\psi$ to $\mathbb{R}^k\times\{0\}$. Since $H_{k+1;r}-H_{k+1}$ restricts to the constant $r\cdot c$ on Z,

$$\mathrm{Crit}\big(H_{k+1;r}|_Z\big)=\mathrm{Crit}\big(H_{k+1}|_Z\big),\quad \mathrm{E}_x^0\big(H_{k+1;r}|_Z\big)=\mathrm{E}_x^0\big(H_{k+1}|_Z\big),\quad n_x^{\pm}\big(H_{k+1;r}|_Z\big)=n_x^{\pm}\big(H_{k+1}|_Z\big).$$

By Lemma 5.8, the closed submanifold $Z\cap\mathrm{Crit}(H_{k+1;r})$ of Z is thus an open subset of $\mathrm{Crit}(H_{k+1})$,

$$T_x\big(Z\cap\mathrm{Crit}(H_{k+1;r})\big)=\mathrm{E}_x^0\big(H_{k+1}|_Z\big),\qquad\text{and}\qquad n_x^{\pm}\big(H_{k+1}|_Z\big)\in2\mathbb{Z}^{\ge0}. \tag{5.14}$$

We conclude that $\mathrm{Crit}(H_{k+1}|_Z)$ is a finite union of submanifolds $Z\cap\mathrm{Crit}(H_{k+1;r})$ of Z with $r\in\mathbb{R}^k$, each of which is a union of the topological components of $\mathrm{Crit}(H_{k+1}|_Z)$ and satisfies (5.14) for all $x\in Z\cap\mathrm{Crit}(H_{k+1;r})$. □

Proof of ($A_k^\star$) on page 4. The claim is trivially true for $k=0$. We assume that it is true for some $k\in\mathbb{Z}^{\ge 0}$ and show that it also holds with k replaced by $k+1$. Let

$$\widetilde{H}\equiv(H,H_{k+1})\colon X\longrightarrow\mathbb{R}^k\times\mathbb{R}=\mathbb{R}^{k+1}$$

be a Hamiltonian for an almost periodic $\mathbb{R}^{k+1}$-action $\widetilde{\psi}$ on (X,ω) and $\widetilde{c}\equiv(c,c_{k+1})\in\mathbb{R}^k\times\mathbb{R}$.

Suppose first that $c\in\mathbb{R}^k$ is a regular value of H and thus $Z\subset H^{-1}(c)$ is a closed submanifold of X. It is connected by the inductive assumption. By Corollary 5.9, $H_{k+1}|_Z$ is a Morse-Bott function with $n_x^\pm(H_{k+1}|_Z)\in 2\mathbb{Z}^{\ge 0}$. By Proposition 4.8,

$$\widetilde{H}^{-1}(\widetilde{c})\equiv\big\{H_{k+1}|_Z\big\}^{-1}\big(c_{k+1}\big)\subset X$$

is thus connected.

Let $\mathbb{R}^k_H\subset\mathbb{R}^k$ and $\mathbb{R}^{k+1}_{\widetilde{H}}\subset\mathbb{R}^{k+1}$ be the subsets of regular values of H and $\widetilde{H}$, respectively. In particular,

$$\mathbb{R}^{k+1}_{\widetilde{H}}=\big\{\widetilde{c}\in\mathbb{R}^{k+1}\colon \mathrm{d}_xH_1,\ldots,\mathrm{d}_xH_{k+1}\in T_x^*X\text{ are lin. independent }\forall\,x\in\widetilde{H}^{-1}(\widetilde{c})\big\}.$$

Since the subset $\widetilde{H}^{-1}(\widetilde{c})\subset X$ is compact for every $\widetilde{c}\in\mathbb{R}^{k+1}$, the subset $\mathbb{R}^{k+1}_{\widetilde{H}}\subset\mathbb{R}^{k+1}$ is open. The function

$$\mathbb{R}^{k+1}_{\widetilde{H}}\longrightarrow\mathbb{Z}^{\ge 0},\qquad \widetilde{c}\longrightarrow\big|\pi_0\big(\widetilde{H}^{-1}(\widetilde{c})\big)\big|,\tag{5.15}$$

is constant on the connected components of $\mathbb{R}^{k+1}_{\widetilde{H}}$ and takes value 0 or 1 on $\mathbb{R}^{k+1}_{\widetilde{H}}\cap(\mathbb{R}^k_H\times\mathbb{R})$. Since the subset $\mathbb{R}^k_H\subset\mathbb{R}^k$ is dense by Sard's Theorem and each connected component of $\mathbb{R}^{k+1}_{\widetilde{H}}$ is open in $\mathbb{R}^{k+1}$, the function (5.15) takes value 0 or 1 on each connected component of $\mathbb{R}^{k+1}_{\widetilde{H}}$, i.e. $\widetilde{H}^{-1}(\widetilde{c})\subset X$ is connected for every $\widetilde{c}\in\mathbb{R}^{k+1}_{\widetilde{H}}$. $\square$

Proof of (A_k). By Exercises 3.10 and 2.4, we can assume that the action ψ is irreducible. Let $(T_0^*\mathbb{R}^k)_\mu\subset T_0^*\mathbb{R}^k$ be the subset of regular values of μ. By (F_k), $\mu^{-1}((T_0^*\mathbb{R}^k)_\mu)\subset X$ is the complement of a finite union of submanifolds of positive codimensions. Thus, the subset

$$P^*\equiv\mu\big(\mu^{-1}\big((T_0^*\mathbb{R}^k)_\mu\big)\big)\subset\mu(X)\equiv P$$

is dense in P. By (D_k), the map $\mu\colon X\longrightarrow P$ is open. By ($\mathrm{A}_k^\star$), $\mu^{-1}(\alpha)\subset X$ is connected for every $\alpha\in P^*$. Thus, (A_k) now follows from Exercise 4.10. $\square$

Proof of (B_k). This claim is trivially true for $k=0$. Suppose $k\in\mathbb{Z}^+$ and $H\colon X\longrightarrow\mathbb{R}^k$ is a Hamiltonian for an almost periodic $\mathbb{R}^k$-action $\widetilde{\psi}$ on (X,ω). For a $k\times(k-1)$ real matrix A, the $\mathbb{R}^{k-1}$-action $\psi_A\equiv\psi\circ A$ is then also almost periodic with Hamiltonian

$$H_A\equiv A^{\mathrm{tr}}\circ H\colon X\longrightarrow\mathbb{R}^{k-1}\,.$$

Suppose $x_0,x_1\in X$. Let A be a $k\times(k-1)$ injective real matrix A so that

$$H(x_1)-H(x_0)\in\ker A^{\mathrm{tr}}.$$

Thus, $x_1\in H_A^{-1}(H_A(x_0))$ and

$$H(x_1)\in H\big(H_A^{-1}\big(H_A(x_0)\big)\big)\subset\big\{H(x_0)+c\colon c\in\ker A^{\rm tr}\big\}.$$

Since $\ker A^{\rm tr}$ is a line and $H_A^{-1}\big(H_A(x_0)\big)\subset X$ is connected by (A_k),

$$H\big(H_A^{-1}\big(H_A(x_0)\big)\big)\subset H(X)$$

contains the line segment between $H(x_0)$ and $H(x_1)$. Thus, $H(X)\subset\mathbb{R}^k$ is convex. $\square$

Proof of (C_k). Let $\rho\colon\mathbb{R}^k\longrightarrow\mathbb{T}$ and ψ' be as in (1.5). We can assume that the image of ρ is dense in $\mathbb{T}$ and so $X^\psi=X^{\psi'}$. The first claim then follows from Proposition 3.14(1). By Proposition 3.8(1) and the first equation in (1.4), $\mathrm{d}\mu$ vanishes along X^ψ; this implies the second claim. By (B_k), $\mu(X)\subset\mathrm{CH}(\mu(X^\psi))$.

Suppose $\eta_0\in T_0^*\mathbb{R}^k-\mathrm{CH}(\mu(X^\psi))$. Thus, there exists $v\in T_0\mathbb{R}^k$ so that

$$\eta_0(v)<\min\big\{\eta(v)\colon\eta\in\mu(X^\psi)\big\}=\min\big\{\eta(v)\colon\eta\in\mathrm{CH}\big(\mu(X^\psi)\big)\big\}. \tag{5.16}$$

Let $y\in X$ be a minimum of the smooth function μ_v as in (5.12) Thus, $\mathrm{d}_y\mu_v=0$, the vector field ζ_v as in (1.2) vanishes at y, and y lies in the fixed locus $X^{\mathbb{R}v}$ of the restriction of the ψ-action to $\mathbb{R}v\subset\mathbb{R}^k$. For a generic choice of $v\in\mathbb{R}^k$ satisfying (5.16), $\rho(\mathbb{R}v)\subset\mathbb{T}$ is dense and thus $X^{\mathbb{R}v}=X^\psi$. It follows that

$$\eta_0(v)<\min\big\{\eta(v)\colon\eta\in\mu(X^\psi)\big\}=\min\big\{\mu_v(x)\colon x\in X\big\}=\min\big\{\eta(v)\colon\eta\in\mu(X)\big\},$$

i.e. $\eta_0\notin\mu(X)$.

Thus, $\mu(X)=\mathrm{CH}(\mu(X^\psi))$. The vertices of this polytope are of the form $\mu(Y)$ with $Y\in\pi_0(X^\psi)$. By (3.25), the number of edges at any such vertex $\mu(Y)$ is at most $|S(Y)|$. Since the real rank of each subbundle $\mathcal{N}_X^\alpha Y\subset TX|_Y$ is at least 2, $|S(Y)|\le(\dim X)/2$. $\square$

Proof of (E_k). Suppose $Y\in\pi_0(X^\psi)$. Let ρ, J, $S(Y)$, $\mathcal{N}_X^\alpha Y$ for each $\alpha\in S(Y)$, and $\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$ be as in Proposition 3.27. If $\rho^*S(Y)$ does not span $T_0^*\mathbb{R}^k$ over $\mathbb{R}$, there exists

$$v\in T_0\mathbb{R}^k-\{0\}\qquad\text{s.t.}\quad\{\rho^*\alpha\}(v)=0\ \ \forall\,\alpha\in S(Y).$$

The subgroup $\mathbb{R}v\subset\mathbb{R}^k$ then acts trivially on $TX|_Y$. By Proposition 3.3(1), this implies that the connected component of the $\mathbb{R}v$-fixed locus $X^{\mathbb{R}v}$ containing Y is a connected component of X, i.e. $\mathbb{R}v$ acts trivially on X (and so the action ψ is reducible), since X is connected. If the action ψ is reducible, then $\mathbb{R}v\subset\mathbb{R}^k$ acts trivially on X and thus on $TX|_Y$ for some $v\in T_0\mathbb{R}^k$ nonzero and thus $\{\rho^*\alpha\}(v)=0$ for every $\alpha\in S(Y)$, i.e. $\rho^*S(Y)$ does not span $T_0^*\mathbb{R}^k$ over $\mathbb{R}$.

Thus, $\rho^*S(Y)$ spans $T_0^*\mathbb{R}^k$ over $\mathbb{R}$ if and only if the action ψ is irreducible. Suppose $\mu(Y)\in\mathrm{Ver}(\mu(X^\psi))$ is a vertex of the polytope $\mu(X)=\mathrm{CH}(\mu(X^\psi))$. By Proposition 3.27, the edges of $\mu(X)$ at $\mu(Y)$ are the edges of the cone $\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$. A subset $S_\mu(Y)$ of $\rho^*S(Y)$ thus forms a collection of edge vectors of the polytope $\mu(X)$ at the vertex $\mu(Y)$, while the elements of $\rho^*S(Y)-S_\mu(Y)$ lie in the span of $S_\mu(Y)$. We conclude that $S_\mu(Y)$ spans $T_0^*\mathbb{R}^k$ over $\mathbb{R}$ if and only if the action ψ is irreducible. $\square$

Proof of Theorem 2(1). Suppose $Y\in\pi_0(X^\psi)$. Let ρ, J, $S(Y)$, and $\mathcal{N}_X^\alpha Y$ for each $\alpha\in S(Y)$ be as in Proposition 3.27 with $(\mathbb{R}^k,0)$ replaced by $(\mathbb{T},\mathbb{1})$ and thus $\rho=\mathrm{id}$. As above, a subset $S_\mu(Y)$ of $\rho^*S(Y)=S(Y)$ forms a collection of edge vectors of the polytope $\mu(X)$ at $\mu(Y)$. Since $S(Y)\subset(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$, all edges of the polytope $\mu(X)$ at $\mu(Y)$ are rational. □

Proof of (0^+) on page 8. Let ρ, J, $S(Y)$ for each $Y\in\pi_0(X^\psi)$, $\mathcal{N}_X^\alpha Y$ for each $\alpha\in S(Y)$, and $\mathcal{C}_{\mu(Y)}(\rho^*S(Y))$ be as in Proposition 3.27 with $(\mathbb{R}^k,0)$ replaced by $(\mathbb{T},\mathbb{1})$ and thus $\rho=\mathrm{id}$. By (3.25) and (1.7), $|S(Y)|$ is at most the dimension of $\mathbb{T}$. Since the action ψ is effective, Proposition 3.6(1) then implies that $S(Y)$ is a $\mathbb{Z}$-basis for $T_{\mathbb{1}}^*\mathbb{T}$ for every $Y\in\pi_0(X^\psi)$; this remains the case if some of the elements of $S(Y)$ are negated. In particular, the polytope $\mu(X)\subset T_{\mathbb{1}}^*\mathbb{T}$ is of full dimension. Furthermore, for every $Y\in\pi_0(X^\psi)$, the cone $\mathcal{C}_{\mu(Y)}(S(Y))$ contains no lines, $\mu(Y)\in\mathrm{Ver}(\mu(X))$, and for every $S\subset S(Y)$ the μ-image of the topological component X_Y^S of the $\psi|_{\mathbb{T}_S}$-fixed locus X^S containing Y lies in the cone $\mathcal{C}_{\mu(Y)}(S)$ of dimension $|S|$ and contains a neighborhood of the vertex $\mu(Y)$ of this cone.

By Proposition 3.14(1), $(X_Y^S,\omega|_{X_Y^S},\psi|_{X_Y^S},\mu|_{X_Y^S})$ is a closed connected Hamiltonian $\mathbb{T}$-manifold for every $Y\in\pi_0(X^\psi)$ and $S\subset S(Y)$. Thus,

$$\mu(X_Y^S)=\mathrm{CH}\big(\mu\big((X_Y^S)^\psi\big)\big)=\mathrm{CH}\big(\mu(X^\psi\cap X_Y^S)\big)$$

by Theorem 1(C_k). Since $\mu(X^\psi\cap X_Y^S)\subset\mathrm{Ver}(\mu(X))$, it follows that $\mu(X_Y^S)$ is the face $F_{\mu(Y);S}(\mu(X))$ of the polytope $\mu(X)$ containing the edges

$$e_{\mu(Y);\alpha}\equiv\mu(X)\cap\{\mu(Y)+t_\alpha\alpha\colon t_\alpha\in\mathbb{R}^{\ge0}\}$$

with $\alpha\in S$. Since $S(Y)$ is a $\mathbb{Z}$-basis for $T_{\mathbb{1}}^*\mathbb{T}$ for every $Y\in\pi_0(X^\psi)$, Exercise 3.24 implies that for each $x\in X$ there exist $Y_x\in\pi_0(X^\psi)$ and $S_x\subset S(Y_x)$ so that the ψ-stabilizer $\mathbb{T}_x(\psi)\subset\mathbb{T}$ of x is the subtorus $\mathbb{T}_{S_x}\subset\mathbb{T}$ and $x\in X_{Y_x}^{S_x}$. It follows that

$$\mu^{-1}\big(F^\circ_{\mu(Y);S}(\mu(X))\big)=\big\{x\in X_Y^{\mathbb{T}_S}\colon\mathbb{T}_x(\psi)=\mathbb{T}_S\big\}\qquad\forall\,Y\in\pi_0(X^\psi),\ S\subset S(Y),\tag{5.17}$$

where $F^\circ_{\mu(Y);S}(\mu(X))\subset F_{\mu(Y);S}(\mu(X))$ is the interior.

Suppose $e\in\mathrm{Edg}(\mu(X))$ is an edge of the polytope $\mu(X)$ and thus $e=e_{\mu(Y);\alpha}$ for some $Y\in\pi_0(X^\psi)$ and $\alpha\in S(Y)$. We then set $\alpha_e=\alpha$. If $Y'\in\pi_0(X^\psi)$ is such that $\mu(Y')$ is the vertex of $e_{\mu(Y);\alpha}$ other than $\mu(Y)$, then $-\alpha\in S(Y')$ and $e=e_{\mu(Y');-\alpha}$. Thus, $(\alpha_e)_{e\in\mathrm{Edg}(\mu(X))}$ is a full tuple of integral edge vectors for the polytope $\mu(X)$ such that for each vertex η of $\mu(X)$ the components α_e with $e\in\mathrm{Edg}_\eta(\mu(X))$ form a $\mathbb{Z}$-basis for $(T_{\mathbb{1}}^*\mathbb{T})_{\mathbb{Z}}$ and

$$\mathbb{T}_S=\bigcap_{\alpha\in S}\mathbb{T}_{\alpha_{e_{\mu(Y);\alpha}}}\qquad\forall\,Y\in\pi_0(X^\psi),\ S\subset S(Y).\tag{5.18}$$

Suppose $F\subset\mu(X)$ is a face of the polytope $\mu(X)$ and thus $F=F_{\mu(Y);S}(\mu(X))$ for some $Y\in\pi_0(X^\psi)$ and $S\subset S(Y)$. We then set $\mathbb{T}_F=\mathbb{T}_S$. Thus, $\mathbb{T}_F\subset\mathbb{T}$ is a subtorus. By (5.17), (0^+b) holds. This implies that $\mathbb{T}_F$ is independent of the choice of $\mu(Y)\in F$. By (5.18), (0^+a) thus holds for all $\eta\in F$.

Let $Y\in\pi_0(X^\psi)$, $S\subset S(Y)$, and $F=F_{\mu(Y);S}(\mu(X))$. Since

$$\dim Y+\sum_{\alpha\in S(Y)}\mathrm{rk}\,\mathcal{N}_X^\alpha Y=\dim X,\quad \mathrm{rk}\,\mathcal{N}_X^\alpha Y\ge2\ \forall\,\alpha\in S(Y),\quad\text{and}\quad |S(Y)|=\dim\mathbb{T}=(\dim X)/2,$$

we conclude that $\dim Y = 0$ and $\mathrm{rk}\,\mathcal{N}_X^{\alpha} Y = 2$ for every $\alpha \in S(Y)$. Along with Proposition 3.6(2), this implies that

$$\dim X_Y^{\mathbb{T}_S} = 2|S| = 2\dim F = 2\dim(\mathbb{T}/\mathbb{T}_F)\,. \tag{5.19}$$

Since $\alpha(v)=0$ for all $\alpha\in S$ and $v\in T_{\mathbb{1}}\mathbb{T}_S$, the affine map

$$\Phi_{Y;S}\colon \big\{\mu(Y)+\sum_{\alpha\in S} t_\alpha\alpha\colon t_\alpha\in\mathbb{R}\big\} \longrightarrow T_{\mathbb{1}}^*(\mathbb{T}/\mathbb{T}_S), \quad \big\{\Phi_{Y;S}(\eta)\big\}(v) = \eta(v) - \big\{\mu(Y)\big\}(v),$$

is a well-defined surjection and thus affine isomorphism by the last equality in (5.19). By (5.17), the torus $\mathbb{T}/\mathbb{T}_F$ acts freely on $\mu^{-1}(F^\circ)\subset X_Y^{\mathbb{T}_S}$ via ψ; we denote this action by $\psi_{Y;S}$. By Theorem 1(A_k), the fibers of the restriction

$$\mu\colon \mu^{-1}(F^\circ) \longrightarrow F^\circ$$

are connected. Since $F^\circ\subset F$ is open, $\mu^{-1}(F^\circ)\subset X_Y^{\mathbb{T}_S}$ is a symplectic submanifold. Thus, (0^+c) follows from Exercise 3.22(a) with $\mathbb{T}$ and (X,ω,ψ,μ) replaced by $\mathbb{T}/\mathbb{T}_S$ and

$$\big(\mu^{-1}(F^\circ), \omega|_{\mu^{-1}(F^\circ)}, \psi_{Y;S}, \Phi_{Y;S}\circ\mu|_{\mu^{-1}(F^\circ)}\big),$$

respectively. □

6 Symplectic Quotient and Cut Constructions

6.1 Symplectic quotient

For a Lie group G, let

$$\big(T_{\mathbb{1}}^*G\big)^G \equiv \big\{\alpha\in T_{\mathbb{1}}^*G\colon \mathrm{Ad}_g^*(\alpha)=\alpha\ \forall\, g\in G\big\}$$

be the fixed locus of the dual of the adjoint action of G on $T_{\mathbb{1}}G$. If ψ is a G-action on a space X, $\mu\colon X\longrightarrow T_{\mathbb{1}}G$ is a map satisfying the second condition in (1.4), and $\alpha\in(T_{\mathbb{1}}^*G)^G$, then ψ restricts to a G-action on $\mu^{-1}(\alpha)\subset X$. If G is abelian, then $(T_{\mathbb{1}}^*G)^G = T_{\mathbb{1}}^*G$.

Theorem 4 ([23, Theorems 3,4])**.** *Suppose G is a compact Lie group, $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$ is a Hamiltonian G-manifold, and $\alpha\in(T_{\mathbb{1}}^*G)^G$ are such that G acts freely on $\widetilde{\mu}^{-1}(\alpha)$. Then,*

(0) $\alpha\in T_{\mathbb{1}}^*G$ is a regular of $\widetilde{\mu}$;

(1) there is a unique smooth structure on $X_\alpha\equiv\widetilde{\mu}^{-1}(\alpha)/G$ so that the quotient projection

$$p_\alpha\colon \widetilde{\mu}^{-1}(\alpha) \longrightarrow X_\alpha$$

is a principal G-bundle;

(2) there exists a unique 2-form ω_α on X_α so that $p_\alpha^*\omega_\alpha=\widetilde{\omega}|_{T\widetilde{\mu}^{-1}(\alpha)}$;

(3) the 2-form ω_α is symplectic.

If G' is another Lie group and $(\widetilde{X},\widetilde{\omega},\widetilde{\psi}',\widetilde{\mu}')$ is a Hamiltonian G'-manifold such that the actions $\widetilde{\psi}$ and $\widetilde{\psi}'$ commute, $\widetilde{\mu}'$ is ψ-invariant, and $\widetilde{\mu}$ is ψ'-invariant, then $\widetilde{\psi}'$ and $\widetilde{\mu}'$ descend to a G'-action ψ'_α on X_α and a smooth map $\mu'_\alpha\colon X_\alpha\longrightarrow T_{\mathbb{1}}G'$, respectively, so that $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ is a Hamiltonian G'-manifold.

The symplectic manifold (X_α,ω_α) of Theorem 4 is called the symplectic quotient of $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$ at α. We will similarly call the Hamiltonian G'-manifold $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ of this theorem the Hamiltonian quotient of $(\widetilde{X},\widetilde{\omega},(\widetilde{\psi},\widetilde{\psi}'),(\widetilde{\mu},\widetilde{\mu}'))$ at α.

Proof of Theorem 4. Exercise 3.21(b) establishes (0). Since G is compact, the quotient projection p_α is a closed map. By [25, Lemma 37.1], the quotient space X_α is thus Hausdorff. By (0) and the Implicit Function Theorem, $\widetilde{\mu}^{-1}(\alpha)\subset\widetilde{X}$ is a smooth submanifold on which the compact Lie G acts smoothly and freely. By the Slice Theorem (equivariant version of the Tubular Neighborhood Theorem), for every $x\in\widetilde{\mu}^{-1}(\alpha)$ there thus exists a submanifold $S_x\subset\widetilde{X}$ so that $x\in S_x$ and the map

$$S_x\times G\longrightarrow\widetilde{X},\qquad (x',u)\longrightarrow\psi_u(x'),$$

is a diffeomorphism onto an open neighborhood $\widetilde{U}_x\subset\widetilde{X}$ of x preserved by G. This submanifold is then transverse to the orbits Gx' of G and thus to $\widetilde{\mu}^{-1}(\alpha)$. The restriction

$$p_\alpha\colon\widetilde{\mu}^{-1}(\alpha)\cap S_x\longrightarrow p_\alpha\big(\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{U}_x\big)\subset X_\alpha$$

of the quotient map is a homeomorphism onto an open subset of X_α and induces a smooth structure on $p_\alpha(\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{U}_x)$ so that the restriction

$$p_\alpha\colon\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{U}_x\longrightarrow p_\alpha\big(\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{U}_x\big)$$

is a (trivial) principle G-bundle. Since there is at most one smooth structure on $p_\alpha(\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{U}_x)$ so that the latter restriction is a submersion, the smooth structures on open subset of X_α obtained in this way overlap smoothly. This establishes (1).

By (1), for every $x\in\widetilde{\mu}^{-1}(\alpha)$ the map

$$\mathrm{d}_xp_\alpha\colon T_x\widetilde{\mu}^{-1}(\alpha)/T_x(Gx)\longrightarrow T_{p_\alpha(x)}X_\alpha$$

is a well-defined isomorphism. For each $v\in T_{\mathbb{1}}G$, let $\zeta_v\in\Gamma(\widetilde{X};T\widetilde{X})$ be as in (1.2) with (X,ψ) replaced by $(\widetilde{X},\widetilde{\psi})$. Thus,

$$\begin{gathered}T_x(Gx)=\big\{\zeta_v(x)\colon v\in T_{\mathbb{1}}G\big\}\quad\forall\,x\in\widetilde{\mu}^{-1}(\alpha)\qquad\text{and}\\ -\big(\iota_{\zeta_v(x)}\widetilde{\omega}\big)\big|_{T_x\widetilde{\mu}^{-1}(\alpha)}=\mathrm{d}_x\big(\{\widetilde{\mu}(\cdot)\}(v)\big|_{T_x\widetilde{\mu}^{-1}(\alpha)}\big)=\mathrm{d}_x\big(\alpha(v)\big|_{T_x\widetilde{\mu}^{-1}(\alpha)}\big)=0\quad\forall\,v\in T_{\mathbb{1}}G,\,x\in\widetilde{\mu}^{-1}(\alpha).\end{gathered}$$

It follows that there is a unique alternating 2-tensor $\omega_\alpha|_{p_\alpha(x)}$ on $T_{p_\alpha(x)}X_\alpha$ for each $x\in\widetilde{\mu}^{-1}(\alpha)$ so that

$$\omega_\alpha|_{p_\alpha(x)}\big(\mathrm{d}_xp_\alpha(w),\mathrm{d}_xp_\alpha(w')\big)=\widetilde{\omega}|_x(w,w')\quad\forall\,w,w'\in T_x\widetilde{\mu}^{-1}(\alpha),$$

i.e. $p_\alpha^*(\omega_\alpha|_{p_\alpha(x)})=\widetilde{\omega}|_{T_x\widetilde{\mu}^{-1}(\alpha)}$. Since ω is G-invariant, $\omega_\alpha|_{p_\alpha(x)}$ does not depend on the choice of x in $p_\alpha^{-1}(p_\alpha(x))$, i.e. ω_α is a well-defined 2-form on X_α. Since p_α is a submersion and the 2-form $\widetilde{\omega}|_{T\widetilde{\mu}^{-1}(\alpha)}$ is smooth and closed, so is the 2-form ω_α. Since α is a regular value of $\widetilde{\mu}$ and $\widetilde{\omega}$ is nondegenerate on $\widetilde{X}$, the first statement of Exercise 3.21 with (ω,μ) replaced by $(\widetilde{\omega},\widetilde{\mu})$ implies that

$$\big(T_x\widetilde{\mu}^{-1}(\alpha)\big)^{\widetilde{\omega}}=\big(\ker\mathrm{d}_x\widetilde{\mu}\big)^{\widetilde{\omega}}=T_x(Gx).$$

Thus, ω_α is nondegenerate.

Suppose G' is another Lie group and $(\widetilde{X},\widetilde{\omega},\widetilde{\psi}',\widetilde{\mu}')$ is a Hamiltonian G'-manifold such that the actions $\widetilde{\psi}$ and $\widetilde{\psi}'$ commute, $\widetilde{\mu}'$ is ψ-invariant, and $\widetilde{\mu}$ is ψ'-invariant. Since $\widetilde{\mu}$ is ψ'-invariant, ψ' preserves $\widetilde{\mu}^{-1}(\alpha)\subset X$. Since the actions $\widetilde{\psi}$ and $\widetilde{\psi}'$ commute and $\widetilde{\mu}'$ is ψ-invariant, the restriction of $\widetilde{\psi}'$ and $\widetilde{\mu}'$ to $\widetilde{\mu}^{-1}(\alpha)$ descend to a G'-action ψ'_α on X_α and a smooth map $\mu'_\alpha\colon X_\alpha\longrightarrow T_{\mathbb{1}}G'$. By Exercise 2.3 with $(\widetilde{\psi},\psi)$ replaced by $(\widetilde{\psi}',\psi')$, $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ is thus a Hamiltonian G'-manifold.
□

Exercise 6.1. Suppose G, $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$, α, (X_α,ω_α), and p_α are as in Theorem 4 and $\widetilde{X}'\subset\widetilde{X}$ is an $\widetilde{\omega}$-symplectic submanifold preserved by the G-action $\widetilde{\psi}$.

(a) Show that the symplectic quotient $(X'_\alpha,\omega'_\alpha)$ of $(\widetilde{X}',\widetilde{\omega}|_{\widetilde{X}'},\widetilde{\psi}|_{\widetilde{X}'},\widetilde{\mu}|_{\widetilde{X}'})$ at α is a symplectic submanifold of (X_α,ω_α) and the bundle homomorphisms

$$\mathcal{N}_{\widetilde{X}}\widetilde{X}'\big|_{\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'}\longleftarrow\mathcal{N}_{\widetilde{\mu}^{-1}(\alpha)}\big(\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'\big)\xrightarrow{\mathrm{d}p_\alpha}\big\{p_\alpha|_{\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'}\big\}^*\mathcal{N}_{X_\alpha}X'_\alpha \tag{6.1}$$

over $\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{X}'$ induced by the inclusions and the quotient projection are isomorphisms.

(b) Suppose that G', $\widetilde{\psi}'$, $\widetilde{\mu}'$, and $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ are also as in Theorem 4 and the submanifold $\widetilde{X}'\subset\widetilde{X}$ is preserved by the G'-action $\widetilde{\psi}'$. Show that the submanifold $X'_\alpha\subset X_\alpha$ is preserved by the G'-action ψ', $(X'_\alpha,\omega'_\alpha,\psi'_\alpha|_{X'_\alpha},\mu'_\alpha|_{X'_\alpha})$ is the Hamiltonian quotient of

$$\big(\widetilde{X}',\widetilde{\omega}|_{\widetilde{X}'},(\widetilde{\psi},\widetilde{\psi}')|_{\widetilde{X}'},(\widetilde{\mu},\widetilde{\mu}')|_{\widetilde{X}'}\big)$$

at α, and the bundle isomorphisms in (6.1) are G'-equivariant.

Exercise 6.2. Suppose G, $(\widetilde{X},\widetilde{\omega},\widetilde{\psi},\widetilde{\mu})$, α, (X_α,ω_α), and p_α are as in Theorem 4 and $\widetilde{J}$ is a ψ-invariant almost complex structure on $\widetilde{X}$ compatible with $\widetilde{\omega}$. Show that

(a) the restriction of the differential

$$\mathrm{d}_xp_\alpha\colon T_x\widetilde{\mu}^{-1}(\alpha)\cap\widetilde{J}\big(T_x\widetilde{\mu}^{-1}(\alpha)\big)\longrightarrow T_{p_\alpha(x)}X_\alpha$$

is an isomorphism for every $x\in\widetilde{\mu}^{-1}(\alpha)$ and thus induces an almost complex structure J_α on X_α compatible with ω_α;

(b) if G', $\widetilde{\psi}'$, $\widetilde{\mu}'$, and $(X_\alpha,\omega_\alpha,\psi'_\alpha,\mu'_\alpha)$ are also as in Theorem 4 and the almost complex structure $\widetilde{J}$ on $\widetilde{X}$ is $\widetilde{\psi}'$-invariant as well, then the almost complex structure J_α is ψ'_α-invariant;

(c) if $\widetilde{X}'\subset\widetilde{X}$ is an almost complex submanifold preserved by the G-action $\widetilde{\psi}$ and $(X'_\alpha,\omega'_\alpha)$ is the symplectic quotient of $(\widetilde{X}',\widetilde{\omega}|_{\widetilde{X}'},\widetilde{\psi}|_{\widetilde{X}'},\widetilde{\mu}|_{\widetilde{X}'})$ at α, then X'_α is an almost complex submanifold of X_α and the composite isomorphism from the left-hand side in (6.1) to the right-hand side is $\mathbb{C}$-linear with respect to the complex structures induced by $\widetilde{J}$ and J_α.

Example 6.3. Let $n\in\mathbb{Z}^+$. By Exercise 2.5,

$$H\colon\mathbb{C}^n\longrightarrow\mathbb{R},\qquad H(z_1,\ldots,z_n)=\pi\sum_{k=1}^n|z_k|^2,$$

is a Hamiltonian for the standard action ψ of S^1 on $\mathbb{C}^n$,

$$\psi_u\colon\mathbb{C}^n\longrightarrow\mathbb{C}^n,\quad \psi_u(z)=uz,\qquad\forall\,u\in S^1\subset\mathbb{C}.$$

For each $r\in\mathbb{R}^+$, the group S^1 acts freely on $H^{-1}(\pi r^2)$, the sphere of radius r centered at the origin. The associated quotient of Theorem 4 is the complex projective space $\mathbb{C}P^{n-1}$ with a symplectic form $\omega_{\mathbb{C}P^{n-1};r}$. By Exercise 2.7(b),

$$\omega_{\mathbb{C}P^{n-1};r} = \pi r^2 \omega_{\mathrm{FS};n-1}\,.$$

Exercise 6.4. Let $n\in\mathbb{Z}^+$ and $q\colon \mathbb{C}^n-\{0\}\longrightarrow\mathbb{C}P^{n-1}$ be the usual quotient projection. The $\mathbb{C}^*$-action on $\mathbb{C}^n$ by the coordinate multiplication restricts to a $\mathbb{C}^*$-action on $\mathbb{C}^n-\{0\}$ and S^1-actions on $\mathbb{C}^n$ and the unit sphere $S^{2n-1}\subset\mathbb{C}^n$. Show that

(a) the quotient topologies on $\mathbb{C}P^{n-1}$ given by $(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ and S^{2n-1}/S^1 are the same (i.e. the map $S^{2n-1}/S^1\longrightarrow(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ induced by inclusions is a homeomorphism);

(b) $\mathbb{C}P^{n-1}$ is a compact topological $2(n-1)$-manifold that admits a complex structure so that the quotient projections

$$q\colon \mathbb{C}^n-\{0\}\longrightarrow \mathbb{C}P^{n-1}=(\mathbb{C}^n-\{0\})/\mathbb{C}^* \quad\text{and}\quad p\colon S^{2n-1}\longrightarrow \mathbb{C}P^{n-1}=S^{2n-1}/S^1$$

are a holomorphic submersion and a smooth submersion, respectively.

(c) the above complex structure is compatible with the Fubini-Study symplectic form $\omega_{\mathrm{FS};n-1}$ of Exercise 2.7(b).

6.2 Hamiltonian symplectic cut

Suppose $\mathbb{T}$ is a torus. For $\upsilon\equiv(v,c)\in T_{\mathbb{1}}\mathbb{T}\times\mathbb{R}$, let

$$c_\upsilon=c, \qquad \mathcal{H}_\upsilon\equiv\big\{\alpha\in T^*_{\mathbb{1}}\mathbb{T}\colon \alpha(v)\ge c\big\}, \qquad\text{and}\qquad \partial\mathcal{H}_\upsilon\equiv\big\{\alpha\in T^*_{\mathbb{1}}\mathbb{T}\colon \alpha(v)=c\big\};$$

the subspaces $\mathcal{H}_\upsilon,\partial\mathcal{H}_\upsilon\subset T^*_{\mathbb{1}}\mathbb{T}$ are a (closed) half-space and an affine hyperplane, respectively, if $v\neq0$. For $\mathscr{H}\subset T_{\mathbb{1}}\mathbb{T}\times\mathbb{R}$ and $\mathscr{H}'\subset\mathscr{H}$, let

$$\langle\mathscr{H}\rangle=\bigcap_{\upsilon\in\mathscr{H}}\mathcal{H}_\upsilon\subset T^*_{\mathbb{1}}\mathbb{T},\quad \langle\mathscr{H}\rangle^\partial=\bigcap_{\upsilon\in\mathscr{H}}\partial\mathcal{H}_\upsilon\subset\langle\mathscr{H}\rangle,\quad \langle\mathscr{H}'\rangle^\partial_{\mathscr{H}}=\langle\mathscr{H}'\rangle^\partial\cap\langle\mathscr{H}\rangle-\bigcap_{\mathscr{H}'\subsetneq\mathscr{H}''\subset\mathscr{H}}\langle\mathscr{H}''\rangle^\partial.$$

In particular, $\langle\emptyset\rangle=\langle\emptyset\rangle^\partial=\langle\emptyset\rangle^\partial_\emptyset=T^*_{\mathbb{1}}\mathbb{T}$, $\langle\mathscr{H}'\rangle^\partial_{\mathscr{H}}\subset\langle\mathscr{H}'\rangle^\partial$ is an open subset,

$$\langle\mathscr{H}'\rangle\cap\langle\mathscr{H}-\mathscr{H}'\rangle=\langle\mathscr{H}\rangle, \qquad\text{and}\qquad \langle\mathscr{H}'\rangle^\partial\cap\langle\emptyset\rangle^\partial_{\mathscr{H}-\mathscr{H}'}=\langle\mathscr{H}'\rangle^\partial_{\mathscr{H}}\,.$$

We call a collection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}-\{0\})\times\mathbb{R}$ minimal if $\partial\mathcal{H}_\upsilon\cap\langle\mathscr{H}\rangle\neq\emptyset$ for every $\upsilon\in\mathscr{H}$. Every collection $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}-\{0\})\times\mathbb{R}$ with $\langle\mathscr{H}\rangle\neq\emptyset$ contains a unique minimal subcollection $\mathscr{H}'$ with $\langle\mathscr{H}'\rangle=\langle\mathscr{H}\rangle$.

If $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}-\{0\})\times\mathbb{R}$ is a finite collection, $\langle\mathscr{H}\rangle$ is a polyhedron by definition. A polytope is easily seen to be a compact polyhedron. The converse, which is not needed for our purposes, follows from the Minkowski-Weyl Theorem [9, Theorem 3.13], which states that a polyhedron is a finitely generated cone on a polytope.

If $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subset, define

$$L_{\mathscr{H}}\colon\mathbb{R}^{\mathscr{H}}\longrightarrow T_{\mathbb{1}}\mathbb{T},\quad L_{\mathscr{H}}\big((r_{v,c})_{(v,c)\in\mathscr{H}}\big)=\sum_{(v,c)\in\mathscr{H}}r_{v,c}v\,,$$

$$\Phi_{\mathscr{H}}\colon\mathbb{T}^{\mathscr{H}}\equiv\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}\longrightarrow\mathbb{T},\quad \Phi_{\mathscr{H}}\big([\mathbf{r}]\big)=\mathrm{e}^{L_{\mathscr{H}}(\mathbf{r})}\,,\qquad \mathbb{T}_{\mathscr{H}}=\operatorname{Im}\Phi_{\mathscr{H}}. \tag{6.2}$$

In particular, $\mathbb{T}_{\mathscr{H}}\subset\mathbb{T}$ is a subtorus. We call a finite subset $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ Delzant if $\Phi_{\mathscr{H}'}$ is injective for every subset $\mathscr{H}'\subset\mathscr{H}$ such that $\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset$. This implies in particular that $v\in(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ is primitive for every element $(v,c)\in\mathscr{H}$ such that $\partial\mathcal{H}_{v,c}\cap\langle\mathscr{H}\rangle\neq\emptyset$ and $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}$ is a subtorus of dimension $|\mathscr{H}'|$ for every subset $\mathscr{H}'\subset\mathscr{H}$ such that $\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset$.

Exercise 6.5. Let $\mathbb{T}$ be a torus and $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a finite subset.

(a) Suppose $\mathbb{T}'\subset\mathbb{T}$ is a subtorus of $\mathbb{T}$ so that $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T}')_{\mathbb{Z}}\times\mathbb{R}$. Show that the images of

$$\langle\mathscr{H}\rangle,\langle\mathscr{H}\rangle^{\partial},\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle,\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\subset T^*_{\mathbb{1}}\mathbb{T}\qquad\text{with}\quad\mathscr{H}'\subset\mathscr{H}$$

under the restriction homomorphism $T^*_{\mathbb{1}}\mathbb{T}\longrightarrow T^*_{\mathbb{1}}\mathbb{T}'$ are the corresponding subsets of $T^*_{\mathbb{1}}\mathbb{T}'$. Conclude that $\mathscr{H}$ is Delzant with respect to $\mathbb{T}$ if and only if $\mathscr{H}$ is Delzant with respect to $\mathbb{T}'$.

(b) Suppose $\langle\mathscr{H}\rangle\neq\emptyset$. Show that there exists a subset $\mathscr{H}'\subset\mathscr{H}$ such that

$$\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\neq\emptyset\subset T^*_{\mathbb{1}}\mathbb{T}\qquad\text{and}\qquad\operatorname{Im}L_{\mathscr{H}'}=\operatorname{Im}L_{\mathscr{H}}\subset T_{\mathbb{1}}\mathbb{T}.$$

(c) Suppose $\mathscr{H}$ is Delzant. Show that $\ker\Phi_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ is a subtorus of codimension equal to the dimension of $\operatorname{Im}L_{\mathscr{H}}\subset T_{\mathbb{1}}\mathbb{T}$.

Exercise 6.6. Suppose $\mathbb{T}$ is a torus and $P\subset T^*_{\mathbb{1}}\mathbb{T}$ is a polytope. Show that P is Delzant if and only if $P=\langle\mathscr{H}\rangle$ for some Delzant subset $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ such that the homomorphism $L_{\mathscr{H}}$ is surjective.

Exercise 6.7. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is finite subset, and (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold. Show that

(a) $\mu^{-1}(\langle\mathscr{H}\rangle^{\partial})\subset X$ is a fiber of a moment map for the restriction of the action ψ to $\mathbb{T}_{\mathscr{H}}\subset\mathbb{T}$;

(b) if $\mathbb{T}_{\mathscr{H}}$ acts freely on $\mu^{-1}(\langle\mathscr{H}\rangle^{\partial})$, then $\mu^{-1}(\langle\mathscr{H}\rangle^{\partial})\subset X$ is a closed submanifold of codimension equal to the dimension of $\mathbb{T}_{\mathscr{H}}$;

(c) if $\mu(X)\subset\langle\mathscr{H}\rangle^{\partial}$, then $\mathbb{T}_{\mathscr{H}}$ acts trivially on X;

(d) if $\mu(X)\subset\langle\mathscr{H}\rangle^{\partial}$, $x\in X$, and $\operatorname{Stab}_x(\psi)=\mathbb{T}_{\mathscr{H}}$, then the differential $\mathrm{d}_x\mu\colon T_xX\longrightarrow T_{\mu(x)}\langle\mathscr{H}\rangle^{\partial}$ is surjective.

Definition 6.8. Let $\mathbb{T}$ be a torus and $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a finite subset. A Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) is

- $\mathscr{H}$-cuttable if for every subset $\mathscr{H}'\subset\mathscr{H}$ such that $\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})\neq\emptyset$ the Lie group homomorphism $\Phi_{\mathscr{H}'}$ as in (6.2) is injective and the subtorus $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}$ acts freely on $\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})$;
- $\mathscr{H}$-cut if $\mu(X)\subset\langle\mathscr{H}\rangle$ and for every $\mathscr{H}'\subset\mathscr{H}$ the subspace $Y_{\mathscr{H}'}\equiv\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})$ is a union of topological components of the fixed locus $X^{\mathbb{T}_{\mathscr{H}'}}\subset X$ of $\psi|_{\mathbb{T}_{\mathscr{H}'}}$ and there is a $\mathbb{T}$-equivariant splitting
$$TX|_{Y_{\mathscr{H}'}}=TY_{\mathscr{H}'}\oplus\bigoplus_{v\in\mathscr{H}'}\mathcal{N}_X^{v}Y_{\mathscr{H}'}\longrightarrow Y_{\mathscr{H}'}\tag{6.3}$$
of $TX|_{Y_{\mathscr{H}'}}$ with a ψ-invariant complex structure J compatible with ω so that
$$\begin{gathered}\operatorname{rk}_{\mathbb{C}}\mathcal{N}_X^{v}Y_{\mathscr{H}'}=1\quad\forall\ v\in\mathscr{H}'\qquad\text{and}\\ \mathrm{d}\psi_{\Phi_{\mathscr{H}'}([(r_v)_{v\in\mathscr{H}'}])}(w)=\mathrm{e}^{2\pi\mathrm{i}r_{v'}}w\quad\forall\ (r_v)_{v\in\mathscr{H}'}\in\mathbb{R}^{\mathscr{H}'},\ w\in\mathcal{N}_X^{v'}Y_{\mathscr{H}'},\ v'\in\mathscr{H}'.\end{gathered}\tag{6.4}$$

Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, (X,ω,ψ,μ) is a Hamiltonian $\mathbb{T}$-manifold, and $\mathscr{H}'\subset\mathscr{H}$ is subset such that $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})\neq\emptyset$. If (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable, then $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})\subset X$ is a closed submanifold of codimension $|\mathscr{H}'|$ by Exercise 6.7(b) and thus

$$\mu^{-1}\big(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\big)=\mu^{-1}\big(\langle\!\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle\big)-\bigcap_{\mathscr{H}'\subsetneq\mathscr{H}''\subset\mathscr{H}}\mu^{-1}\big(\langle\!\langle\mathscr{H}''\rangle^{\partial}\big)$$

is an open subset. If (X,ω,ψ,μ) is $\mathscr{H}$-cut, then $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})\subset X$ is a closed ω-symplectic submanifold of codimension $2|\mathscr{H}'|$ and the Lie group homomorphism $\Phi_{\mathscr{H}'}$ as in (6.2) is injective by Proposition 3.14(1), (6.3), and (6.4). Thus, $\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})\subset\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})$ is again an open subset; it is dense in this case, since $\mu(X)\subset\langle\mathscr{H}\rangle$. If in addition $\mathscr{H}_1',\mathscr{H}_2'\subset\mathscr{H}$ are disjoint subsets, then the restriction of μ to the symplectic submanifold $\mu^{-1}(\langle\!\langle\mathscr{H}_2'\rangle^{\partial})\subset X$ is transverse to $\langle\mathscr{H}_1'\rangle^{\partial}\subset T^*_{\mathbb{1}}\mathbb{T}$ by (6.3) and (6.4).

Exercise 6.9. Suppose (X,ω,ψ,μ) is a symplectic toric $\mathbb{T}$-manifold, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a Delzant subset so that $\mu(X)=\langle\mathscr{H}\rangle$, and $\mathscr{H}'\subset\mathscr{H}$. Show that $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}$ is the subtorus $\mathbb{T}_{\langle\mathscr{H}'\rangle^{\partial}\cap\langle\mathscr{H}\rangle}\subset\mathbb{T}$ as in (0^+) on page 8, $Y_{\mathscr{H}'}\equiv\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial})$ is a connected component of the fixed locus $X^{\mathbb{T}_{\mathscr{H}'}}\subset X$ of $\psi|_{\mathbb{T}_{\mathscr{H}'}}$, and $TX|_{Y_{\mathscr{H}'}}$ splits as in (6.3) and (6.4). Conclude that the Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) is $\mathscr{H}$-cut.

Exercise 6.10. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of a finite subset of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, and (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold. Show that

(a) (X,ω,ψ,μ) is $\mathscr{H}_1$-cuttable;

(b) for all $\mathscr{H}_1'\subset\mathscr{H}_1$ and $\mathscr{H}_2'\subset\mathscr{H}_2$ such that $\mu^{-1}(\langle\!\langle\mathscr{H}_2'\rangle^{\partial})\neq\emptyset$ the Lie group homomorphism $\Phi_{\mathscr{H}_2'}$ as in (6.2) is injective and the subtorus $\mathbb{T}_{\mathscr{H}_2'}\subset\mathbb{T}$ acts freely on $\mu^{-1}(\langle\!\langle\mathscr{H}_1'\sqcup\mathscr{H}_2'\rangle^{\partial})/\mathbb{T}_{\mathscr{H}_1'}$.

Theorem 5 ([19, Proposition 2.4])**.** *Suppose $\mathbb{T}$ is a torus, $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a finite subset, and (X,ω,ψ,μ) is an $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold. There exists a unique $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold*

$$(X,\omega,\psi,\mu)_{\mathscr{H}}\equiv\big(X_{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}}\big)\tag{6.5}$$

so that

(1) $X_{\mathscr{H}}=\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)/\!\sim$ with $x\sim x'$ if there exist $\mathscr{H}'\subset\mathscr{H}$ and $u\in\mathbb{T}_{\mathscr{H}'}$ so that $\mu(x)\in\langle\mathscr{H}'\rangle^{\partial}$ and $x'=\psi_u(x)$;

(2) the quotient projection $p_{\mathscr{H}}\colon\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)\longrightarrow X_{\mathscr{H}}$ is $\mathbb{T}$-equivariant and $\mu|_{\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)}=\mu_{\mathscr{H}}\circ p_{\mathscr{H}}$;

(3) for every $\mathscr{H}'\subset\mathscr{H}$, $p_{\mathscr{H}}:\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})\longrightarrow\mu^{-1}_{\mathscr{H}}(\langle\!\langle\mathscr{H}'\rangle^{\partial})$ is a submersion (onto a dense open subset) and

$$\big\{p_{\mathscr{H}}|_{\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})}\big\}^*\big(\omega_{\mathscr{H}}|_{T\mu^{-1}_{\mathscr{H}}(\langle\!\langle\mathscr{H}'\rangle^{\partial})}\big)=\omega|_{T(\mu^{-1}(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}))}\,.\tag{6.6}$$

For any partition $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$, $(X,\omega,\psi,\mu)_{\mathscr{H}_1}$ is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold and

$$(X,\omega,\psi,\mu)_{\mathscr{H}}=\big((X,\omega,\psi,\mu)_{\mathscr{H}_1}\big)_{\mathscr{H}_2}\,.\tag{6.7}$$

Proof. Let $\mathbf{c}=(c_v)_{v\in\mathscr{H}}\in\mathbb{R}^{\mathscr{H}}$. With $\omega_{\mathscr{H}}$ denoting the standard symplectic form on $\mathbb{C}^{\mathscr{H}}$, analogously to (2.10), define

$$\widetilde{\omega}\equiv\pi_1^*\omega\oplus\pi_2^*\omega_{\mathscr{H}}\,,$$

where $\pi_1,\pi_2\colon X\times\mathbb{C}^{\mathscr{H}}\longrightarrow X,\mathbb{C}^{\mathscr{H}}$ are the component projections. Let $\widetilde{\psi}$ be the $\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$-action on $X\times\mathbb{C}^{\mathscr{H}}$ given by

$$\widetilde{\psi}_{[(r_v)_{v\in\mathscr{H}}]}\big(x,(z_v)_{v\in\mathscr{H}}\big)=\big(\psi_{\Phi_{\mathscr{H}}([(r_v)_{v\in\mathscr{H}}])}(x),\big(\mathrm{e}^{-2\pi\mathrm{i}r_v}z_v\big)_{v\in\mathscr{H}}\big). \tag{6.8}$$

This action commutes with the $\mathbb{T}$-action ψ on the first component and preserves its moment map

$$\mu\circ\pi_1\colon X\times\mathbb{C}^{\mathscr{H}}\longrightarrow T^*_{\mathbb{1}}\mathbb{T} \tag{6.9}$$

with respect to $\widetilde{\omega}$. By Exercises 2.2 and 2.5, the smooth function

$$\widetilde{H}\colon X\times\mathbb{C}^{\mathscr{H}}\longrightarrow T^*_{\mathbb{1}}(\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}})=\mathbb{R}^{\mathscr{H}},\quad \widetilde{H}\big(x,(z_{v,c})_{(v,c)\in\mathscr{H}}\big)=\big(\mu_v(x)-\pi|z_{v,c}|^2\big)_{(v,c)\in\mathscr{H}}, \tag{6.10}$$

is a Hamiltonian for the action $\widetilde{\psi}$ with respect to $\widetilde{\omega}$. It is preserved by the $\mathbb{T}$-action ψ on the first component.

Suppose $(x,(z_v)_{v\in\mathscr{H}})\in\widetilde{H}^{-1}(\mathbf{c})$, $u\in\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$, and $\widetilde{\psi}_u(x,(z_v)_{v\in\mathscr{H}})=(x,(z_v)_{v\in\mathscr{H}})$. Let

$$\mathscr{H}'=\big\{v\in\mathscr{H}\colon z_v=0\big\}.$$

From (6.10) and (6.8), we then obtain

$$\mu(x)\in\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\subset T^*_{\mathbb{1}}\mathbb{T}\qquad\text{and}\qquad u\in\mathbb{R}^{\mathscr{H}'}/\mathbb{Z}^{\mathscr{H}'}\subset\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}. \tag{6.11}$$

Since (X,ω,ψ,μ) is $\mathscr{H}$-cuttable, it follows that $u=\mathbb{1}$. Thus, $\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$ acts freely on $\widetilde{H}^{-1}(\mathbf{c})$ via (6.8). Let

$$(X_{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}})\equiv\big(X_\alpha,\omega_\alpha,\psi_\alpha,\mu_\alpha\big)$$

be the associated quotient Hamiltonian $\mathbb{T}$-manifold of Theorem 4 and

$$p\colon\widetilde{H}^{-1}(\mathbf{c})\longrightarrow X_{\mathscr{H}}\equiv\widetilde{H}^{-1}(\mathbf{c})/(\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}})$$

be the quotient projection.

The map

$$p_{\mathscr{H}}\colon\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)\longrightarrow X_{\mathscr{H}}\equiv\widetilde{H}^{-1}(\mathbf{c})/(\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}),\quad p_{\mathscr{H}}(x)=\big[x,\big(\sqrt{(\mu_v(x)-c_v)/\pi}\big)_{(v,c)\in\mathscr{H}}\big],$$

is well-defined, continuous, surjective, and $\mathbb{T}$-equivariant and satisfies the last condition in (2). In particular, $\mu_{\mathscr{H}}(X_{\mathscr{H}})\subset\langle\mathscr{H}\rangle$. By the first statement in (6.11), $p_{\mathscr{H}}$ induces an injective map from the quotient $\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)/\!\sim$ in (1) to $X_{\mathscr{H}}$. Since the map

$$\widetilde{p}_{\mathscr{H}}\colon\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)\longrightarrow\widetilde{H}^{-1}(\mathbf{c}),\qquad\widetilde{p}_{\mathscr{H}}(x)=\big(x,\big(\sqrt{(\mu_v(x)-c_v)/\pi}\big)_{(v,c)\in\mathscr{H}}\big), \tag{6.12}$$

is closed and the group $\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$ is compact, $p_{\mathscr{H}}$ is a closed map and thus so is the induced map from $\mu^{-1}(\langle\!\langle\mathscr{H}\rangle\!\rangle)$. This confirms (1) and (2).

Let J be a ψ-invariant almost complex structure on X compatible with ω, $J_{\mathscr{H}}$ be the standard complex structure on $\mathbb{C}^{\mathscr{H}}$, and $\widetilde{\psi}'$ be the action of $\mathbb{T}_{\mathscr{H}}$ on $X\times\mathbb{C}^{\mathscr{H}}$ given by

$$\begin{split}\widetilde{\psi}'_{\Phi_{\mathscr{H}}([(r_v)_{v\in\mathscr{H}}])}\big(x,(z_v)_{v\in\mathscr{H}}\big) &= \widetilde{\psi}_{[(-r_v)_{v\in\mathscr{H}}]}\big(\psi_{\Phi_{\mathscr{H}}([(r_v)_{v\in\mathscr{H}}])}(x),(z_v)_{v\in\mathscr{H}}\big)\\ &= \big(x,\big(\mathrm{e}^{2\pi\mathrm{i}r_v}z_v\big)_{v\in\mathscr{H}}\big).\end{split}\tag{6.13}$$

This action commutes with the $\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$-action $\widetilde{\psi}$ and thus induces a $\mathbb{T}_{\mathscr{H}}$-action on $X_{\mathscr{H}}$. By the middle expression in (6.13), the latter is the restriction of the $\mathbb{T}$-action $\psi_{\mathscr{H}}$ to $\mathbb{T}_{\mathscr{H}}$. The almost complex structure $\widetilde{J}\equiv J\oplus J_{\mathscr{H}}$ on $X\times\mathbb{C}^{\mathscr{H}}$ is ψ-, $\widetilde{\psi}$-, and $\widetilde{\psi}'$-invariant and compatible with $\widetilde{\omega}$. By Exercise 6.2, $\widetilde{J}$ thus descends to a $\psi_{\mathscr{H}}$-invariant almost complex structure $J_{\mathscr{H}}$ on $X_{\mathscr{H}}$ which is compatible with $\omega_{\mathscr{H}}$.

Let $\mathscr{H}'\subset\mathscr{H}$. By the first statement in (6.11),

$$\begin{split}\widetilde{Y}_{\mathscr{H}'} \equiv \big(\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})\times\mathbb{C}^{\mathscr{H}}\big)\cap\widetilde{H}^{-1}(\mathbf{c}) &= \big\{\widetilde{H}|_{X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}}\big\}^{-1}(\mathbf{c})\\ &\subset X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}=\big(X\times\mathbb{C}^{\mathscr{H}}\big)^{\widetilde{\psi}'|_{\mathbb{T}_{\mathscr{H}'}}}.\end{split}\tag{6.14}$$

Since the moment map $\mu_{\mathscr{H}}\colon X_{\mathscr{H}}\longrightarrow T^*_{\mathbb{1}}\mathbb{T}$ is induced by (6.9),

$$Y_{\mathscr{H}'}\equiv\mu_{\mathscr{H}}^{-1}\big(\langle\mathscr{H}'\rangle^{\partial}\big)=p\big(\widetilde{Y}_{\mathscr{H}'}\big)=p_{\mathscr{H}}\big(\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial})\big)\subset X_{\mathscr{H}}^{\mathbb{T}_{\mathscr{H}'}}\subset X_{\mathscr{H}};$$

the first inclusion above follows from the last equality in (6.14). The $\widetilde{\omega}$-symplectic submanifold $X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}\subset X\times\mathbb{C}^{\mathscr{H}}$ is preserved by the $\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$-action $\widetilde{\psi}$, the $\mathbb{T}$-action ψ, and the $\mathbb{T}_{\mathscr{H}}$-action $\widetilde{\psi}'$. By Exercise 6.1, $Y_{\mathscr{H}'}\subset X_{\mathscr{H}}$ is thus an $\omega_{\mathscr{H}}$-symplectic submanifold preserved by the $\mathbb{T}$-action $\psi_{\mathscr{H}}$. By (6.13), the natural splitting

$$\mathcal{N}_{X\times\mathbb{C}^{\mathscr{H}}}\big(X\times\mathbb{C}^{\mathscr{H}-\mathscr{H}'}\big)=\bigoplus_{v\in\mathscr{H}'}\big(X\times\mathbb{C}^{\{v\}}\big)$$

is $\mathbb{T}-$, $\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$-, and $\mathbb{T}_{\mathscr{H}}$-equivariant with respect to the actions $\mathrm{d}\psi$, $\mathrm{d}\widetilde{\psi}$, and $\mathrm{d}\widetilde{\psi}'$ on the left-hand side and the actions

$$\begin{gathered}u\cdot\big(x,z_v\big)=\big(\psi_u(x),z_v\big),\quad \big[(r_{v'})_{v'\in\mathscr{H}}\big]\cdot\big(x,z_v\big),=\big(\psi_{\Phi_{\mathscr{H}}([(r_{v'})_{v'\in\mathscr{H}}])}(x),\mathrm{e}^{-2\pi\mathrm{i}r_v}z_v\big),\\ \text{and}\qquad \Phi_{\mathscr{H}}\big([(r_{v'})_{v'\in\mathscr{H}}]\big)\cdot\big(x,z_v\big)=\big(x,\mathrm{e}^{2\pi\mathrm{i}r_v}z_v\big)\end{gathered}$$

on the summand $X\times\mathbb{C}^{\{v\}}$ on the right-hand side. By Exercises 6.1 and 6.2, $TX_{\mathscr{H}}|_{Y_{\mathscr{H}'}}$ thus splits $\mathbb{T}$-equivariantly as in (6.3) and (6.4). It follows that $Y_{\mathscr{H}'}\subset X_{\mathscr{H}}$ is a union of topological components of the fixed locus $X_{\mathscr{H}}^{\mathbb{T}_{\mathscr{H}'}}$ of the restriction of the $\mathbb{T}$-action $\psi_{\mathscr{H}}$ to $\mathbb{T}_{\mathscr{H}'}\subset\mathbb{T}_{\mathscr{H}}$. Thus, (6.5) is an $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold.

The restriction of the map $\widetilde{p}_{\mathscr{H}}$ in (6.12) to $\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})$ is a smooth embedding; its image is $\widetilde{Y}_{\mathscr{H}'}\cap(X\times(\mathbb{R}^+)^{\mathscr{H}-\mathscr{H}'})$. Thus,

$$\big\{\widetilde{p}_{\mathscr{H}}|_{\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}})}\big\}^*\big(\widetilde{\omega}|_{T\widetilde{Y}_{\mathscr{H}'}}\big)=\omega\big|_{T(\mu^{-1}(\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}))}\,.\tag{6.15}$$

Since $p^*(\omega_{\mathscr{H}}|_{TY_{\mathscr{H}'}})=\widetilde{\omega}|_{T\widetilde{Y}_{\mathscr{H}'}}$ by Theorem 4(2) and Exercise 6.1, (6.6) follows from (6.15). The map

$$\begin{gathered}\widetilde{P}_{\mathscr{H};\mathscr{H}'}\colon\big(\mathbb{R}^{\mathscr{H}-\mathscr{H}'}/\mathbb{Z}^{\mathscr{H}-\mathscr{H}'}\big)\times\mu^{-1}\big(\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\big)\longrightarrow\widetilde{Y}_{\mathscr{H}'}\cap\big(X\times(\mathbb{C}^*)^{\mathscr{H}-\mathscr{H}'}\big),\\ \widetilde{P}_{\mathscr{H};\mathscr{H}'}(u,x)=\widetilde{\psi}_u\big(\widetilde{p}_{\mathscr{H}}(x)\big),\end{gathered}$$

is a diffeomorphism. Since the map $p\colon \widetilde{Y}_{\mathscr{H}'} \longrightarrow Y_{\mathscr{H}'}$ is a principal $\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}$-bundle (and thus a submersion) by Theorem 4(2),

$$\mu_{\mathscr{H}}^{-1}\big(\langle\!\langle \mathscr{H}'\rangle\!\rangle_{\mathscr{H}}^{\partial}\big) = \widetilde{Y}_{\mathscr{H}'}\cap\big(X\times(\mathbb{C}^*)^{\mathscr{H}-\mathscr{H}'}\big),$$

and $\mathrm{d}p$ vanishes on $\mathrm{d}\widetilde{P}_{\mathscr{H};\mathscr{H}'}(T(\mathbb{R}^{\mathscr{H}-\mathscr{H}'}/\mathbb{Z}^{\mathscr{H}-\mathscr{H}'}))$, the composition

$$p_{\mathscr{H}} = p\circ\widetilde{p}_{\mathscr{H}}\colon \mu^{-1}\big(\langle\!\langle \mathscr{H}'\rangle\!\rangle_{\mathscr{H}}^{\partial}\big) \longrightarrow \mu_{\mathscr{H}}^{-1}\big(\langle\!\langle \mathscr{H}'\rangle\!\rangle_{\mathscr{H}}^{\partial}\big) \subset \mu_{\mathscr{H}}^{-1}\big(\langle\!\langle \mathscr{H}'\rangle^{\partial}\big)$$

is a submersion. This confirms (3). The conditions (1)-(3) ensure the uniqueness of $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold satisfying these properties.

Suppose $\mathscr{H} = \mathscr{H}_1\sqcup\mathscr{H}_2$. By Exercise 6.10, the Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,μ) is $\mathscr{H}_1$-cuttable. Let

$$(X,\omega,\psi,\mu)_{\mathscr{H}_1} \equiv \big(X_{\mathscr{H}_1},\omega_{\mathscr{H}_1},\psi_{\mathscr{H}_1},\mu_{\mathscr{H}_1}\big) \tag{6.16}$$

be the corresponding $\mathscr{H}_1$-cut Hamiltonian $\mathbb{T}$-manifold as in (6.5). If $\mathscr{H}_2'\subset\mathscr{H}_2$,

$$\mu_{\mathscr{H}_1}^{-1}\big(\langle\!\langle \mathscr{H}_2'\rangle^{\partial}\big) = \bigcup_{\mathscr{H}_1''\subset\mathscr{H}_1}\ \bigcup_{\mathscr{H}_2'\subset\mathscr{H}_2''\subset\mathscr{H}_2}\mu_{\mathscr{H}_1}^{-1}\big(\langle\!\langle \mathscr{H}_1''\sqcup\mathscr{H}_2''\rangle\!\rangle_{\mathscr{H}}^{\partial}\big) = \bigcup_{\mathscr{H}_1''\subset\mathscr{H}_1}\ \bigcup_{\mathscr{H}_2'\subset\mathscr{H}_2''\subset\mathscr{H}_2}\mu^{-1}\big(\langle\!\langle \mathscr{H}_1''\sqcup\mathscr{H}_2''\rangle\!\rangle_{\mathscr{H}}^{\partial}\big)/\mathbb{T}_{\mathscr{H}_1''}\,;$$

the last equality holds by (1) with $\mathscr{H}$ replaced by $\mathscr{H}_1$. If $\mu_{\mathscr{H}_1}^{-1}(\langle\!\langle \mathscr{H}_2'\rangle^{\partial})\neq\emptyset$, Exercise 6.10 thus implies that the Lie group homomorphism $\Phi_{\mathscr{H}_2'}$ as in (6.2) is injective and $\mathbb{T}_{\mathscr{H}_2'}$ acts freely on $\mu_{\mathscr{H}_1}^{-1}(\langle\!\langle \mathscr{H}_2'\rangle^{\partial})$. We conclude that the Hamiltonian $\mathbb{T}$-manifold (6.16) is $\mathscr{H}_2$-cuttable.

By (1) and (2) with $\mathscr{H}$ replaced by $\mathscr{H}_1$ and $\mathscr{H}_2$,

$$\big(X_{\mathscr{H}_1}\big)_{\mathscr{H}_2} = \mu_{\mathscr{H}_1}^{-1}\big(\langle\!\langle \mathscr{H}_2\rangle\!\rangle\big)/\!\sim_{\mathscr{H}_2} = p_{\mathscr{H}_1}\big(\mu^{-1}(\langle\!\langle \mathscr{H}_1\rangle\!\rangle)\cap\mu^{-1}(\langle\!\langle \mathscr{H}_2\rangle\!\rangle)\big)/\!\sim_{\mathscr{H}_2} = \big(\mu^{-1}(\langle\!\langle \mathscr{H}\rangle\!\rangle)/\!\sim_{\mathscr{H}_1}\big)/\!\sim_{\mathscr{H}_2}\,,$$

with $x,x'\in\mu^{-1}(\langle\!\langle \mathscr{H}\rangle\!\rangle)$ being equivalent in the double quotient if there exist $\mathscr{H}_1'\subset\mathscr{H}_1$, $\mathscr{H}_2'\subset\mathscr{H}_2$, and u in the subgroup generated by $\mathbb{T}_{\mathscr{H}_1'},\mathbb{T}_{\mathscr{H}_2'}\subset\mathbb{T}$ such that

$$\mu(x)\in\langle \mathscr{H}_1'\rangle^{\partial}\cap\langle \mathscr{H}_2'\rangle^{\partial} = \langle \mathscr{H}_1'\sqcup\mathscr{H}_2'\rangle^{\partial} \qquad\text{and}\qquad x' = \psi_u(x).$$

By definition, the subgroup generated by $\mathbb{T}_{\mathscr{H}_1'},\mathbb{T}_{\mathscr{H}_2'}\subset\mathbb{T}$ is $\mathbb{T}_{\mathscr{H}_1'\sqcup\mathscr{H}_2'}$. Thus,

$$p_{\mathscr{H}_2}\circ p_{\mathscr{H}_1} = p_{\mathscr{H}}\colon X\longrightarrow \big(X_{\mathscr{H}_1}\big)_{\mathscr{H}_2} = X_{\mathscr{H}}, \qquad (\mu_{\mathscr{H}_1})_{\mathscr{H}_2} = \mu_{\mathscr{H}}\colon X_{\mathscr{H}}\longrightarrow T_{\mathbb{1}}^*\mathbb{T}\,,$$

and the $\mathbb{T}$-actions $(\psi_{\mathscr{H}_1})_{\mathscr{H}_2}$ and $\psi_{\mathscr{H}}$ on $X_{\mathscr{H}}$ are the same (as they are induced by the same $\mathbb{T}$-action ψ on X). Since $\langle\emptyset\rangle_{\mathscr{H}}^{\partial} = \langle\emptyset\rangle_{\mathscr{H}_1}^{\partial}\cap\langle\emptyset\rangle_{\mathscr{H}_2}^{\partial}$ is an open subset of $T_{\mathbb{1}}^*\mathbb{T}$, (6.6) gives

$$\begin{aligned}\big\{p_{\mathscr{H}}|_{\mu^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*(\omega_{\mathscr{H}_1})_{\mathscr{H}_2} &= \big\{p_{\mathscr{H}_1}|_{\mu^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*\big\{p_{\mathscr{H}_2}|_{\mu_{\mathscr{H}_1}^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}_2}^{\partial})}\big\}^*(\omega_{\mathscr{H}_1})_{\mathscr{H}_2}\\ &= \big\{p_{\mathscr{H}_1}|_{\mu^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*\omega_{\mathscr{H}_1} = \omega\big|_{\mu^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}}^{\partial})} = \big\{p_{\mathscr{H}}|_{\mu^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}}^{\partial})}\big\}^*\omega_{\mathscr{H}}\,.\end{aligned}$$

Since $p_{\mathscr{H}}$ is a submersion on $\mu^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}}^{\partial})$, it follows that $(\omega_{\mathscr{H}_1})_{\mathscr{H}_2} = \omega$ on the dense open subset $\mu_{\mathscr{H}}^{-1}(\langle\!\langle\emptyset\rangle_{\mathscr{H}}^{\partial})\subset X_{\mathscr{H}}$ and thus everywhere on $X_{\mathscr{H}}$. This establishes (6.7). $\square$

6.3 Hamiltonian symplectic uncut

In this section, we show that the Hamiltonian symplectic cut construction is reversible and complete the proof of Theorem 3. This formalizes the argument sketched in the proof of [22, Theorem 7.5.10].

Exercise 6.11. Suppose $\mathbb{T}$ is a torus, $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of a finite subset of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, (X,ω,ψ,μ) is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold so that $(X,\omega,\psi,\mu)_{\mathscr{H}_2}$ is an $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold, and $Y\subset X_{\mathscr{H}_2}$ is a topological component of $\mu_{\mathscr{H}_2}^{-1}(\langle\mathscr{H}_1\rangle^{\partial})$. Show that

(a) there exists a (unique) topological component $\widetilde{Y}\subset X$ of $\mu^{-1}(\langle\mathscr{H}_1\rangle^{\partial})$ which contains $p_{\mathscr{H}_2}^{-1}(Y)$;

(b) $\widetilde{Y}$ is a topological component of the fixed locus $X^{\mathbb{T}_{\mathscr{H}_1}}\subset X$ of $\psi|_{\mathbb{T}_{\mathscr{H}_1}}$ with its normal bundle admitting a splitting as in (6.3) and (6.4) with $(Y_{\mathscr{H}'},\mathscr{H}')$ replaced by $(\widetilde{Y},\mathscr{H}_1)$.

Corollary 6.12. *Suppose $\mathbb{T}$ is a torus, $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of a finite subset of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, and (X,ω,ψ,μ) is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold. If $(X,\omega,\psi,\mu)_{\mathscr{H}_2}$ is an $\mathscr{H}$-cut Hamiltonian $\mathbb{T}$-manifold, then there exists an open $\mathbb{T}$-invariant subset $X'\subset X$ so that*

$$\big(X',\omega|_{X'},\psi|_{X'},\mu|_{X'}\big)_{\mathscr{H}_2}=(X,\omega,\psi,\mu)_{\mathscr{H}_2} \tag{6.17}$$

and the Hamiltonian $\mathbb{T}$-manifold $(X',\omega|_{X'},\psi|_{X'},\mu|_{X'})$ is $\mathscr{H}_1$-cut. If $X_{\mathscr{H}_2}$ is connected and/or the restriction

$$\mu\colon \mu^{-1}\big(\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\big)\longrightarrow\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu(X) \tag{6.18}$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for every $\mathscr{H}_1'\subset\mathscr{H}_1$, then (X',ω',ψ',μ') can be chosen so that X' is also connected and/or the restriction

$$\mu\colon \mu^{-1}\big(\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\big)\cap X'\longrightarrow\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu(X') \tag{6.19}$$

is also a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for every $\mathscr{H}_1'\subset\mathscr{H}_1$, respectively.

Proof. Suppose $\mathscr{H}_1'\subset\mathscr{H}_1$ and $Y\subset X_{\mathscr{H}_2}$ is a topological component of $\mu_{\mathscr{H}_2}^{-1}(\langle\mathscr{H}_1'\rangle^{\partial})$. It is thus contained in a topological component $Y_{\mathscr{H}_1''}\subset X_{\mathscr{H}_2}$ of $\mu_{\mathscr{H}_2}^{-1}(\langle\mathscr{H}_1''\rangle^{\partial})$ for every $\mathscr{H}_1''\subset\mathscr{H}_1'$. By Exercise 6.11(a), there exists a (unique) topological component $\widetilde{Y}_{\mathscr{H}_1''}\subset X$ of $\mu^{-1}(\langle\mathscr{H}_1''\rangle^{\partial})$ which contains $p_{\mathscr{H}_2}^{-1}(Y_{\mathscr{H}_1''})$. In particular, $\widetilde{Y}_{\mathscr{H}_1'}$ is disjoint from the closed subsets $\mu^{-1}(\langle\mathscr{H}_1''\rangle^{\partial})-\widetilde{Y}_{\mathscr{H}_1''}$ of X with $\mathscr{H}_1''\subset\mathscr{H}_1'$.

By Exercise 6.11(b) and the first part of Proposition 3.27 with $\mathbb{T}$ replaced by $\mathbb{T}_{\mathscr{H}_1'}$, there thus exists a $\mathbb{T}$-invariant neighborhood $\widetilde{U}_Y$ of $\widetilde{Y}_{\mathscr{H}_1'}$ in X so that

$$\mu(\widetilde{U}_Y)\subset\langle\mathscr{H}_1'\rangle\qquad\text{and}\qquad\widetilde{U}_Y\cap\mu^{-1}\big(\langle\mathscr{H}_1''\rangle^{\partial}\big)\subset\widetilde{Y}_{\mathscr{H}_1''}\quad\forall\,\mathscr{H}_1''\subset\mathscr{H}_1'\,. \tag{6.20}$$

The $\mathbb{T}$-invariant neighborhood

$$\widetilde{U}_Y'\equiv\widetilde{U}_Y\cap\mu^{-1}\big(\langle\emptyset\rangle^{\partial}_{\mathscr{H}_1-\mathscr{H}_1'}\big)\subset\mu^{-1}\big(\langle\mathscr{H}_1'\rangle\big)\cap\mu^{-1}\big(\langle\mathscr{H}_1-\mathscr{H}_1'\rangle\big)=\mu^{-1}\big(\langle\mathscr{H}_1\rangle\big)$$

of $\widetilde{Y}_{\mathscr{H}_1'}\cap\mu^{-1}(\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1})$ in X is then disjoint from $\mu^{-1}(\langle\mathscr{H}_1''\rangle^{\partial})$ for every subset $\mathscr{H}_1''\subset\mathscr{H}_1$ not contained in $\mathscr{H}_1'$. Thus,

$$X'\equiv\bigcup_{\mathscr{H}_1'\subset\mathscr{H}_1}\;\bigcup_{Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\mathscr{H}_1'\rangle^{\partial}))}\widetilde{U}_Y'\quad\subset\mu^{-1}\big(\langle\mathscr{H}_1\rangle\big) \tag{6.21}$$

is a $\mathbb{T}$-invariant neighborhood of

$$\bigcup_{\mathscr{H}_1'\subset\mathscr{H}_1}\ \bigcup_{Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}))}\big(\widetilde{Y}_{\mathscr{H}_1'}\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1})\big)\supset\bigcup_{\mathscr{H}_1'\subset\mathscr{H}_1}\ \bigcup_{Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}))}p_{\mathscr{H}_2}^{-1}(Y)=p_{\mathscr{H}_2}^{-1}(X_{\mathscr{H}_2})=\mu^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle\!\rangle\big)$$

in X. By Theorem 5(1) with $\mathscr{H}$ replaced by $\mathscr{H}_2$, the above inclusion implies (6.17).

By (6.21), $\mu(X')\subset\langle\mathscr{H}_1\rangle$. Let $\mathscr{H}_1''\subset\mathscr{H}_1$ be such that $\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})\neq\emptyset$. Since $\widetilde{U}_Y'\subset\widetilde{U}_Y$ with $Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}))$ is disjoint from $\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})$ whenever $\mathscr{H}_1''\not\subset\mathscr{H}_1'$, the second statement in (6.20) implies that

$$X'\cap\mu^{-1}\big(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}\big)\subset\bigcup_{\mathscr{H}_1''\subset\mathscr{H}_1'\subset\mathscr{H}_1}\ \bigcup_{Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}))}\widetilde{Y}_{\mathscr{H}_1''}\quad=\quad\bigcup_{Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}))}\widetilde{Y}_{\mathscr{H}_1''}\ .\tag{6.22}$$

Since $(X,\omega,\psi,\mu)_{\mathscr{H}_2}$ is $\mathscr{H}$-cut, it follows that the Lie group homomorphism $\Phi_{\mathscr{H}_1''}$ as in (6.2) is injective. By (6.22), $X'\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})\subset X$ is the disjoint union of the open subspaces $X'\cap\widetilde{Y}_{\mathscr{H}_1''}$ of $\widetilde{Y}_{\mathscr{H}_1''}$ with $Y\in\pi_0(\mu_{\mathscr{H}_2}^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}))$. By Exercise 6.11(b), $X'\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial})$ is thus a union of topological components of the fixed locus $X'^{\mathbb{T}_{\mathscr{H}_1''}}$ of the restriction of the $\mathbb{T}$-action ψ to $\mathbb{T}_{\mathscr{H}_1''}\subset\mathbb{T}$ and $X'\subset X$ with its normal bundles admitting a splitting as in (6.3) and (6.4) with $(Y_{\mathscr{H}'},\mathscr{H}')$ replaced by $(X'\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1''\rangle^{\partial}),\mathscr{H}_1'')$. Thus, $(X',\omega|_{X'},\psi|_{X'},\mu|_{X'})$ is $\mathscr{H}_1$-cut.

Since $\mu_{\mathscr{H}_2}(X_{\mathscr{H}_2})\subset\langle\mathscr{H}_2\rangle$, the subspaces $\widetilde{Y}_{\mathscr{H}_1'}\cap\mu^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1})\subset X$ above intersect $\mu^{-1}(\langle\!\langle\mathscr{H}_2\rangle\!\rangle)$ and thus so do their neighborhoods $\widetilde{U}_Y'\subset X'\subset X$. If $X_{\mathscr{H}_2}$ is connected, then so is $\mu^{-1}(\langle\!\langle\mathscr{H}_2\rangle\!\rangle)\subset X$ by Theorem 5(1) with $\mathscr{H}$ replaced by $\mathscr{H}_2$. It follows that $X'\subset X$ is then connected. If the restriction (6.18) is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for some $\mathscr{H}_1'\subset\mathscr{H}_1$, then $\mathbb{T}$ acts transitively on the fibers of μ over $\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}}\cap\mu(X)$. Since $X'\subset X$ is a $\mathbb{T}$-invariant subset, (6.19) is the restriction of principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle (6.18) to $\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu(X')\subset\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu(X)$ and thus is still a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. $\square$

Proposition 6.13. *Suppose $\mathbb{T}$ is a torus, $\mathscr{H}=\mathscr{H}_1\sqcup\mathscr{H}_2$ is a partition of a finite subset of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$, and (X,ω,ψ,μ) is an $\mathscr{H}_2$-cut Hamiltonian $\mathbb{T}$-manifold. Then,*

$$(X,\omega,\psi,\mu)=(X',\omega',\psi',\mu')_{\mathscr{H}_2}\tag{6.23}$$

for some $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold (X',ω',ψ',μ'). If

(a) X is connected and/or

(b) (X,ω,ψ,μ) is $\mathscr{H}$-cut and the restriction

$$\mu\colon\mu^{-1}\big(\langle\!\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\big)\longrightarrow\langle\mathscr{H}'\rangle^{\partial}_{\mathscr{H}}\cap\mu(X)\tag{6.24}$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}'}$-bundle for every $\mathscr{H}'\subset\mathscr{H}$,

then (X',ω',ψ',μ') can be chosen so that

(a') X' is connected and/or

(b') (X',ω',ψ',μ') is $\mathscr{H}_1$-cut and the restriction

$$\mu'\colon \mu'^{-1}\big(\langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\big) \longrightarrow \langle\mathscr{H}_1'\rangle^{\partial}_{\mathscr{H}_1}\cap\mu'(X') \tag{6.25}$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle for every $\mathscr{H}_1'\subset\mathscr{H}_1$,

respectively.

Proof. By (6.7) and Corollary 6.12, it is sufficient to establish this proposition with $\mathscr{H}_2$ consisting of a single element $v\equiv(v,c)$ of $(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ with $v\neq 0$. We assume that the closed codimension 2 symplectic submanifold

$$Y\equiv\mu^{-1}\big(\langle\mathscr{H}_2\rangle^{\partial}\big)\equiv\big\{x\in X\colon \mu_v(x)=c\big\}$$

of (X,ω) is nonempty; otherwise, we can take $(X',\omega',\psi',\mu')=(X,\omega,\psi,\mu)$. Since (X,ω,ψ,μ) is $\mathscr{H}_2$-cut, the Lie group homomorphism $\Phi_{\mathscr{H}_2}$ as in (6.2) is then injective. We establish the proposition by removing Y and continuing the radial directions in the normal bundle of Y in X into the negative values without them coming together at 0. Since (X,ω,ψ,μ) is $\mathscr{H}_2$-cut,

$$X-Y=\big\{x\in X\colon \mu_v(x)>c\big\}=\mu^{-1}\big(\langle\emptyset\rangle^{\partial}_{\mathscr{H}_2}\big)\,. \tag{6.26}$$

Let J be a $\mathbb{T}^n$-invariant almost complex structure on X compatible with ω and $g(\cdot,\cdot)\equiv\omega(\cdot,J\cdot)$ be the associated $\mathbb{T}^n$-invariant metric compatible with J.

Since (X,ω,ψ,μ) is $\mathscr{H}_2$-cut, Y is a union of topological components of the fixed locus $X^{\mathbb{T}_v}$ of the restriction of the $\mathbb{T}$-action ψ to the circle $\mathbb{T}_v\subset\mathbb{T}$ generated by $v\in(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}$ and

$$\pi\colon TY^{\omega}\equiv\big\{w\in TX|_Y\colon \omega(w,w')=0\ \forall\,w'\in TY\big\}\longrightarrow Y$$

is a complex line bundle complementary to TY. It is preserved by the $\mathbb{T}^n$-action $\mathrm{d}\psi$ and

$$\mathrm{d}\psi_{\mathrm{e}^{tv}}(w)=\mathrm{e}^{2\pi\mathrm{i}t}w \qquad \forall\,t\in\mathbb{R},\ w\in TY^{\omega} \tag{6.27}$$

by (6.4). Let

$$\zeta_v\in\Gamma\big(TY^{\omega};T(TY^{\omega})\big),\qquad \zeta_v(w)=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\psi_{\mathrm{e}^{tv}}(w)\Big|_{t=0}=2\pi\mathrm{i}w,$$

be the (vertical) vector field on TY^{ω} generating the S^1-action (6.27). Since this action preserves the unit circle bundle of TY^{ω},

$$\pi\colon S\big(TY^{\omega}\big)\equiv\big\{w\in TY^{\omega}\colon g(w,w)=1\big\}\longrightarrow Y, \tag{6.28}$$

the vector field $\zeta_v|_{S(TY^{\omega})}$ is tangent to $S(TY^{\omega})$. The maps

$$\begin{gathered}\widetilde{\iota}\colon S\big(TY^{\omega}\big)\times\mathbb{R}^+\longrightarrow S\big(TY^{\omega}\big)\times\mathbb{C},\quad \widetilde{\iota}(w,t)=\big(w,\sqrt{2t}\big),\\ p\colon S\big(TY^{\omega}\big)\times\mathbb{C}\longrightarrow TY^{\omega},\quad p(w,z)=zw,\end{gathered}$$

are smooth. The map p descends to a $\mathbb{T}$-equivariant diffeomorphism from the quotient $S(TY^{\omega})\times_{S^1}\mathbb{C}$ of $S(TY^{\omega})\times\mathbb{C}$ by the S^1-action

$$S^1\times\big(S(TY^{\omega})\times\mathbb{C}\big)\longrightarrow S(TY^{\omega})\times\mathbb{C},\qquad u\cdot(w,z)=\big(uw,u^{-1}z\big), \tag{6.29}$$

to TY^ω. The generating vector field for this action is $\widetilde{\zeta}_v \equiv (\zeta_v, -2\pi\partial_\theta)$. The map

$$\iota \equiv p\circ\widetilde{\iota}\colon S\big(TY^\omega\big)\times\mathbb{R}^+ \longrightarrow TY^\omega$$

is a $\mathbb{T}$-equivariant diffeomorphism onto $TY^\omega - Y$.

Let λ be a $\mathbb{T}$-invariant connection 1-form on the principal S^1-bundle $S(TY^\omega)\longrightarrow Y$, i.e. λ is a 1-form on $S(TY^\omega)$ so that

$$\lambda(\zeta_v) = 2\pi \qquad \text{and} \qquad \iota_{\zeta_v}(\mathrm{d}\lambda) = 0\,. \tag{6.30}$$

In light of the first condition above, the second condition is equivalent to λ being S^1-invariant. Let $\widetilde{\omega}$ and $\widetilde{\omega}'$ be the $\mathbb{T}$-invariant closed 2-forms on $S(TY^\omega)\times\mathbb{C}$ and $S(TY^\omega)\times\mathbb{R}$, respectively, given by

$$\widetilde{\omega} = \pi^*\omega + \omega_{\mathbb{C}} + \frac{1}{2}\mathrm{d}\big(|z|^2\lambda\big) \qquad \text{and} \qquad \widetilde{\omega}' = \pi^*\omega + \mathrm{d}(t\lambda),$$

where $\omega_{\mathbb{C}}$ is the standard symplectic form on $\mathbb{C}$ as in Example 2.5, z is the standard coordinate on $\mathbb{C}$, and t is the standard coordinate on $\mathbb{R}$. Since

$$\widetilde{\omega}'_{(w,0)} = \omega_{\pi(w)} + \mathrm{d}_0 t\wedge\lambda_w \qquad \forall\ w\in S(TY^\omega), \tag{6.31}$$

the 2-form $\widetilde{\omega}'$ is nondegenerate (and thus symplectic) on some neighborhood $\mathcal{U}'\subset S(TY^\omega)\times\mathbb{R}$ of $S(TY^\omega)\times\{0\}$.

By (6.30) and the last coordinate of the map $\widetilde{\iota}$ taking only real values,

$$\iota_{\zeta_v}\widetilde{\omega}' = -2\pi\mathrm{d}t \qquad \text{and} \qquad \widetilde{\omega}'\big|_{S(TY^\omega)\times\mathbb{R}^+} = \widetilde{\iota}^*\widetilde{\omega}\,, \tag{6.32}$$

respectively. With (r,θ) denoting the standard radius-angle coordinates on $\mathbb{C}$ so that $\omega_{\mathbb{C}} = r\mathrm{d}r\wedge\mathrm{d}\theta$,

$$\big(\iota_{\widetilde{\zeta}_v}\widetilde{\omega}\big)_{(w,re^{\mathrm{i}\theta})} = 0 - 2\pi\iota_{\partial_\theta}\omega_{\mathbb{C}} + \iota_{\zeta_v}\big(r\mathrm{d}r\wedge\lambda + \frac{1}{2}r^2\mathrm{d}\lambda\big) = 2\pi r\mathrm{d}r - 2\pi r\mathrm{d}r + 0 = 0\,; \tag{6.33}$$

the middle equality above follows again from (6.30). Since the S^1-action (6.29) preserves the 2-form $\widetilde{\omega}$, (6.33) implies that there is a unique 2-form ω_{TY^ω} on (the total space of) TY^ω so that $p^*\omega_{TY^\omega} = \widetilde{\omega}$. This form is $\mathbb{T}$-equivariant and closed and satisfies

$$\widetilde{\omega}'\big|_{S(TY^\omega)\times\mathbb{R}^+} = \iota^*\omega_{TY^\omega}; \tag{6.34}$$

see the second equation in (6.32).

Let λ_S be the 1-form on $L_S = TY^\omega$ determined by λ as in Exercise A.8(a). Thus,

$$\omega_{TY^\omega} = \pi^*\omega + \frac{1}{2}\mathrm{d}\lambda_S.$$

Along with Exercise A.8(b), this implies that the closed 2-form ω_{TY^ω} on TY^ω satisfies (3.11) with $TY^c = TY^\omega$. By Proposition 3.14(2), there thus exists a $\mathbb{T}$-equivariant tubular neighborhood identification $\Phi\colon\mathcal{U}\longrightarrow U$ for Y in X such that $\mathcal{U}\subset TY^\omega$ and $\Phi^*\omega = \omega_{TY^\omega}|_{\mathcal{U}}$. Along with (6.34), the last identity gives

$$\widetilde{\omega}'\big|_{\iota^{-1}(\mathcal{U})} = \iota^*\Phi^*\omega. \tag{6.35}$$

Let $\mathcal{U}''\subset\mathcal{U}'\subset S(TY^\omega)\times\mathbb{R}$ be a $\mathbb{T}$-invariant tubular neighborhood of $S(TY^\omega)\times\{0\}$ so that

$$\mathcal{U}^+\equiv\mathcal{U}''\cap\big(S(TY^\omega)\times\mathbb{R}^+\big)\subset\iota^{-1}(\mathcal{U}).$$

Since the diffeomorphism $\Phi\circ\iota\colon\mathcal{U}^+\longrightarrow\Phi(\iota(\mathcal{U}^+))$ is $\mathbb{T}$-equivariant and satisfies (6.35) with $\iota^{-1}(\mathcal{U})$ replaced by $\mathcal{U}^+$,

$$\mu\circ\Phi\circ\iota\colon\mathcal{U}^+\longrightarrow T^*_{\mathbb{1}}\mathbb{T} \tag{6.36}$$

is a moment map for the $\mathbb{T}$-action on $(\mathcal{U}^+,\widetilde{\omega}'|_{\mathcal{U}^+})$. By Exercise 3.26, it extends to a moment map $\widetilde{\mu}'\colon\mathcal{U}''\longrightarrow T^*_{\mathbb{1}}\mathbb{T}$ for the $\mathbb{T}$-action on $(\mathcal{U}'',\widetilde{\omega}'|_{\mathcal{U}''})$. Since

$$\lim_{t\longrightarrow0^+}\iota(w,t)=\pi(w)\in Y\subset TY^\omega,\quad \lim_{t\longrightarrow0^+}\mu\big(\Phi\big(\iota(w,t)\big)\big)=\mu\big(\pi(w)\big)\qquad\forall\,w\in S(TY^\omega),$$

the first equation in (6.32) implies that

$$\widetilde{\mu}'_v(w,t)=2\pi t+c,\qquad \widetilde{\mu}'^{-1}_v\big(\langle\mathscr{H}\rangle^\partial\big)=S(TY^\omega)\times\{0\}, \tag{6.37}$$

and $\widetilde{\mu}'=\mu\circ\pi$ on $S(TY^\omega)\times\{0\}=S(TY^\omega)$.

We define

$$X'=\big((X-Y)\sqcup\mathcal{U}''\big)/\sim,\quad \mathcal{U}^+\ni(w,t)\sim\Phi\big(\iota(w,t)\big)\in X-Y, \tag{6.38}$$
$$\omega'_{[x]}=\begin{cases}\omega_x, & \text{if } x\in X-Y;\\ \widetilde{\omega}'_x, & \text{if } x\in\mathcal{U}'';\end{cases}\qquad \mu'([x])=\begin{cases}\mu(x), & \text{if } x\in X-Y;\\ \widetilde{\mu}'(x), & \text{if } x\in\mathcal{U}''.\end{cases}$$

Suppose $x\in X-Y-\Phi(\iota(\mathcal{U}^+))$ and $x'\in\mathcal{U}''-\mathcal{U}^+$. If x' does not lie in the closure $\mathrm{Cl}_{\mathcal{U}''}\mathcal{U}^+$ of $\mathcal{U}^+$ in $\mathcal{U}''$, then the images of $X-Y$ and $\mathcal{U}''-\mathrm{Cl}_{\mathcal{U}''}\mathcal{U}^+$ in X' under the quotient projection

$$q\colon(X-Y)\sqcup\mathcal{U}''\longrightarrow X'$$

are disjoint open subsets around $[x]$ and $[x']$, respectively. If

$$x'\in\mathrm{Cl}_{\mathcal{U}''}\mathcal{U}^+-\mathcal{U}^+=Y\times\{0\}$$

and $U,U'\subset X$ are disjoint open neighborhoods of x and Y, respectively, then

$$q(U),q\big(\big(\mathcal{U}''\cap(S(TY^\omega)\times\mathbb{R}^{\le0})\big)\cup\iota^{-1}\big(\Phi^{-1}(U')\big)\big)\subset X'$$

are disjoint open subsets around $[x]$ and $[x']$, respectively. Since the restrictions of q to the Hausdorff spaces $X-Y$ and $\mathcal{U}''$ are homeomorphisms onto open subsets of X', it follows that X' is a Hausdorff space and a smooth manifold. By (6.35) and the assumption on $\mathcal{U}'$ below (6.31), ω' is a well-defined symplectic form on X'. Since the smooth map $\widetilde{\mu}'$ is an extension of the map (6.36), $\mu'\colon X'\longrightarrow T^*_{\mathbb{1}}\mathbb{T}$ is a well-defined smooth map.

Since the identification of the spaces $X-Y$ and $\mathcal{U}''$ over $\mathcal{U}^+\subset\mathcal{U}''$ in (6.38) is $\mathbb{T}$-equivariant, the $\mathbb{T}$-action ψ on $X-Y\subset X$ and the $\mathbb{T}$-action $\mathrm{d}\psi$ on $\mathcal{U}''\subset S(TY^\omega)\times\mathbb{R}$ induce a smooth $\mathbb{T}$-action ψ' on X' which preserves ω'. Since $\mu|_{X-Y}$ and $\widetilde{\mu}'$ are moment maps for the $\mathbb{T}$-actions on $(X-Y,\omega|_{X-Y})$ and $(\mathcal{U}'',\widetilde{\omega}'|_{\mathcal{U}''})$, μ' is a moment map for the $\mathbb{T}$-action ψ' on (X',ω'). By (6.26) and (6.37),

$$\mu'^{-1}\big(\langle\mathscr{H}_2\rangle^\partial\big)=q\big(S(TY^\omega)\times\{0\}\big)\subset X'.$$

Since the restriction of q to $\mathcal{U}''$ is a diffeomorphism onto $q(\mathcal{U}'')$ and $\mathbb{T}_\upsilon$ acts freely on $S(TY^\omega)\times\{0\}$, $\mathbb{T}_\upsilon$ acts freely on $\mu'^{-1}(\langle\!\langle\mathscr{H}_2\rangle^\partial)$ as well. Thus, (X',ω',ψ',μ') is an $\mathscr{H}_2$-cuttable Hamiltonian $\mathbb{T}$-manifold with

$$\begin{gathered}\mu'^{-1}\big(\langle\!\langle\emptyset\rangle^\partial_{\mathscr{H}_2}\big)=q(X-Y)\approx Y,\quad \mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big)=q\big(S(TY^\omega)\times\{0\}\big)\subset X',\\ X=\mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle\!\rangle\big)/\!\sim,\quad x\sim\psi'_u(x)\;\forall\,x\in\mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big),\;u\in\mathbb{T}_\upsilon,\quad Y=\mu'^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big)/\mathbb{T}_\upsilon,\\ \{q|_{X-Y}\}^*p^*_{\mathscr{H}_2}\omega=\omega|_{X-Y},\quad \{q|_{S(TY^\omega)\times\{0\}}\}^*\{p_{\mathscr{H}_2}|_{\mu'^{-1}(\langle\!\langle\mathscr{H}_2\rangle^\partial)}\}^*(\omega|_{TY})=\widetilde{\omega}'|_{S(TY^\omega)\times\{0\}},\end{gathered}$$

where $p_{\mathscr{H}_2}\colon\mu'^{-1}(\langle\!\langle\mathscr{H}_2\rangle\!\rangle)\longrightarrow X$ is the quotient projection; the two identities on the first line above follow from (6.26) and the second statement in (6.37). Furthermore, the map $p_{\mathscr{H}_2}$ is $\mathbb{T}$-equivariant and the compositions

$$p_{\mathscr{H}_2}\circ q\colon X-Y\longrightarrow\mu^{-1}\big(\langle\!\langle\emptyset\rangle^\partial_{\mathscr{H}_2}\big)\quad\text{and}\quad p_{\mathscr{H}_2}\circ q\colon S(TY^\omega)\times\{0\}\longrightarrow\mu^{-1}\big(\langle\!\langle\mathscr{H}_2\rangle^\partial\big)$$

are submersions. By the uniqueness statement of Theorem 5, (6.23) thus holds.

Every topological component of the tubular neighborhood $q(\mathcal{U}'')\subset X'$ of $q(Y\times\{0\})$ intersects $q(X-Y)$. If X is connected, then so is $X-Y$ (because $Y\subset X$ is a submanifold of codimension 2). It then follows that X' is also connected. If (X,ω,ψ,μ) is $\mathscr{H}$-cut, then (X',ω',ψ',μ') is $\mathscr{H}_1$-cut by Corollary 6.12 if $\mathcal{U}''$ is sufficiently small.

Suppose both conditions in (b) hold and $\mathscr{H}_1'\subset\mathscr{H}_1=\mathscr{H}-\{\upsilon\}$. Let $\mathscr{H}_1'\upsilon=\mathscr{H}_1'\sqcup\{\upsilon\}$. By the above, $\mu'^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^\partial)\subset X'$ is an ω'-symplectic submanifold consisting of components of the fixed locus $X'^{\mathbb{T}_{\mathscr{H}_1'}}$ of the restriction of the action ψ' to the subtorus $\mathbb{T}_{\mathscr{H}_1'}\subset\mathbb{T}$. Since the restriction of the quotient projection q above to $\mathcal{U}''$ is a $\mathbb{T}$-equivariant diffeomorphism onto the open subset $q(\mathcal{U}'')\subset X'$ and $\widetilde{\mu}'=\mu'\circ q$ on $\mathcal{U}''$, it follows that $\widetilde{\mu}'^{-1}(\langle\!\langle\mathscr{H}_1'\rangle^\partial)\subset\mathcal{U}''$ is an $\widetilde{\omega}'$-symplectic submanifold consisting of components of $\mathcal{U}''^{\mathbb{T}_{\mathscr{H}_1'}}$. By (6.24) with $\mathscr{H}'=\mathscr{H}_1'\upsilon$, the restriction

$$\mu\colon\mu^{-1}\big(\langle\!\langle\mathscr{H}_1'\upsilon\rangle^\partial_{\mathscr{H}}\big)=\mu^{-1}\big(\langle\!\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\big)\cap Y\longrightarrow\langle\mathscr{H}_1'\upsilon\rangle^\partial_{\mathscr{H}}\cap\mu(X)=\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\cap\mu(Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'\upsilon}$-bundle. Since $\mathbb{T}_\upsilon$ acts freely on the fibers of the circle bundle (6.28), it follows that the restriction

$$\widetilde{\mu}'=\mu\circ\pi\colon S(TY^\omega)\big|_{\mu^{-1}(\langle\!\langle\mathscr{H}_1'\upsilon\rangle^\partial_{\mathscr{H}})}\times\{0\}=\widetilde{\mu}'^{-1}\big(\langle\!\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\big)\cap\big(S(TY^\omega)\times\{0\}\big)\longrightarrow\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\cap\mu(Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. By Exercise 6.7(d) with $\mathscr{H}$ replaced by $\mathscr{H}_1'$ and the submanifold $\langle\mathscr{H}_1'\upsilon\rangle^\partial\subset\langle\mathscr{H}_1'\rangle^\partial$ being closed, the moment map

$$\widetilde{\mu}'\colon\widetilde{\mu}'^{-1}\big(\langle\!\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\big)\longrightarrow\langle\mathscr{H}_1'\rangle^\partial$$

is then a submersion and thus also a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle over its image if $\mathcal{U}''$ is sufficiently small. Since $q|_{\mathcal{U}''}$ is a $\mathbb{T}$-equivariant diffeomorphism onto $q(\mathcal{U}'')\subset X'$ and $\widetilde{\mu}'=\mu'\circ q$ on $\mathcal{U}''$, it follows that

$$\mu'\colon\mu'^{-1}\big(\langle\!\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\big)\cap q(\mathcal{U}'')\longrightarrow\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\cap\widetilde{\mu}'(\mathcal{U}'')=\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\cap\mu'(q(\mathcal{U}'')).$$

By (6.24) with $\mathscr{H}'=\mathscr{H}_1'$ and (6.26), the restriction

$$\mu\colon\mu^{-1}\big(\langle\!\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}}\big)\cap(X-Y)=\mu^{-1}\big(\langle\!\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\big)\cap(X-Y)\longrightarrow\langle\mathscr{H}_1'\rangle^\partial_{\mathscr{H}_1}\cap\mu(X-Y)$$

is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. Since the restriction of q to $X-Y$ is a $\mathbb{T}$-equivariant diffeomorphism onto the open subset $q(X-Y)\subset X'$ and $\mu=\mu'\circ q$ on $X-Y$, it follows that the restriction

$$\mu'\colon \mu'^{-1}\big(\langle\mathscr{H}_1'\rangle_{\mathscr{H}_1}^{\partial}\big)\cap q(X-Y) \longrightarrow \langle\mathscr{H}_1'\rangle_{\mathscr{H}_1}^{\partial}\cap\mu(X-Y) = \langle\mathscr{H}_1'\rangle_{\mathscr{H}_1}^{\partial}\cap\mu'\big(q(X-Y)\big)$$

is also a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. Since $X'=q(\mathcal{U}'')\cup q(X-Y)$, we conclude that the restriction (6.25) is a principal $\mathbb{T}/\mathbb{T}_{\mathscr{H}_1'}$-bundle. $\square$

Proof of Theorem 3. Since (0) in the statement of this theorem follows from (0^+) on page 8, which was established in Section 5.2, it remains to establish (1) and (2). Suppose $P\subset T_{\mathbb{1}}^*\mathbb{T}$ is a Delzant polytope and $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ is a Delzant subset so that $\mu(X)=\langle\mathscr{H}\rangle$, and $\mathscr{H}'\subset\mathscr{H}$. Let $(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}})$ be the Hamiltonian $\mathbb{T}$-manifold of Exercise 2.11, with $k=n$. By Theorem 5, the Hamiltonian $\mathbb{T}$-manifold

$$(X,\omega,\psi,\mu) \equiv \big(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}}\big)_{\mathscr{H}}$$

as in (6.5) is then a closed connected Hamiltonian $\mathbb{T}$-manifold so that (1.7) holds, the $\mathbb{T}$-action ψ is effective (it is free on $\mu^{-1}(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})$, and $\mu(X)=P$. This gives (1).

Suppose (X,ω,ψ,μ) is any symplectic toric $\mathbb{T}$-manifold with $\mu(X)=P$. In particular, X is connected. By Exercise 6.9 and (0^+c) on page 8, (X,ω,ψ,μ) also satisfies (b) in the statement of Proposition 6.13. By Proposition 6.13 with $\mathscr{H}_1=\emptyset$ and $\mathscr{H}_2=\mathscr{H}$,

$$(X,\omega,\psi,\mu) = (X',\omega',\psi',\mu')_{\mathscr{H}} \tag{6.39}$$

for some $\mathscr{H}$-cuttable Hamiltonian $\mathbb{T}$-manifold (X',ω',ψ',μ') so that X' is connected and

$$\mu'\colon X'=\mu'^{-1}\big(\langle\emptyset\rangle_{\emptyset}^{\partial}\big) \longrightarrow \langle\emptyset\rangle_{\emptyset}^{\partial}\cap\mu'(X')=\mu'(X')$$

is a principal $\mathbb{T}$-bundle. By Exercise 3.21(b) and (6.39), $\mu'(X')\subset T_{\mathbb{1}}^*\mathbb{T}$ is an open neighborhood of the polytope P. By replacing X' with the preimage of a contractible neighborhood of P in $\mu'(X')$, we can assume $\mu'(X')$ is contractible. By Proposition 3.30, (X',ω',ψ',μ') is then isomorphic to $(U,\omega_{\mathbb{T}}|_U,\psi_{\mathbb{T}}|_U,\mu_{\mathbb{T}}|_U)$ for an open neighborhood $U\subset T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T}$ of $\mu_{\mathbb{T}}^{-1}(P)$. Along with (6.39) and Theorem 5, this implies that

$$(X,\omega,\psi,\mu) \approx \big(U,\omega_{\mathbb{T}}|_U,\psi_{\mathbb{T}}|_U,\mu_{\mathbb{T}}|_U\big)_{\mathscr{H}} = \big(T_{\mathbb{1}}^*\mathbb{T}\times\mathbb{T},\omega_{\mathbb{T}},\psi_{\mathbb{T}},\mu_{\mathbb{T}}\big)\,.$$

This gives (2). $\square$

7 Symplectic Toric Manifolds

In this chapter, we describe a construction of toric symplectic manifolds along the lines of [11, Section 3.2] and use it to obtain key properties of these manifolds. Example 2.7 is a special case of this construction. The structure of this chapter is motivated by [28, Chapter 2], which efficiently summarizes these properties from a more concrete perspective. We fix a torus $\mathbb{T}$ of dimension n and continue with the notation and terminology introduced at the beginning of Section 6.2.

7.1 Delzant's construction

Let $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ be a Delzant subset so that the homomorphism $L_{\mathscr{H}}$ in (6.2) is surjective. In this section, we use the symplectic reduction of Theorem 4 to construct a connected Hamiltonian $\mathbb{T}$-manifold (X,ω,ψ,ω) with an effective $\mathbb{T}$-action ψ and $\mu(X)=\langle\mathscr{H}\rangle$. The manifold X obtained in this way is compact if and only if $\langle\mathscr{H}\rangle$ is compact. By Exercise 6.6, every Delzant polytope P equals $\langle\mathscr{H}\rangle$ for some Delzant subset $\mathscr{H}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ so that the homomorphism $L_{\mathscr{H}}$ in (6.2) is surjective. Thus, the construction of this section provides another proof of Theorem 3(1).

By Exercise 6.5(c), the kernel of the homomorphism $\Phi_{\mathscr{H}}$ in (6.2),

$$\mathcal{K}_{\mathscr{H}}\equiv\ker\Phi_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}\equiv\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}},$$

is then a codimension n subtorus. Let

$$\iota_{\mathscr{H}}^*:\mathbb{R}^{\mathscr{H}}=T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}\longrightarrow T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}$$

be the composition of the standard identification of $\mathbb{R}^{\mathscr{H}}$ with $T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}$ and the homomorphism induced by the inclusion $\mathcal{K}_{\mathscr{H}}\longrightarrow\mathbb{T}^{\mathscr{H}}$. We note that the sequence

$$0\longrightarrow T_{\mathbb{1}}^*\mathbb{T}\xrightarrow{L_{\mathscr{H}}^*}T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}=\mathbb{R}^{\mathscr{H}}\xrightarrow{\iota_{\mathscr{H}}^*}T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}\longrightarrow 0 \tag{7.1}$$

of vector spaces is exact and

$$L_{\mathscr{H}}^*\big(\langle\mathscr{H}\rangle\big)=\big\{(s_v)_{v\in\mathscr{H}}\in\ker\iota_{\mathscr{H}}^*:\,s_v\ge c_v\ \forall\,v\in\mathscr{H}\big\}. \tag{7.2}$$

Denote by $\omega_{\mathscr{H}}$ the standard symplectic form on $\mathbb{C}^{\mathscr{H}}$ as in Example 2.5. By this example, the map

$$H_{\mathscr{H}}:\mathbb{C}^{\mathscr{H}}\longrightarrow\mathbb{R}^{\mathscr{H}},\qquad H_{\mathscr{H}}\big((z_v)_{v\in\mathscr{H}}\big)=\big(\pi|z_v|^2+c_v\big)_{v\in\mathscr{H}}, \tag{7.3}$$

is a Hamiltonian with respect to $\omega_{\mathscr{H}}$ for the standard action $\psi_{\mathscr{H}}$ of $\mathbb{T}^{\mathscr{H}}$ on $\mathbb{C}^{\mathscr{H}}$,

$$\psi_{\mathscr{H};[(r_v)_{v\in\mathscr{H}}]}\big((z_v)_{v\in\mathscr{H}}\big)=\big(\mathrm{e}^{2\pi\mathrm{i}r_v}z_v\big)_{v\in\mathscr{H}}. \tag{7.4}$$

Thus,

$$\mu_{\mathscr{H}}\equiv\iota_{\mathscr{H}}^*\circ H_{\mathscr{H}}:\mathbb{C}^{\mathscr{H}}\longrightarrow T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}$$

is a moment map with respect to $\omega_{\mathscr{H}}$ for the restriction $\widetilde{\psi}$ of the action $\psi_{\mathscr{H}}$ to $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$. By (7.2),

$$H_{\mathscr{H}}(\mathbb{C}^{\mathscr{H}})\cap\iota_{\mathscr{H}}^{*\,-1}(0)=L_{\mathscr{H}}^*\big(\langle\mathscr{H}\rangle\big). \tag{7.5}$$

Lemma 7.1. *The subspace $\mu_{\mathscr{H}}^{-1}(0)\subset\mathbb{C}^{\mathscr{H}}$ is preserved by the $\mathbb{T}^{\mathscr{H}}$-action (7.4); if $\langle\mathscr{H}\rangle$ is compact, then so is $\mu_{\mathscr{H}}^{-1}(0)$. The codimension n subtorus $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ acts freely on $\mu_{\mathscr{H}}^{-1}(0)$. For some $z\in\mu_{\mathscr{H}}^{-1}(0)$, $\mathrm{Stab}_z(\psi_{\mathscr{H}})=\{\mathbb{1}\}$.*

Proof. Since the Hamiltonian $\widetilde{H}$ is $\mathbb{T}^{\mathscr{H}}$-invariant, the $\mathbb{T}^{\mathscr{H}}$-action (7.4) preserves $\mu_{\mathscr{H}}^{-1}(0)$. Since the subspace in (7.5) and the fibers of H are path-connected, so is the subspace

$$\mu_{\mathscr{H}}^{-1}(0)=H_{\mathscr{H}}^{-1}\big(\iota_{\mathscr{H}}^{*\,-1}(0)\big).$$

If $\langle\mathscr{H}\rangle$ is compact, then so is the subspace in (7.5). Since the map H is proper, it follows that the subspace $\mu_{\mathscr{H}}^{-1}(0)$ is also compact if $\langle\mathscr{H}\rangle$ is compact.

Suppose $(z_v)_{v\in\mathscr{H}}\in\mu_{\mathscr{H}}^{-1}(0)$, $u\in\mathcal{K}_{\mathscr{H}}$, and $\psi_{\mathscr{H};u}\big((z_v)_{v\in\mathscr{H}}\big)=(z_v)_{v\in\mathscr{H}}$. Let

$$\mathscr{H}'=\big\{v\in\mathscr{H}\colon z_v=0\big\}.$$

From (7.3), (7.4), and (7.5), we then obtain

$$H(z)\in L_{\mathscr{H}}^*\big(\langle\mathscr{H}'\rangle_{\mathscr{H}}^{\partial}\big)\subset\mathbb{R}^{\mathscr{H}}\quad\text{and}\quad u\in\mathbb{R}^{\mathscr{H}'}/\mathbb{Z}^{\mathscr{H}'}\subset\mathbb{R}^{\mathscr{H}}/\mathbb{Z}^{\mathscr{H}}.$$

Since $\mathscr{H}$ is a Delzant subset, it follows from the first statement above that the homomorphism $\Phi_{\mathscr{H}'}$ as in (6.2) is injective and thus from the second that $u=\mathbb{1}$. We conclude that $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ acts freely on $\mu_{\mathscr{H}}^{-1}(0)$. By (7.5), there exists $z\equiv(z_v)_{v\in\mathscr{H}}\in\mu_{\mathscr{H}}^{-1}(0)$ such that $H(z)\in L_{\mathscr{H}}^*(\langle\emptyset\rangle_{\mathscr{H}}^{\partial})$ and thus $z_v\neq0$ for any $v\in\mathscr{H}$. It follows that $\mathrm{Stab}_z(\psi_{\mathscr{H}})=\{\mathbb{1}\}$. $\square$

Thus, $(\mathbb{C}^{\mathscr{H}},\omega_{\mathscr{H}},\psi_{\mathscr{H}},\mu_{\mathscr{H}})$ is a Hamiltonian $\mathcal{K}_{\mathscr{H}}$-manifold such that $\mathcal{K}_{\mathscr{H}}$ acts freely on $\mu_{\mathscr{H}}^{-1}(0)$. Let $(X,\omega)\equiv(X_0,\omega_0)$ be the quotient symplectic manifold provided by the first part of Theorem 4. By (0) and (1) in this theorem,

$$\dim X=\dim_{\mathbb{R}}\mathbb{C}^{\mathscr{H}}-2\dim\mathcal{K}_{\mathscr{H}}=2|\mathscr{H}|-2\big(|\mathscr{H}|-n\big)=2n.$$

By Lemma 7.1, $X\equiv\mu_{\mathscr{H}}^{-1}(0)/\mathcal{K}_{\mathscr{H}}$ is connected; if $\langle\mathscr{H}\rangle\subset T_{\mathbb{1}}^*\mathbb{T}$ is compact, then so is X.

The torus actions $\psi_{\mathscr{H}}|_{\mathcal{K}_{\mathscr{H}}}$ and $\psi_{\mathscr{H}}$ commute, the Hamiltonian $\widetilde{H}$ for $\psi_{\mathscr{H}}$ is $\psi_{\mathscr{H}}|_{\mathcal{K}_{\mathscr{H}}}$-invariant, and the moment map $\mu_{\mathscr{H}}$ for $\psi_{\mathscr{H}}|_{\mathcal{K}_{\mathscr{H}}}$ is $\psi_{\mathscr{H}}$-invariant. Let ψ_0' and μ_0' be the $\mathbb{T}^{\mathscr{H}}$-action on X induced by $\psi_{\mathscr{H}}$ and its Hamiltonian with respect to ω induced by $H_{\mathscr{H}}$, as provided by the last part of Theorem 4.

By Exercise 6.5(b), there exists a subset $\mathscr{H}'\subset\mathscr{H}$ be such that $\langle\mathscr{H}'\rangle^{\partial}$ is a vertex of $\langle\mathscr{H}\rangle$, i.e. $\langle\mathscr{H}'\rangle^{\partial}\subset\langle\mathscr{H}\rangle$ is a single point. Let $\iota_{\mathscr{H};\mathscr{H}'}\colon\mathbb{T}^{\mathscr{H}'}\longrightarrow\mathbb{T}^{\mathscr{H}}$ be the inclusion. Since $\mathscr{H}$ is Delzant, the Lie homomorphism group $\Phi_{\mathscr{H}'}$ as in (6.2) is an isomorphism and thus so is the Lie group homomorphism

$$\mathcal{K}_{\mathscr{H}}\times\mathbb{T}^{\mathscr{H}'}\longrightarrow\mathbb{T}^{\mathscr{H}},\qquad(u,u')\longrightarrow uu'.$$

By the last statement of Lemma 7.1 above and Exercise 7.2 below, the composition ψ of the $\mathbb{T}^{\mathscr{H}}$-action ψ_0' on X with the homomorphism

$$\mathbb{T}\xrightarrow{\Phi_{\mathscr{H}'}^{-1}}\mathbb{T}^{\mathscr{H}'}\xrightarrow{\iota_{\mathscr{H};\mathscr{H}'}}\mathbb{T}^{\mathscr{H}}$$

is therefore an effective $\mathbb{T}$-action on X. The composition

$$\mu\colon X\xrightarrow{\mu_0'}T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}}\xrightarrow{\iota_{\mathscr{H};\mathscr{H}'}^*}T_{\mathbb{1}}^*\mathbb{T}^{\mathscr{H}'}\xrightarrow{L_{\mathscr{H}'}^{*\,-1}}T_{\mathbb{1}}^*\mathbb{T}$$

is its moment map with respect to ω. Thus, (X,ω,ψ,μ) is a connected Hamiltonian $\mathbb{T}$-manifold with an effective $\mathbb{T}$-action ψ and moment polytope

$$\begin{aligned}\mu(X)=L_{\mathscr{H}'}^{*\,-1}\big(\iota_{\mathscr{H};\mathscr{H}'}^*\big(\mu_0'(X)\big)\big)&=L_{\mathscr{H}'}^{*\,-1}\big(\iota_{\mathscr{H};\mathscr{H}'}^*\big(H_{\mathscr{H}}(\mu_{\mathscr{H}}^{-1}(0))\big)\big)\\&=L_{\mathscr{H}'}^{*\,-1}\big(\iota_{\mathscr{H};\mathscr{H}'}^*\big(L_{\mathscr{H}}^*(\langle\mathscr{H}\rangle)\big)\big)=L_{\mathscr{H}'}^{*\,-1}\big(L_{\mathscr{H}'}^*(\langle\mathscr{H}\rangle)\big)=\langle\mathscr{H}\rangle\,;\end{aligned}$$

the second equality above holds by (7.5).

Exercise 7.2. Suppose ψ_1 and ψ_2 are commuting actions of groups G_1 and G_2 on a set Z and $z \in Z$ is a point such that $\mathrm{Stab}_z(\psi_1 \times \psi_2) = \{1\}$. Let $\overline{\psi}_2$ be the induced G_2-action on the quotient Z/G_1 and $G_1 z \in Z/G_1$ be the G_1-orbit of z. Show that $\mathrm{Stab}_{G_1 z}(\overline{\psi}_2) = \{1\}$.

Exercise 7.3. Show that the Hamiltonian $\mathbb{T}$-manifold (X, ω, ψ, μ) constructed above does not depend on the choice of subset $\mathscr{H}' \subset \mathscr{H}$ such that $\langle \mathscr{H}' \rangle^\partial$ is a vertex of $\mathscr{H}$.

Exercise 7.4. Suppose $\widetilde{\mathscr{H}} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ is a Delzant subset such that $\langle \widetilde{\mathscr{H}} \rangle = \langle \mathscr{H} \rangle$ and $\widetilde{\mathscr{H}} \supset \mathscr{H}$. Show that

(a) there exists a group homomorphism $\phi_{\mathscr{H};\widetilde{\mathscr{H}}} \colon \mathbb{T}^{\widetilde{\mathscr{H}} - \mathscr{H}} \longrightarrow \mathbb{T}^{\mathscr{H}}$ so that the group homomorphism

$$\mathcal{K}_{\mathscr{H}} \times \mathbb{T}^{\widetilde{\mathscr{H}} - \mathscr{H}} \longrightarrow \mathbb{T}^{\mathscr{H}} \times \mathbb{T}^{\widetilde{\mathscr{H}} - \mathscr{H}} = \mathbb{T}^{\widetilde{\mathscr{H}}}, \qquad (u, u') \longrightarrow \big(u \phi_{\mathscr{H};\widetilde{\mathscr{H}}}(u), u'\big),$$

is an isomorphism onto $\mathcal{K}_{\widetilde{\mathscr{H}}}$;

(b) there exist (unique) linear functionals $\ell_\upsilon \colon \mathrm{Im}\, L^*_{\mathscr{H}} \longrightarrow \mathbb{R}$ with $\upsilon \in \widetilde{\mathscr{H}} - \mathscr{H}$ such that

$$\mathrm{Im}\, L^*_{\widetilde{\mathscr{H}}} = \big\{\big(s, (\ell_{\mathscr{H}}(s))_{\upsilon \in \widetilde{\mathscr{H}} - \mathscr{H}}\big) \colon s \in \mathrm{Im}\, L^*_{\mathscr{H}}\big\} \subset \mathbb{R}^{\mathscr{H}} \times \mathbb{R}^{\widetilde{\mathscr{H}} - \mathscr{H}} = \mathbb{R}^{\widetilde{\mathscr{H}}};$$

(c) $\ell_\upsilon(s) > c_\upsilon$ for all $\upsilon \in \widetilde{\mathscr{H}} - \mathscr{H}$ and $s \in L^*_{\mathscr{H}}(\langle \mathscr{H} \rangle)$ and

$$L^*_{\widetilde{\mathscr{H}}}\big(\langle \widetilde{\mathscr{H}} \rangle\big) = \big\{\big(s, (\ell_{\mathscr{H}}(s))_{\upsilon \in \widetilde{\mathscr{H}} - \mathscr{H}}\big) \colon s \in L^*_{\mathscr{H}}\big(\langle \mathscr{H} \rangle\big)\big\} \subset \mathbb{R}^{\mathscr{H}} \times \mathbb{R}^{\widetilde{\mathscr{H}} - \mathscr{H}} = \mathbb{R}^{\widetilde{\mathscr{H}}};$$

(d) the projection $\mathbb{C}^{\widetilde{\mathscr{H}}} \longrightarrow \mathbb{C}^{\mathscr{H}}$ restricts to a principal $\mathbb{T}^{\widetilde{\mathscr{H}} - \mathscr{H}}$-bundle $\mu^{-1}_{\widetilde{\mathscr{H}}}(0) \longrightarrow \mu^{-1}_{\mathscr{H}}(0)$ with smooth $\mathbb{T}^{\mathscr{H}}$-equivariant section

$$s_{\widetilde{\mathscr{H}},\mathscr{H}} \colon \mu^{-1}_{\mathscr{H}}(0) \longrightarrow \mu^{-1}_{\widetilde{\mathscr{H}}}(0) \subset \mathbb{C}^{\mathscr{H}} \times \mathbb{C}^{\widetilde{\mathscr{H}} - \mathscr{H}}, \quad s_{\widetilde{\mathscr{H}},\mathscr{H}}(z) = \Big(z, \big(\sqrt{(\ell_\upsilon(H_{\mathscr{H}}(z)) - c_\upsilon)/\pi}\big)_{\upsilon \in \widetilde{\mathscr{H}} - \mathscr{H}}\Big);$$

(e) $s^*_{\widetilde{\mathscr{H}},\mathscr{H}} \omega_{\widetilde{\mathscr{H}}} = \omega_{\mathscr{H}}|_{T\mu^{-1}_{\mathscr{H}}(0)}$ and

$$\iota^*_{\widetilde{\mathscr{H}};\mathscr{H}'} \circ H_{\widetilde{\mathscr{H}}} \circ s^*_{\widetilde{\mathscr{H}},\mathscr{H}} = \iota^*_{\mathscr{H};\mathscr{H}'} \circ H_{\mathscr{H}} \colon \mu^{-1}_{\mathscr{H}}(0) \longrightarrow \mathrm{Im}\, L^*_{\mathscr{H}'} \subset \mathbb{R}^{\mathscr{H}'} \quad \forall\, \mathscr{H}' \subset \mathscr{H}.$$

Conclude the Hamiltonian $\mathbb{T}$-manifold (X, ω, ψ, μ) constructed above does not depend on the choice of Delzant subset $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ with $\langle \mathscr{H} \rangle$ fixed.

7.2 Kähler structure

in preparation

As before, let $\mathscr{H} \subset (T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}} \times \mathbb{R}$ be a Delzant subset so that the homomorphism $L_{\mathscr{H}}$ in (6.2) is surjective. In this section, we show that the Hamiltonian $\mathbb{T}$-manifold $(X, \omega, \psi, \omega)$ constructed in in Section 7.1, admits a compatible (integrable) complex structure J, i.e. J is compatible with the symplectic form ω and is preserved by the effective $\mathbb{T}$-action ψ on X. We continue with the notation introduced in Section 7.1.

Let $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\subset\mathbb{T}_{\mathbb{C}}^{\mathscr{H}}$ be the complexification of $\mathcal{K}_{\mathscr{H}}\subset\mathbb{T}^{\mathscr{H}}$ and $(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}\subset(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$ be the purely imaginary subgroup (it corresponds to a subgroup of $(\mathbb{R}^*)^{\mathscr{H}}\subset(\mathbb{C}^*)^{\mathscr{H}}$ via $\mathrm{e}^{2\pi\mathfrak{i}\cdot}$). The group $\mathbb{T}_{\mathbb{C}}^{\mathscr{H}}\approx(\mathbb{C}^*)^{\mathscr{H}}$ acts on $\mathbb{C}^{\mathscr{H}}$ by the coordinate-wise multiplication in the usual way, i.e. as in (7.4); we denote this complexified action in the same way. Define

$$\widetilde{X}_{\mathscr{H}}=\mathbb{C}^{\mathscr{H}}-\bigcup_{\substack{\mathscr{H}'\subset\mathscr{H}\\ \mathbb{C}^{\mathscr{H}'}\cap\mu_{\mathscr{H}}^{-1}(0)=\emptyset}}\mathbb{C}^{\mathscr{H}'},\qquad X_{\mathscr{H}}\equiv\widetilde{X}_{\mathscr{H}}/(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}\,.$$

In particular, $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$ is a $\mathbb{T}_{\mathbb{C}}^{\mathscr{H}}$-invariant path-connected open subset containing $\mu_{\mathscr{H}}^{-1}(0)$.

Exercise 7.5. Show that

(a) $\widetilde{X}_{\widetilde{\mathscr{H}}}=\widetilde{X}_{\mathscr{H}}\times(\mathbb{C}^*)^{\widetilde{\mathscr{H}}-\mathscr{H}}$ for any Delzant subset $\widetilde{\mathscr{H}}\subset(T_{\mathbb{1}}\mathbb{T})_{\mathbb{Z}}\times\mathbb{R}$ with $\langle\widetilde{\mathscr{H}}\rangle=\langle\mathscr{H}\rangle$ and $\widetilde{\mathscr{H}}\supset\mathscr{H}$;

(b) $(\mathcal{K}_{\mathscr{H}})_{\mathbb{C}}$ acts freely on the subspace $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$;

(c) the subspace $\widetilde{X}_{\mathscr{H}}\subset\mathbb{C}^{\mathscr{H}}$ is simply connected if $\mathscr{H}$ is minimal.

Hint: see the proof of Lemma 7.1 for (b).

Lemma 7.6. *The smooth map*

$$\Psi_{\mathscr{H}}\colon(\mathcal{K}_{\mathscr{H}})_{\mathfrak{i}}\times\mu_{\mathscr{H}}^{-1}(0)\longrightarrow\widetilde{X}_{\mathscr{H}},\qquad\Psi_{\mathscr{H}}(u,z)=\psi_{\mathscr{H};u}(z),$$

is a diffeomorphism.

Proof. □

7.3 Line bundles and projectivity

Define

$$\alpha_{\mathscr{H}}=-\iota_{\mathscr{H}*}\big((c_v)_{v\in\mathscr{H}}\big)\in T_{\mathbb{1}}^*\mathcal{K}_{\mathscr{H}}\,.$$

$$P_A^\alpha=\big\{s\in\big(\mathbb{R}^{\ge0}\big)^N\colon\iota_A^*(s)=\alpha\big\},\quad\mathscr{V}_A^\alpha=\big\{J\subset[N]\colon|J|=n,\,P_A^\tau\cap(\mathbb{R}^{\ge0})^{[N]-J}\neq\emptyset\big\},$$

$$\widetilde{\mu}_{\mathbb{C}^N}=\sum_{j=1}^N\big(-y_j\mathrm{d}x_j+x_j\mathrm{d}y_j\big).$$

Let $P\equiv\big\{r\in\mathbb{R}^n\colon v_k\cdot r\ge c_k\ \forall\,k\in[N]\big\}$ be a Delzant polytope with the inward normals $v_1,\ldots,v_N$ to the facets (codimension 1 faces) meeting at each vertex of P forming a $\mathbb{Z}$-basis for $\mathbb{Z}^N$. Define

$$A=\big(v_1\,\ldots\,v_N\big)\colon(\mathbb{R}^N,\mathbb{Z}^N)\longrightarrow(\mathbb{R}^n,\mathbb{Z}^n),\quad c=(c_1,\ldots,c_N)\in\mathbb{R}^N,\quad\alpha=-\iota_A^*(c)\in T_{\mathbb{1}}^*\mathbb{T}_A.$$

Thus, $\widetilde{Z}_A^\alpha\subset\mathbb{C}^N$ is the preimage of the regular value $\alpha\in T_{\mathbb{1}}\mathbb{T}^*$ of $\widetilde{H}_A$, $\mathbb{T}_A$ acts freely on $\widetilde{Z}_A^\alpha$, and

$$(M_P,\omega_P)\equiv\big(\widetilde{Z}_A^\alpha,\omega_{\mathbb{C}^N}|_{T\widetilde{Z}_A^\alpha}\big)/\mathbb{T}_A$$

is the compact connected symplectic manifold obtained from P via the Delzant construction in class. Show that

(a) the inclusions $\widetilde{Z}^\alpha_A \longrightarrow \widetilde{M}^\alpha_A$ and $\mathbb{T}_A \longrightarrow (\mathbb{T}_A)_\mathbb{C}$ induce a homeomorphism

$$M^\alpha_A \equiv \widetilde{M}^\alpha_A/(\mathbb{T}_A)_\mathbb{C} \longrightarrow \widetilde{Z}^\alpha_A/\mathbb{T}_A \equiv M_P$$

with respect to the quotient topologies;

(b) the smooth manifold $M^\alpha_A = M_P$ is simply connected and admits a complex manifold structure, compatible with the smooth and symplectic structures, so that the quotient projection $q\colon \widetilde{M}^\alpha_A \longrightarrow M^\alpha_A$ is a holomorphic submersion and $(\mathbb{T}_A)_\mathbb{C}$ acts on M^α_A by biholomorphisms.

A Bundle Connections

A.1 Connections and splittings

Suppose X is a smooth manifold and $\pi_E\colon E \longrightarrow X$ is a (smooth) real vector bundle. We identify X with the zero section of E. Denote by

$$\mathfrak{a}\colon E\oplus E \longrightarrow E \qquad \text{and} \qquad \pi_{E\oplus E}\colon E\oplus E \longrightarrow X$$

the associated addition map and the induced projection map, respectively. For $f\in C^\infty(X;\mathbb{R})$, define

$$m_f\colon E \longrightarrow E \qquad \text{by} \qquad m_f(v) = f\big(\pi_E(v)\big)\cdot v \quad \forall\, v\in E. \tag{A.1}$$

In particular,

$$\pi_{E\oplus E} = \pi_E\circ\mathfrak{a}, \qquad \pi_E = \pi_E\circ m_f \quad \forall\, f\in C^\infty(X;\mathbb{R}).$$

The total spaces of the vector bundles

$$\pi_{E\oplus E}\colon E\oplus E \longrightarrow X \qquad \text{and} \qquad \pi_E^*E \longrightarrow E$$

consist of the pairs (v,w) in $E\times E$ such that $\pi_E(v)=\pi_E(w)$.

Define a smooth bundle homomorphism

$$\iota_E\colon \pi_E^*E \longrightarrow TE, \qquad \iota_E(v,w) = \frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0}. \tag{A.2}$$

Since the restriction of ι_E to the fiber over $v\in E$ is the composition of the isomorphism

$$E_{\pi_E(v)} \longrightarrow T_vE_{\pi_E(v)}, \qquad w \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0},$$

with the differential of the embedding of the fiber $E_{\pi_E(v)}$ into E, ι_E is an injective bundle homomorphism. Furthermore,

$$\begin{gathered}\mathrm{d}\pi_E\circ\iota_E = 0\colon \pi_E^*E \longrightarrow \pi_E^*TX, \quad m_f^*\iota_E\circ\pi_E^*m_f = \mathrm{d}m_f\circ\iota_E\colon \pi_E^*E \longrightarrow m_f^*TE,\\ \mathfrak{a}^*\iota_E\circ\pi_{E\oplus E}^*\mathfrak{a} = \mathrm{d}\mathfrak{a}\circ\iota_{E\oplus E}\colon \pi_{E\oplus E}^*(E\oplus E) \longrightarrow \mathfrak{a}^*TE\,,\end{gathered} \tag{A.3}$$

$$TE|_X = TX\oplus\iota_E\big(\pi_E^*E|_X\big) = TX\oplus\iota_E(E)\,. \tag{A.4}$$

Let

$$\zeta_E \in \Gamma(E;TE), \qquad \zeta_E(v) = \iota_E(v,v) \in T_vE, \tag{A.5}$$

be the canonical vertical vector field on E.

Exercise A.1. Suppose $p\in\mathbb{Z}^+$, $\pi_E\colon E\longrightarrow X$ is a real vector bundle, $\mathcal{U}\subset E$ is a tubular neighborhood of X in E, i.e. $tv\in\mathcal{U}$ whenever $v\in\mathcal{U}$ and $t\in[0,1]$, and ϖ is a closed p-form on $\mathcal{U}$. Show that the family $(m_t^*\varpi)_{t\in[0,1]}$ of p-forms on $\mathcal{U}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}m_t^*\varpi = m_t^*\big(\mathcal{L}_{t^{-1}\zeta_E}\varpi\big) = \mathrm{d}\big(m_t^*(\iota_{t^{-1}\zeta_E}\varpi)\big) \qquad \forall\, t\in[0,1],$$

where $\mathcal{L}$ is the Lie derivative.

By the first statement in (A.3), the injectivity of ι_E, and surjectivity of $\mathrm{d}\pi_E$,

$$0\longrightarrow \pi_E^*E \xrightarrow{\iota_E} TE \xrightarrow{\mathrm{d}\pi_E} \pi_E^*TX \longrightarrow 0 \tag{A.6}$$

is an exact sequence of real vector bundles over E. By the second statement in (A.3), the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_E^*E \arrow[r, "\iota_E"] \arrow[d, "\pi_E^*m_f"] & TE \arrow[r, "\mathrm{d}\pi_E"] \arrow[d, "\mathrm{d}m_f"] & \pi_E^*TX \arrow[r] \arrow[d, "\pi_E^*\mathrm{id}_{TX}"] & 0 \\ 0 \arrow[r] & \pi_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{d}\pi_E"] & \pi_E^*TX \arrow[r] & 0 \end{tikzcd} \tag{A.7}$$

of real vector bundle homomorphisms over E commutes. By the third statement in (A.3), the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_{E\oplus E}^*(E\oplus E) \arrow[r, "\iota_{E\oplus E}"] \arrow[d, "\pi_{E\oplus E}^*\mathfrak{a}"] & T(E\oplus E) \arrow[r, "\mathrm{d}\pi_{E\oplus E}"] \arrow[d, "\mathrm{d}\mathfrak{a}"] & \pi_{E\oplus E}^*TX \arrow[r] \arrow[d, "\pi_{E\oplus E}^*\mathrm{id}_{TX}"] & 0 \\ 0 \arrow[r] & \pi_{E\oplus E}^*E \arrow[r, "\mathfrak{a}^*\iota_E"] & \mathfrak{a}^*TE \arrow[r, "\mathfrak{a}^*\mathrm{d}\pi_E"] & \pi_{E\oplus E}^*TX \arrow[r] & 0 \end{tikzcd} \tag{A.8}$$

of real vector bundle homomorphisms over $E\oplus E$ commutes.

A connection in E is an $\mathbb{R}$-linear map

$$\begin{gathered} \nabla\colon \Gamma(X;E)\longrightarrow \Gamma(X;T^*X\otimes_{\mathbb{R}}E) \qquad \text{s.t.} \\ \nabla(fs) = \mathrm{d}f\otimes s + f\nabla s \quad \forall\ f\in C^\infty(X),\ s\in\Gamma(X;E). \end{gathered} \tag{A.9}$$

The Leibnitz property (A.9) implies that any two connections in E differ by a 1-form on X. In other words, if ∇ and ∇' are connections in E there exists

$$\begin{gathered} \theta\in\Gamma\big(X;T^*X\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{R}}(E,E)\big) \qquad \text{s.t.} \\ \nabla'_v s = \nabla_v s + \big\{\theta(v)\big\}\big(s(x)\big) \qquad \forall\ s\in\Gamma(X;E),\ v\in T_xX,\ x\in X. \end{gathered} \tag{A.10}$$

If U is a neighborhood of $x\in X$ and f is a smooth function on X supported in U such that $f(x)=1$, then

$$\nabla s\big|_x = \nabla\big(fs\big)\big|_x - \mathrm{d}_xf\otimes s(x) \tag{A.11}$$

by (A.9). The right-hand side of (A.11) depends only on $s|_U$. Thus, a connection ∇ in E is a local operator, i.e. the value of $\nabla\xi$ at a point $x\in X$ depends only on the restriction of s to any neighborhood U of x.

Exercise A.2. Suppose ∇, ∇' are connections in real vector bundles $E, E' \longrightarrow X$, respectively. Show that the map

$$\nabla\oplus\nabla' \colon \Gamma\big(X; E\oplus E'\big) \longrightarrow \Gamma\big(X; T^*X\otimes_{\mathbb{R}}(E\oplus E')\big), \quad \nabla\oplus\nabla'(s,s') = \big(\nabla s, \nabla' s'\big),$$

is a connection in the real vector bundle $E\oplus E' \longrightarrow X$.

Exercise A.3. Suppose ∇ is a connection in a real vector bundle $\pi_E \colon E \longrightarrow X$. Show that

(a) the linear map ∇ extends to a linear map on the E-valued p-forms by

$$\nabla \colon \Gamma\big(X; \Lambda^p(T^*X)\otimes_{\mathbb{R}} E\big) \longrightarrow \Gamma\big(X; \Lambda^{p+1}(T^*X)\otimes_{\mathbb{R}} E\big), \quad \nabla(\eta\otimes s) = (\mathrm{d}\eta)\otimes s + (-1)^p \eta\otimes(\nabla s)\,;$$

(b) there exists $\kappa_\nabla \in \Gamma(X; \Lambda^2(T^*X)\otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{R}}(E))$ so that

$$\nabla(\nabla\widetilde{\eta}) = \kappa_\nabla \wedge \widetilde{\eta} \quad \forall\, \widetilde{\eta}\in\Gamma\big(X; \Lambda^p(T^*X)\otimes_{\mathbb{R}} E\big),\ p\in\mathbb{Z}^{\geq 0}.$$

Note: the bundle section κ_∇ above is called the curvature of ∇.

Suppose U is an open subset of X and $s_1, \ldots, s_n \in \Gamma(U; E)$ is a frame for E on U, i.e.

$$s_1(x), \ldots, s_n(x) \in E_x$$

is a basis for E_x for all $x\in U$. By definition of ∇, there exist

$$\theta_l^k \in \Gamma(U; T^*U) \qquad \text{s.t.} \qquad \nabla s_\ell = \sum_{k=1}^{k=n} s_k \theta_\ell^k \equiv \sum_{k=1}^{k=n} \theta_\ell^k \otimes s_k \quad \forall\ \ell = 1, \ldots, n.$$

We call

$$\theta \equiv \big(\theta_\ell^k\big)_{k,\ell=1,\ldots,n} \in \Gamma\big(U; T^*U\otimes_{\mathbb{R}} \mathrm{Mat}_n\mathbb{R}\big)$$

the connection 1-form of ∇ with respect to the frame $(s_k)_k$.

For an arbitrary section

$$s \equiv \sum_{\ell=1}^{\ell=n} f^\ell s_\ell \in \Gamma(U; E),$$

by (A.9) we have

$$\nabla s = \sum_{k=1}^{k=n} s_k \Big(\mathrm{d}f^k + \sum_{\ell=1}^{\ell=n} \theta_\ell^k f^\ell\Big), \qquad \text{i.e.} \qquad \nabla\big(\underline{s}\cdot \underline{f}^t\big) = \underline{s}\cdot \big\{\mathrm{d}+\theta\big\}\underline{f}^t, \tag{A.12}$$

$$\text{where} \qquad \underline{s} = (s_1, \ldots, s_n), \quad \underline{f} = (f^1, \ldots, f^n). \tag{A.13}$$

This implies that

$$\nabla s\big|_x = \pi_2|_x \circ \mathrm{d}_x s \colon T_xX \longrightarrow E_x \qquad \forall\ x\in X,\ s\in\Gamma(X;E) \text{ s.t. } s(x)=0, \tag{A.14}$$

where $\pi_2|_x \colon T_xE \longrightarrow E_x$ is the projection to the second component in (A.4).

Lemma A.4. *Suppose X is a smooth manifold and $\pi_E: E \longrightarrow X$ is a real vector bundle. A connection ∇ in E induces a splitting*

$$TE \approx \pi_E^* TX \oplus \pi_E^* E \tag{A.15}$$

of the exact sequence (A.6) extending the splitting (A.4) such that

$$\nabla s\big|_x = \pi_\nabla \circ \mathrm{d}_x s \colon T_x X \longrightarrow E_x \qquad \forall\, s \in \Gamma(X;E),\ x \in X, \tag{A.16}$$

where $\pi_\nabla: TE \longrightarrow \pi_E^* E$ is the projection onto the second component in (A.15). Furthermore,

$$\mathrm{d}m_t \approx \pi_E^*\mathrm{id} \oplus \pi_E^* m_t \ \ \forall\, t \in \mathbb{R} \qquad \textit{and} \qquad \mathfrak{a} \approx \pi_{E\oplus E}^*\mathrm{id} \oplus \pi_{E\oplus E}^*\mathfrak{a}, \tag{A.17}$$

with respect to the splitting (A.15) and the corresponding splitting for the connection $\nabla\oplus\nabla$ in the real vector bundle $E\oplus E \longrightarrow X$, i.e. these splittings are consistent with the commutative diagrams (A.7) and (A.8).

Proof. Given $x \in X$ and $v \in E_x$, choose $s \in \Gamma(X;E)$ such that $s(x)=v$ and let

$$T_v E^{\mathrm{h}} = \big\{\mathrm{d}_x s(w) - \iota_E\big(\nabla_w s\big) \colon w \in T_x X\big\} \subset T_v E.$$

Since $\pi_E \circ s = \mathrm{id}_X$ and $\mathrm{d}\pi_E \circ \iota_E = 0$,

$$\mathrm{d}_v \pi_E \circ \big\{\mathrm{d}s - \iota_E \circ \nabla s\big\}\big|_x = \mathrm{id}_{T_x X} \qquad \Longrightarrow \qquad T_v E \approx T_v E^{\mathrm{h}} \oplus E_x \approx T_x X \oplus E_x.$$

This splitting of $T_v E$ satisfies (A.16) at $v = s(x)$.

With the notation as in (A.12),

$$\big\{\mathrm{d}s - \iota_E \circ \nabla s\big\}\big|_x = \Big(\mathrm{d}_x \mathrm{id}_X, -\sum_{\ell=1}^{\ell=n} f^\ell(x)\theta_\ell^1|_x, \ldots -\sum_{\ell=1}^{\ell=n} f^\ell(x)\theta_\ell^n|_x\Big) \colon T_x X \longrightarrow T_x X \oplus \mathbb{R}^n \tag{A.18}$$

with respect to the identification $E|_U \approx U \times \mathbb{R}^n$ determined by the frame $(s_k)_k$. Thus, $T_v E^{\mathrm{h}}$ is independent of the choice of s. Since $T_x E^{\mathrm{h}} = T_x X$ for every $x \in X$, the resulting splitting (A.15) of (A.6) extends (A.4). By (A.18), it also satisfies (A.17). $\square$

Exercise A.5. Suppose $p \in \mathbb{Z}^+$, $\pi_E: E \longrightarrow X$ is a real vector bundle, Ω is a fiberwise p-form on E, and ∇ is a connection in E with the associated projection $\pi_\nabla: TE \longrightarrow \pi_E^* E$ as in Lemma A.4. Thus, $\Omega_\nabla \equiv \pi_\nabla^* \Omega$ is a p-form on the total space of E. Let $\zeta_E \in \Gamma(E;TE)$ be the canonical vertical vector field on E as in (A.5). Show that

$$\iota_E^*\big(\mathrm{d}(\iota_{\zeta_E}\Omega_\nabla)\big) = p\Omega \qquad \text{and} \qquad \big(\mathrm{d}(\iota_{\zeta_E}\Omega_\nabla)\big)\big|_{T_x X} = 0 \ \ \forall\, x \in X. \tag{A.19}$$

Suppose g is a metric on a real vector bundle $E \longrightarrow X$, i.e.

$$g \in \Gamma\big(X; E^* \otimes_{\mathbb{R}} E^*\big) \qquad \text{s.t.} \qquad g(v,w) = g(w,v), \ \ g(v,v) > 0 \quad \forall\, v,w \in E_x,\ v \neq 0,\ x \in X.$$

A connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(s,s')\big) = g(\nabla s, s') + g(s, \nabla s') \in \Gamma(X; T^*X) \qquad \forall\, s, s' \in \Gamma(X;E).$$

Suppose U is an open subset of X and $s_1,\ldots,s_n\in\Gamma(U;E)$ is a frame for E on U. For $i,j=1,\ldots,n$, let

$$g_{ij}=g(s_i,s_j)\in C^\infty(U).$$

If ∇ is a connection in E and $\theta_{k\ell}$ is the connection 1-form for ∇ with respect to the frame $\{s_k\}_k$, then ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n}\big(g_{ik}\theta_j^k+g_{jk}\theta_i^k\big)=\mathrm{d}g_{ij}\qquad\forall\, i,j=1,2,\ldots,n. \tag{A.20}$$

A.2 Complex vector bundles

Suppose X is a smooth manifold and $\pi_E: E\longrightarrow X$ is a complex vector bundle. Similarly to Section A.1, there is an exact sequence

$$0\longrightarrow\pi_E^*E\xrightarrow{\iota_E}TE\xrightarrow{\mathrm{d}\pi_E}\pi_E^*TX\longrightarrow 0 \tag{A.21}$$

of complex vector bundles over E. The homomorphism ι_E is now $\mathbb{C}$-linear. If $f\in C^\infty(X;\mathbb{C})$ and $m_f\colon E\longrightarrow E$ is defined as in (A.1), there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_E^*E \arrow[r, "\iota_E"] \arrow[d, "\pi_E^*m_f"] & TE \arrow[r, "\mathrm{d}\pi_E"] \arrow[d, "\mathrm{d}m_f"] & \pi_E^*TX \arrow[r] \arrow[d, "\pi_E^*\mathrm{id}_{TX}"] & 0 \\
0 \arrow[r] & \pi_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{d}\pi_E"] & \pi_E^*TX \arrow[r] & 0
\end{tikzcd} \tag{A.22}$$

of complex vector bundle maps over E.

Suppose ∇ is a ($\mathbb{C}$-linear) connection in the complex vector bundle $\pi_E\colon E\longrightarrow X$, i.e.

$$\nabla_v(\mathfrak{i}s)=\mathfrak{i}(\nabla_v s)\qquad\forall\, s\in\Gamma(X;E),\ v\in TX.$$

If U is an open subset of X and $s_1,\ldots,s_n\in\Gamma(U;E)$ is a $\mathbb{C}$-frame for E on U, then there exist

$$\theta_\ell^k\in\Gamma\big(U;T^*U\times_{\mathbb{R}}\mathbb{C}\big)\qquad\text{s.t.}\qquad\nabla\xi_\ell=\sum_{k=1}^{k=n}s_k\theta_\ell^k\equiv\sum_{k=1}^{k=n}\theta_\ell^k\otimes s_k\quad\forall\;\ell=1,\ldots,n.$$

For an arbitrary section

$$s=\sum_{\ell=1}^{\ell=n}f^\ell s_\ell\in\Gamma(U;E),$$

by (A.9) and $\mathbb{C}$-linearity of ∇ we have

$$\nabla\xi=\sum_{k=1}^{k=n}s_k\Big(\mathrm{d}f^k+\sum_{\ell=1}^{\ell=n}\theta_\ell^k f^\ell\Big),\qquad\text{i.e.}\qquad\nabla\big(\underline{s}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\big\{\mathrm{d}+\theta\big\}\underline{f}^t, \tag{A.23}$$

where $\underline{s}$ and $\underline{f}$ are as (A.13).

Let g be a Hermitian metric on E, i.e.

$$g \in \Gamma\big(X; \mathrm{Hom}_{\mathbb{C}}(E\otimes_{\mathbb{C}}\overline{E}, \mathbb{C})\big) \quad \text{s.t.} \quad g(v,w) = \overline{g(w,v)}, \quad g(v,v) > 0 \quad \forall\, v,w \in E_x,\ v\neq 0,\ x\in X.$$

A ($\mathbb{C}$-linear) connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(s,s')\big) = g(\nabla s, s') + g(s, \nabla s') \in \Gamma(X; T^*X\otimes_{\mathbb{R}}\mathbb{C}) \qquad \forall\, s,s' \in \Gamma(X;E).$$

With the notation as in the previous paragraph, let

$$g_{ij} = g(s_i, s_j) \in C^{\infty}(U;\mathbb{C}) \qquad \forall\, i,j=1,\ldots,n.$$

Then ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n} \big(g_{ik}\theta_j^k + \bar{g}_{jk}\bar{\theta}_i^k\big) = \mathrm{d}g_{ij} \qquad \forall\, i,j = 1,2,\ldots,n. \tag{A.24}$$

Exercise A.6. Suppose ∇ is a connection in a complex vector bundle $\pi_E : E \longrightarrow X$. Let $\kappa_{\nabla}\in\Gamma(X;\Lambda^2(T^*X)\otimes_{\mathbb{R}}\mathrm{End}_{\mathbb{R}}(E))$ be the curvature of ∇ as in Exercise A.3 and $TE^{\mathrm{h}}\subset TE$ be the complement of $\iota_E(\pi_E^*E)\subset TE$ determined by ∇ as in the proof of Lemma A.4.

(a) Show that the splitting (A.15) satisfies the first property in (A.17) for all $t \in \mathbb{C}$ and that $\kappa_{\nabla}\in\Gamma(X;\Lambda^2(T^*X)\otimes_{\mathbb{R}}\mathrm{End}_{\mathbb{C}}(E))$;

(b) Suppose ∇ is compatible with a Hermitian metric g on E and

$$v \in S_{E,g} \equiv \big\{w\in E\colon g(w,w)=1\big\}.$$

Show that $T_vE^{\mathrm{h}}\subset T_vS_{E,g}$.

(c) Suppose in addition that $\mathrm{rk}_{\mathbb{C}}E=1$. Show that κ_{∇} is a 1-form on X with values in $\mathfrak{i}\mathbb{R}$.

A.3 Principal S^1-bundles

Suppose X is a smooth manifold and $\pi_S\colon S\longrightarrow X$ is a (smooth) principal S^1-bundle. Let

$$\zeta_S \in \Gamma(S;TS), \qquad \zeta_S(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{2\pi\mathrm{i}t}\cdot v\big)\Big|_{t=0}, \tag{A.25}$$

be the vector field generating the S^1-action. This vector field generates the vertical tangent bundle of π_S, i.e.

$$TS^{\mathrm{ver}} \equiv \ker \mathrm{d}\pi_S = \big\{t\zeta_S(v)\colon v\in S,\ t\in\mathbb{R}\big\} \longrightarrow S.$$

A connection 1-form on S is an S^1-invariant 1-form λ on (the total space of) S such that $\lambda(\zeta_S)=2\pi$. Such a form determines an S^1-equivariant splitting of the exact sequence

$$0 \longrightarrow TS^{\mathrm{ver}} \longrightarrow TS \xrightarrow{\mathrm{d}\pi_S} \pi_S^*TX \longrightarrow 0$$

of real vector bundles over S with

$$TS = TS^{\mathrm{ver}}\oplus(\ker\lambda).$$

Exercise A.7. Suppose $\pi_S\colon S\longrightarrow X$ is a principal S^1-bundle.

(a) Show that the S^1-invariance condition on 1-form λ being a connection 1-form can be equivalently replaced by the condition $\iota_{\zeta_S}\mathrm{d}\lambda=0$;

(b) Let λ be a connection 1-form on S. Show that there exists a 2-form κ_λ on X so that $\mathrm{d}\lambda=\pi_S^*\kappa_\lambda$.

Note: the 2-form κ_λ above is called the curvature of λ.

A principal S^1-bundle $\pi_S\colon S\longrightarrow X$ determines a complex line bundle

$$\pi_{L_S}\colon L_S\equiv\big(S\times_{S^1}\mathbb{C}\big)/\!\sim\longrightarrow X,$$
$$(v,z)\sim\big(u\cdot v,u^{-1}\cdot z\big)\ \forall\ (v,z)\in S\times\mathbb{C},\ u\in S^1,\qquad \pi_{L_S}\big([v,z]\big)=\pi_S(v),$$

with a Hermitian metric specified by

$$g_S\big([v,z],[v,z']\big)=z\overline{z'}\,.$$

Conversely, a complex line bundle $\pi_L\colon L\longrightarrow X$ with a Hermitian metric g determines a principal S^1-bundle, the unit circle bundle of L,

$$\pi_{S_{L,g}}\colon S_{L,g}\equiv\big\{v\in L\colon g(v,v)=1\big\}\longrightarrow X,\quad \pi_{S_{L,g}}(v)=\pi_L(v).$$

With S and (L,g) as above, the maps

$$S\longrightarrow S_{L_S,g_S},\quad v\longrightarrow[v,1],\qquad\text{and}\qquad L_{S_{L_S,g_S}}\longrightarrow L,\quad [v,z]\longrightarrow zv,\tag{A.26}$$

are isomorphism of principal S^1-bundles over X and of complex line bundles with Hermitian metrics over X . Thus, we have constructed a bijective correspondence between the isomorphism classes of principal S^1-bundles over X and the isomorphism classes of complex line bundles with Hermitian metrics over X.

Exercise A.8. Suppose λ is a connection 1-form on a principal S^1-bundle $\pi_S:S\longrightarrow X$ and $p\colon S\times\mathbb{C}\longrightarrow L_S$ is the quotient projection. Show that

(a) there is a unique 1-form λ_S on L_S so that

$$p^*\lambda_S\big|_{(v,z)}=|z|^2\lambda_v+\frac{\mathfrak{i}}{2}\big(z\mathrm{d}\overline{z}-\overline{z}\mathrm{d}z\big)\qquad\forall\ (v,z)\in L\times\mathbb{C};$$

(b) $\iota_{L_S}^*(\mathrm{d}\lambda_S)=2\,\mathrm{Re}\,g_S(\mathfrak{i}\cdot,\cdot)$ and $\iota_v(\mathrm{d}\lambda_S)=0$ for all $v\in T_xX\subset T_xL_S$, $x\in X$;

(c) there is a unique ($\mathbb{C}$-linear) connection ∇^λ in the complex line bundle $L_S\longrightarrow X$ compatible with the Hermitian metric g_S so that the 1-form λ_S vanishes on the complement $TL^{\mathrm{h}}\subset TL$ of $\iota_L(\pi_L^*L)\subset TL$ determined by ∇^λ as in the proof of Lemma A.4.

Thus, a connection 1-form λ in a principal S^1-bundle $\pi_S:S\longrightarrow X$ determines a connection ∇^λ in the associated complex line bundle $\pi_{L_S}\colon L_S\longrightarrow X$ compatible with the Hermitian metric g_S on L_S. Suppose $\pi_L\colon L\longrightarrow X$ is a complex line bundle with a Hermitian metric g and ∇ is a ($\mathbb{C}$-linear) connection in L compatible with g. Let $\zeta_L\in\Gamma(L;TL)$ be the canonical vertical vector field as in (A.5) and $TL^{\mathrm{h}} \subset TL$ be the complement of $\iota_L(\pi_L^* L) \subset TL$ determined by ∇ as in the proof of Lemma A.4. In particular, $2\pi\mathfrak{i}\zeta_L \in \Gamma(L; TL)$ is the vector field generating the S^1-action on L by scalar multiplication. By Exercise A.6, the S^1-action on L preserves the subbundle $TL^{\mathrm{h}}|_{S_{L,g}}$ of $TS_{L,g}$. Thus, the 1-form λ_∇ on $S_{L,g}$ defined by

$$\lambda_\nabla\big(2\pi\mathfrak{i}\zeta_L(v)\big) = 2\pi, \quad \lambda_\nabla\big|_{T_v L^{\mathrm{h}}} = 0 \qquad \forall\, v \in S_{L,g}$$

is a connection 1-form on the principal S^1-bundle $S_{L,g} \longrightarrow X$. By Exercise A.9 below, we have constructed a bijective correspondence between the isomorphism classes of principal S^1-bundles over X with connection 1-forms and the isomorphism classes of complex line bundles with Hermitian metrics over X and compatible connections.

Exercise A.9. Suppose λ is a connection 1-form on a principal S^1-bundle $\pi_S\colon S \longrightarrow X$ and ∇ is a connection in a complex line bundle $\pi_L\colon L \longrightarrow X$ compatible with a Hermitian metric g. Show that

$$\lambda = \lambda_{\nabla^\lambda} \qquad \text{and} \qquad \nabla^{\lambda_\nabla} = \nabla$$

under the isomorphisms (A.26) and that $\kappa_{\nabla^\lambda} = \mathfrak{i}\kappa_\lambda$.

Remark A.10. By a Čech cohomology computation [13, p141] and Exercise A.9,

$$c_1(L) = \frac{\mathfrak{i}}{2\pi}\big[\kappa_\nabla\big] = -\frac{1}{2\pi}\big[\kappa_\lambda\big] \in H^2_{\mathrm{deR}}(X),$$

if ∇ is a connection in a complex line bundle $L \longrightarrow X$ and λ is a connection 1-form in an associated principal S^1-bundle.

References

[1] M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. LMS 14 (1982), no. 1, 1—15

[2] M. Audin, *Torus Actions on Symplectic Manifolds*, Progr. Math. 93, 2nd revised Ed., Birkhäuser, 2004

[3] D. Austin and P. Braam, *Morse-Bott theory and equivariant cohomology*, Progr. Math. 133, 123—183, Birkhäuser Verlag, 1995

[4] P. Birtea, J-P. Ortega, and T. Ratiu, *A local-to-global principle for convexity in metric spaces*, J. Lie Theory 18 (2008), no. 2, 445—469

[5] P. Birtea, J-P. Ortega, and T. Ratiu, *Openness and convexity for momentum maps*, Trans. AMS 361 (2009), no. 2, 603—630

[6] C. Bjorndahl and Y. Karshon, *Revisiting Tietze-Nakajima: local and global convexity for maps*, Canad. J. Math. 62 (2010), no. 5, 975—993

[7] R. Bott, *Nondegenerate critical manifolds*, Ann. Math. 60 (1954), no. 2, 248-–261

[8] A. Canas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Math. 1764, Springer-Verlag, 2001 (revised 2006)

[9] M. Conforti, G. Cornuéjols, and G. Zambelli, *Integer Programming*, GTM 271, Springer, 2014

[10] M. Condevaux, P. Dazord, and P. Molino, *Geometrie du moment*, Sem. Sud-Rhodanien, 1988

[11] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France 116 (1988), no. 3, 315–339

[12] I. Gelbukh, *On the topology of the Reeb graph*, Publ. Math. Debrecen 104 (2024), no. 3-4, 343—365

[13] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Willey & Sons, 1994

[14] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), no. 2, 307—347

[15] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. 67 (1982), no. 3, 491–513

[16] J. Hilgert, K. Neeb, and W. Plank, *Symplectic convexity theorems*, Sem. Sophus Lie 3 (1993), no. 2, 123-–135

[17] J. Hilgert, K. Neeb, and W. Plank, *Symplectic convexity theorems and coadjoint orbits*, Compositio Math. 94 (1994), no. 2, 129-–180

[18] E. Lerman, *Symplectic cuts*, MRL 22 (1995), no. 3, 247—258

[19] E. Lerman, E. Meinrenken, S. Tolman, and C. Woodward, *Nonabelian convexity by symplectic cuts*, Topology 37 (1998), no. 2, 245-–259

[20] D. McDuff, *Examples of symplectic structures*, Invent. Math. 89 (1987), no. 1, 13-–36

[21] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 3rd Ed., Oxford University Press, 2017

[22] E. Meinrenken, *Symplectic geometry*, math.toronto.edu/mein/teaching/LectureNotes/sympl.pdf

[23] K. Meyer, *Symmetries and integrals in mechanics*, in *Dynamical Systems* (*Proc. Sympos., Univ. Bahia, Salvador, 1971*), 259—272, Academic Press, 1973

[24] M. Morse, *The Calculus of Variations in the Large*, AMS Colloquium Publ. 18, 1934

[25] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, 1984

[26] S. Nakajima, *Über konvexe Kurven and Flächen*, Tôhoku Math. J. 29 (1928), 227-–230

[27] A. Pires, *Convexity in symplectic geometry: the Atiyah-Guillemin-Sternberg Theorem*, https://faculty.fordham.edu/apissarrapires/convexitytalk.pdf

[28] A. Popa, *Two-Point Gromov-Witten Formulas for Symplectic Toric Manifolds*, PhD thesis, Stony Brook University, 2012

[29] H. Tietze, *Über Konvexheit im kleinen und im großen und über gewisse den Punkter einer Menge zugeordete Dimensionszahlen*, Math. Z. 28 (1928), 697—707

[30] F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, GTM 94, Springer-Verlag, 1983

[31] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982), no. 4, 661—692

[32] A. Zinger, *Notes on J-Holomorphic Maps*, 1706.00331v2