Seminar on Symplectic Toric Manifolds

– sketch under revision–

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Ana Cannas[1]

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[1] E-mail: acannas@math.ethz.ch

Contents

Foreword

These notes aimed to serve as a rough guideline for the student seminar on *symplectic toric manifolds* – abbreviated *STMs* – at ETH Zurich in the Spring of 2019. This seminar comprised twelve lectures and was an introduction to STMs, i.e., smooth toric varieties from the symplectic viewpoint. It started from basic notions in symplectic geometry, went over the classification of STMs and closed with some advanced topics.

Geometry of manifolds was the basic prerequisite for this seminar, hence also for these notes. Some familiarity with symplectic geometry is useful to get through faster, though most of the needed definitions and results are stated here.

The study of toric manifolds has many different entrances and has been scoring a wide spectrum of applications. For symplectic geometers, they provide examples of extremely symmetric and completely integrable hamiltonian spaces. In order to distinguish the algebraic from the symplectic approach, we say *symplectic toric manifolds* when focusing on the symplectic and smooth properties.

Native to algebraic geometry, the theory of toric varieties has been around for about thirty years. It was introduced by Demazure in [20] who used toric varieties for classifying some algebraic subgroups. Since 1970 many nice surveys of the theory of toric varieties have appeared (see, for instance, [18, 25, 35, 52]). For the last thirty years, toric geometry became an important tool in physics in connection with mirror symmetry [17] where research has been intensive.

In this text we emphasize the geometry of the *moment map* whose image, the so-called *moment polytope*, determines the STM by the celebrated classification theorem of Delzant [19]. The notion of a moment map associated to a group action generalizes that of a hamiltonian function associated to a vector field. Either of these notions formalizes the Noether principle, which states that to every symmetry (such as a group action) in a mechanical system, there corresponds a conserved quantity. The concept of a moment map was introduced by Souriau [56] under the French name *application moment* (besides the more widespread English translation to *moment map*, the alternative *momentum map* is also used). Moment maps have been asserting themselves as a main tool to study problems in geometry and topology when there is a suitable symmetry, as illustrated in the book by Gelfand, Kapranov and Zelevinsky [26].

For their contributions, comments, corrections and interesting questions – some of which have already been incorporated in these notes – I am thankful to the participants of the seminar, namely: Giovanni Ambrosioni, Yannis Bähni, Joël Beimler, Valentin Bosshard, Gilles Englebert, Alessandro Fasse, Simon Grüning, Amanda Jenny, Shengxuan Liu, Yefei Ma, Angela Maennel, Benjamin Pollitt, Marcella Storino, and Johannes Weidenfeller.

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Chapter 1

Symplectic Preliminaries

We begin by introducing the basic objects in symplectic/hamiltonian geometry which lead to symplectic toric manifolds.

1.1 Symplectic Manifolds

Definition 1.1.1. *A **symplectic form**[1] on a manifold M is a closed 2-form on M which is nondegenerate at every point of M. A **symplectic manifold** is a pair (M, ω) where M is a manifold and ω is a symplectic form on M.*

A 2-form ω gives at each point $p \in M$ a skew-symmetric bilinear pairing of tangent vectors at that point,

$$\omega_p : T_pM \times T_pM \to \mathbb{R} .$$

Nondegeneracy means that, for any nonzero tangent vector $u \in T_pM$, there is $v \in T_pM$ such that $\omega_p(u, v) \neq 0$. By a skew-symmetric version of the Gram-Schmidt process (see Theorem 1.2.2), we can then conclude that T_pM must be even-dimensional. It follows that a symplectic manifold is necessarily *even-dimensional*. When the manifold M has dimension $2n$, the nondegeneracy of a 2-form ω amounts to the top wedge power, ω^n, being nonzero, i.e., a volume form. It follows that a symplectic manifold is *oriented* by its symplectic form, a volume form being ω^n. For more details on these assertions, see for instance [15, Chapter 1].

[1] If you consult a major English dictionary, you are likely to find that *symplectic* is the name for a bone in a fish's head. However, as clarified in [58], the word *symplectic* in mathematics was coined by Weyl [59, p.165] who substituted the Latin root in *complex* by the corresponding Greek root, in order to label the symplectic group. (In linguistics, a word created this way is called a *calque*.) Weyl thus avoided that this group connote the complex numbers, and also spared us from much confusion that would have arisen, had the name remained the former one in honor of Abel: *abelian linear group*.

Examples.

1. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The **standard symplectic form on** $\mathbb{R}^{2n}$ is

$$\omega_0 = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

2. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, \ldots, z_n$. The form

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is a symplectic form on $\mathbb{C}^n$. In fact, this form equals that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + iy_k$.

3. Let X be any n-dimensional manifold and $M = T^*X$ its cotangent bundle. If the manifold structure on X is described by coordinate charts $(\mathcal{U}, x_1, \ldots, x_n)$ with $x_k : \mathcal{U} \to \mathbb{R}$, then at any $x \in \mathcal{U}$, the differentials $(dx_1)_x, \ldots (dx_n)_x$ form a basis of T_x^*X. Namely, if $\xi \in T_x^*X$, then $\xi = \sum_{k=1}^{n} \xi_k (dx_k)_x$ for some real coefficients $\xi_1, \ldots, \xi_n$. This induces a map

$$\begin{array}{ccc} T^*\mathcal{U} & \longrightarrow & \mathbb{R}^{2n} \\ (x, \xi) & \longmapsto & (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) \end{array}$$

and $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ is a coordinate chart for T^*X; the coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ are the **cotangent coordinates** associated to the coordinates $x_1, \ldots, x_n$ on $\mathcal{U}$. The **canonical symplectic form** on T^*X is the 2-form given on the coordinate chart $T^*\mathcal{U}$ by

$$\omega = \sum_{k=1}^{n} dx_k \wedge d\xi_k \ .$$

One can check that this form is well-defined and can be intrinsically defined in terms of the so-called *tautological 1-form*. This case will probably be discussed further in Chapter 3.

4. Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to S^2 at p may then be identified with vectors orthogonal to p. The **euclidean symplectic form on** S^2 is the form induced by the inner and exterior products:

$$\omega_p(u, v) := \langle p, u \times v \rangle \ , \qquad \text{for } u, v \in T_pS^2 = \{p\}^\perp \ .$$

This form is closed because it is of top degree; it is nondegenerate because $\langle p, u \times v \rangle \neq 0$ when $u \neq 0$ and we take, for instance, $v = u \times p$.

$\diamondsuit$

Exercise 1.1.2. *Check that, in cylindrical coordinates away from the poles ($0 \leqslant \theta < 2\pi$ and $-1 < h < 1$), the euclidean symplectic form on S^2 is the area form given by*

$$\omega_{\text{eucl}} = d\theta \wedge dh \ .$$

This confirms that the total area is 4π.

The natural notion of equivalence in the symplectic category is expressed by a *symplectomorphism*:

Definition 1.1.3. *Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a* ***symplectomorphism*** *if $\varphi^* \omega_2 = \omega_1$. The set of all symplectomorphisms from a symplectic manifold (M, ω) to itself equipped with composition is called the* ***group of symplectomorphisms*** *of (M, ω) and denoted* $\text{Sympl}(M, \omega)$.

We would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem (Section 1.2) takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional *symplectic* manifold looks locally like $(\mathbb{R}^{2n}, \omega_0)$. More precisely, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. In other words, the prototype of a local piece of a $2n$-dimensional symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$.

A key feature of symplectic forms is that they provide the mechanism to associate to any smooth real function on the underlying manifold $H : M \to \mathbb{R}$ a nontrivial (eventually local) flow that preserves both the symplectic form and the given function. This is the *hamiltonian flow* associated to a *(hamiltonian) function*.

Let (M, ω) be a symplectic manifold.

Definition 1.1.4. *A vector field X on M is* ***symplectic*** *if the contraction $\imath_X \omega$ is closed. A vector field X on M is* ***hamiltonian*** *if the contraction $\imath_X \omega$ is exact.*

By Poincaré's Lemma, locally on every contractible open set, every symplectic vector field is hamiltonian. If the first de Rham cohomology group is trivial, then globally every symplectic vector field is hamiltonian; in general, $H^1_{\text{deRham}}(M)$ measures the obstruction for symplectic vector fields to be hamiltonian.

The flow of a symplectic vector field X preserves the symplectic form:

$$\mathcal{L}_X \omega = d \underbrace{\imath_X \omega}_{\text{closed}} + \imath_X \underbrace{d\omega}_{0} = 0 \ .$$

If a vector field X is hamiltonian with[2] $\imath_X \omega = -dH$ for some smooth function $H : M \to \mathbb{R}$, then the flow of X also preserves the function H:

$$\mathcal{L}_X H = \imath_X dH = -\imath_X \imath_X \omega = 0 \ .$$

[2]The sign here is included just to be consistent with Definition 1.1.5.

Therefore, each integral curve $\{\rho_t(x) \mid t \in \mathbb{R}\}$ of X must be contained in a level set of H:

$$H(x) = (\rho_t^* H)(x) = H(\rho_t(x)) \ , \quad \forall t \ .$$

Definition 1.1.5. *A* ***hamiltonian function*** *for a hamiltonian vector field X on M is a smooth function $H : M \to \mathbb{R}$ such that $\imath_X \omega = -dH$.*

Note the above sign convention, chosen to produce more *positive* pictures later on in this text.

By nondegeneracy of ω, any function $H \in C^\infty(M)$ is a hamiltonian function for some hamiltonian vector field because the equation $\imath_X \omega = -dH$ can be always solved for a smooth vector field X. A hamiltonian vector field X defines a hamiltonian function *up to* a locally constant function.

Examples.

1. On the symplectic manifold $(\mathbb{C}^n, \omega_0)$, we translate from linear coordinates $z_1, \ldots, z_n$ to polar coordinates r_k, θ_k on each factor-plane, so that

   $$\omega_0 = \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k = \sum_{k=1}^n r_k dr_k \wedge d\theta_k \ .$$

   Then it is easier to see that the vector field $X = \sum_{k=1}^n \frac{\partial}{\partial \theta_k}$ corresponding to a diagonal rotation is hamiltonian with hamiltonian function given by half of the square of the radius, $H := \frac{1}{2}(|z_1|^2 + \ldots + |z_n|^2)$:

   $$\imath_X \omega_0 = -d \underbrace{\left(\tfrac{1}{2}(|z_1|^2 + \ldots + |z_n|^2)\right)}_{H} \ .$$

   Indeed the diagonal rotation preserves the spheres centered at the origin, as well as the area on each factor-plane.

2. On the euclidean symplectic 2-sphere $(S^2, d\theta \wedge dh)$, the vector field $X = \frac{\partial}{\partial \theta}$ is hamiltonian with hamiltonian function $H = -h$ given by the negative of the height function:

   $$\imath_X (d\theta \wedge dh) = dh = -d(-h) \ .$$

   The motion generated by this vector field is rotation about the vertical axis, which of course preserves both area and height.

3. On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields $X_1 = \frac{\partial}{\partial \theta_1}$ and $X_2 = \frac{\partial}{\partial \theta_2}$ are symplectic but not hamiltonian.

$\diamondsuit$

1.2 Darboux's Theorem

Let (M, ω) be a symplectic manifold of dimension $2n$.

Definition 1.2.1. *A* ***Darboux chart*** *for M is a chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that*

$$\omega|_{\mathcal{U}} = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

By the Darboux theorem (Theorem 1.2.4), there exists a Darboux chart centered at each point of a symplectic manifold. The modern proof of the Darboux theorem was first noted by Moser and can be broken into the following two key facts, one from linear algebra and the other based on *Moser's argument*.

Theorem 1.2.2. ***(Standard Form for Skew-symmetric Bilinear Maps)*** *Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. Assume that the map Ω is* skew-symmetric, *i.e., $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.*

Then there is a basis $u_1, \ldots, u_\ell, e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that

$$\begin{array}{ll} \Omega(u_j, v) = 0 \ , & \text{for all } i \text{ and all } v \in V, \\ \Omega(e_j, e_k) = 0 = \Omega(f_j, f_k) \ , & \text{for all } i, j, \text{ and} \\ \Omega(e_j, f_k) = \delta_{ij} \ , & \text{for all } i, j. \end{array}$$

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.

Let $U := \{u \in V \mid \Omega(u, v) = 0 \text{ for all } v \in V\}$. Choose a basis $u_1, \ldots, u_k$ of U, and choose a complementary space W to U in V,

$$V = U \oplus W \ .$$

Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Assume that $\Omega(e_1, f_1) = 1$. Let

$$\begin{array}{rcl} W_1 & = & \text{span of } e_1, f_1 \\ W_1^\Omega & = & \{w \in W \mid \Omega(w, v) = 0 \text{ for all } v \in W_1\} \ . \end{array}$$

Claim. $W_1 \cap W_1^\Omega = \{0\}$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\left.\begin{array}{l} 0 = \Omega(v, e_1) = -b \\ 0 = \Omega(v, f_1) = \quad a \end{array}\right\} \quad \Longrightarrow \quad v = 0 \ .$$

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega} \ .$$

Go on: let $e_2 \in W_1^\Omega$, $e_2 \neq 0$. There is $f_2 \in W_1^\Omega$ such that $\Omega(e_2, f_2) \neq 0$. Assume that $\Omega(e_2, f_2) = 1$. Let $W_2 =$ span of e_2, f_2. Etc.

This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

where all summands are orthogonal with respect to Ω, and where W_j has basis e_j, f_j with $\Omega(e_j, f_j) = 1$. $\square$

The dimension of the subspace $U = \{u \in V \mid \Omega(u, v) = 0, \text{ for all } v \in V\}$ does not depend on the choice of basis. That is thus an invariant of (V, Ω), $k := \dim U$. Since $k + 2n = m = \dim V$, the number n is also an invariant of (V, Ω) and this is called the **rank** of Ω.

Theorem 1.2.3. ***(Moser Theorem – Relative Version)*** *Let M be a manifold, X a compact submanifold of M, $i : X \hookrightarrow M$ the inclusion map, ω_0 and ω_1 symplectic forms in M. Suppose that $\omega_0|_p = \omega_1|_p$, $\forall p \in X$.*

Then there exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes and $\varphi^*\omega_1 = \omega_0$.

Theorem 1.2.4. ***(Darboux's Theorem)*** *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M.*
Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on $\mathcal{U}$

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

Proof. Apply the Moser relative theorem (Theorem 1.2.3) to $X = \{p\}$:

Use any symplectic basis for T_pM to construct coordinates $(x'_1, \ldots, x'_n, y'_1, \ldots y'_n)$ centered at p and valid on some neighborhood $\mathcal{U}'$, so that

$$\omega_p = \sum dx'_j \wedge dy'_j \Big|_p \ .$$

There are two symplectic forms on $\mathcal{U}'$: the given $\omega_0 = \omega$ and $\omega_1 = \sum dx'_j \wedge dy'_j$. By the Moser theorem, there are neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\varphi(p) = p \quad \text{and} \quad \varphi^*(\textstyle\sum dx'_j \wedge dy'_j) = \omega \ .$$

Since $\varphi^*(\sum dx'_j \wedge dy'_j) = \sum d(x'_j \circ \varphi) \wedge d(y'_j \circ \varphi)$, we only need to set new coordinates $x_j = x'_j \circ \varphi$ and $y_j = y'_j \circ \varphi$. $\square$

As a consequence of Theorem 1.2.4, if we prove for $(\mathbb{R}^{2n}, \sum dx_j \wedge dy_j)$ a local assertion which is invariant under symplectomorphisms, then that assertion holds for any symplectic manifold.

1.3 Moment Maps

We start by recalling notions from Lie group actions.

Definition 1.3.1. *An **action** of a Lie group G on a manifold M is a group homomorphism*

$$\begin{array}{rccl} \psi : & G & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \psi_g \ , \end{array}$$

where $\mathrm{Diff}(M)$ *is the group of diffeomorphisms of M. The **evaluation map** associated with an action $\psi : G \to \mathrm{Diff}(M)$ is*

$$\begin{array}{rccl} \mathrm{ev}_\psi : & M \times G & \longrightarrow & M \\ & (p, g) & \longmapsto & \psi_g(p) \ . \end{array}$$

*The action ψ is **smooth** if* ev_ψ *is a smooth map.*

We will always assume that an action is smooth.

Example. Complete vector fields[3] on a manifold M are in one-to-one correspondence with actions of $\mathbb{R}$ on M. The diffeomorphism $\psi_t : M \to M$ associated to $t \in \mathbb{R}$ is the time-t map $\exp tX$ defined by the flow of the vector field X. ♢

Let (M, ω) be a symplectic manifold, and G a Lie group with an action $\psi : G \to \mathrm{Diff}(M)$.

Definition 1.3.2. *The action ψ is a **symplectic action** if it is by symplectomorphisms, i.e.,*

$$\psi : G \longrightarrow \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M) \ ,$$

where $\mathrm{Sympl}(M, \omega)$ *is the group of symplectomorphisms of* (M, ω).

Examples.

1. On the symplectic 2-sphere $(S^2, d\theta \wedge dh)$ in cylindrical coordinates, the one-parameter group of diffeomorphisms given by rotation around the vertical axis, $\psi_t(\theta, h) = (\theta + t, h)$ $(t \in \mathbb{R})$ is a symplectic action of the group $S^1 \simeq \mathbb{R}/\langle 2\pi \rangle$, as it preserves the area form $d\theta \wedge dh$.

2. On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the one-parameter groups of diffeomorphisms given by rotation around each circle, $\psi_{1,t}(\theta_1, \theta_2) = (\theta_1 + t, \theta_2)$ $(t \in \mathbb{R})$ and $\psi_{2,t}$ similarly defined, are symplectic actions of S^1.

♢

[3] A vector field is **complete** if its integral curves through each point exist for *all* time.

Let (M, ω) be a symplectic manifold, G a Lie group with an action $\psi : G \to \mathrm{Diff}(M)$, and $\mathfrak{g}$ the Lie algebra of G with dual vector space $\mathfrak{g}^*$.

Definition 1.3.3. *The action ψ is a* ***hamiltonian action*** *if there exists a map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying the following two conditions:

- *For each $X \in \mathfrak{g}$, let $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$, be the component of μ along X, and let $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$. Then*

  $$d\mu^X = -\imath_{X^\#}\omega$$

  i.e., the function μ^X is a hamiltonian function for the vector field $X^\#$.

- *The map μ is* equivariant *with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:*

  $$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu \ , \qquad \text{for all } g \in G \ .$$

Then (M, ω, G, μ) is called a ***hamiltonian G-space*** *and μ is called a* ***moment map****. When G is a torus, we will call (M, ω, G, μ) a* ***hamiltonian torus space****.*

Exercise 1.3.4. *Check that complete symplectic vector fields on M are in one-to-one correspondence with symplectic actions of $\mathbb{R}$ on M, and that, similarly, complete hamiltonian vector fields on M are in one-to-one correspondence with hamiltonian actions of $\mathbb{R}$ on M.*

Examples. Consider the previous set of two examples The first – regarding S^2 – is an example of a hamiltonian action of S^1 with moment map given by the negative of the height function, under a suitable identification of the dual of the Lie algebra of S^1 with $\mathbb{R}$. The second example – regarding $\mathbb{T}^2$ – is not hamiltonian since the one-forms $d\theta_1$ and $d\theta_2$ are not exact. ♢

Exercise 1.3.5. *Let G be a Lie group and H a closed subgroup of G, with $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras. The projection $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the map dual to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. Suppose that (M, ω, G, ϕ) is a hamiltonian G-space. Show that the restriction of the G-action to H is hamiltonian with moment map*

$$i^* \circ \phi : M \longrightarrow \mathfrak{h}^* \ .$$

Exercise 1.3.6. *Suppose that a Lie group G acts in a hamiltonian way on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$. The product manifold $M_1 \times M_2$ has a natural product symplectic structure given by the sum of the pull-backs of the symplectic forms on each factor, via the two projections. Prove that the diagonal action of G on $M_1 \times M_2$ is hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by*

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2) \ , \quad \text{for } p_j \in M_j \ .$$

From now on, we concentrate on actions of a *standard torus* or rank $n \geqslant 1$ defined to be the product of n copies of S^1:

$$\mathbb{T}^n := (S^1)^n \ .$$

We write elements of $\mathbb{T}^n$ as n-tuples,

$$\left(e^{i\theta_1}, \dots, e^{i\theta_n}\right) ,$$

of complex numbers of absolute value 1. This now identifies $\mathbb{T}^n$ with the quotient

$$\mathbb{T}^n \simeq \mathbb{R}^n/(2\pi\mathbb{Z})^n \simeq (\mathbb{R}/2\pi\mathbb{Z})^n$$

and we view this as the standard identification of a torus Lie group T with its Lie algebra[4] $\mathfrak{t}$ modulo the *integral lattice* Γ via the *exponential map*:

$$\exp_T : \mathfrak{t} \longrightarrow T \quad \text{has kernel } \Gamma \qquad \Longrightarrow \qquad T \simeq \mathfrak{t}/\Gamma \ ,$$

where here

$$\exp : \mathbb{R}^n \longrightarrow \mathbb{T}^n \ \text{ has kernel } (2\pi\mathbb{Z})^n \ , \quad \exp(\theta_1, \dots, \theta_n) = \left(e^{i\theta_1}, \dots, e^{i\theta_n}\right) \ .$$

Implicitly, we use the standard basis of $\mathbb{R}^n$ as the chosen basis $X_1, \dots, X_n$ of the Lie algebra. This also yields global coordinates (mod 2π) θ_k on $\mathbb{T}^n$. The element

$$[\theta] := [\theta_1, \dots, \theta_n] = \left(e^{i\theta_1}, \dots, e^{i\theta_n}\right) \in \mathbb{T}^n$$

can also be viewed as the element achieved from the *identity* element

$$\mathbb{1} = [0, \dots, 0] = (1, \dots, 1) \in \mathbb{T}^n$$

by flowing along X_1 for time θ_1, along X_2 for time θ_2, ..., and along X_n for time θ_n.

Because the adjoint and coadjoint actions are trivial for a torus $\mathbb{T}^n$ and we are already indentifying the Lie algebra with $\mathbb{R}^n$, the dual of the Lie algebra gets also naturally identified with $\mathbb{R}^n$ via the standard pairing (standard inner product). A moment map for an action of $\mathbb{T}^n$ on (M, ω) is simply a map

$$\mu : M \longrightarrow \mathbb{R}^n \ ,$$

whose coordinate functions $\mu_1, \dots \mu_n$ all satisfy:

- μ_k is $\mathbb{T}^n$-invariant, i.e.:

$$\mu_k([\theta] \cdot p) = \mu_k(p) \text{ for all } [\theta] \in \mathbb{T}^n, \ p \in M, \ k = 1, \dots, n, \qquad \text{and}$$

[4]Since T is abelian, the Lie algebra $\mathfrak{t}$ of a T is defined as the set of (say left-)invariant vector fields on T (equivalently, as the tangent space at the identity) and the Lie bracket is trivial in this case.

- μ_k is a *hamiltonian function* for the vector field $X_k^\sharp$ on M induced by the k-th standard basis vector of $\mathbb{R}^n$, i.e.:

$$d\mu_k = -\imath_{X_k^\sharp}\omega \ , \quad k = 1, \ldots, n \ .$$

If $\mu : M \to \mathbb{R}^n$ is a moment map for a torus action, then clearly any of its translations $\mu + c$ ($c \in \mathbb{R}^n$) is also a moment map for that action. Reciprocally, any two moment maps for a given hamiltonian torus action differ by a constant.

Example. On $(\mathbb{C}, \omega_0 = \frac{i}{2} dz \wedge d\bar{z})$, consider the action of the circle $S^1 = \{t \in \mathbb{C} : |t| = 1\}$ by rotations

$$\psi_t(z) = t^\ell z \ , \qquad t \in S^1 \ ,$$

where $\ell \in \mathbb{Z}$ is fixed. The action $\psi : S^1 \to \mathrm{Diff}(\mathbb{C})$ is hamiltonian with moment map (or hamiltonian function) $\mu : \mathbb{C} \to \mathbb{R}$ given by

$$\mu(z) = \tfrac{1}{2}\ell|z|^2 \ .$$

This can be easily checked in polar coordinates, since $\omega_0 = r\,dr \wedge d\theta$, $\mu(re^{i\theta}) = \frac{1}{2}\ell r^2$ and the vector field on $\mathbb{C}$ corresponding to the generator 1 of the Lie algebra $\mathbb{R}$ is $X^\# = \ell\frac{\partial}{\partial\theta}$. ♢

Exercise 1.3.7. *Let $\mathbb{T}^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n : |t_j| = 1, \text{ for all } j\}$ be a torus acting diagonally on $\mathbb{C}^n$ by*

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1^{\ell_1} z_1, \ldots, t_n^{\ell_n} z_n) \ ,$$

where $\ell_1, \ldots, \ell_n \in \mathbb{Z}$ are fixed. Check that this action is hamiltonian with a moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n$ given by

$$\mu(z_1, \ldots, z_n) = \tfrac{1}{2}(\ell_1|z_1|^2, \ldots, \ell_n|z_n|^2) \ \ (\ +\ \textit{constant}\ ) \ .$$

Exercise 1.3.8. *Suppose that $\mathbb{T}^m$ acts linearly on $(\mathbb{C}^n, \omega_0)$ as follows:*

$$(e^{i\theta_1}, \ldots, e^{i\theta_m}) \cdot (z_1, \ldots, z_n) = \left(e^{i\langle\lambda^{(1)},\theta\rangle} z_1 \ , \ldots, e^{i\langle\lambda^{(n)},\theta\rangle} z_n\right) \ ,$$

for some weights $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$.

Show that, this action is hamiltonian with a moment map $\mu : \mathbb{C}^n \to \mathbb{R}^m$ given by

$$\mu(z_1, \ldots, z_n) = \tfrac{1}{2}\sum_{j=1}^{n} \lambda^{(j)}|z_j|^2 \ \ (\ +\ \textit{constant}\ ) \ .$$

It is a remarkable feature of compact connected hamiltonian torus spaces that the image of a moment map is a convex polytope. This was discovered and proved independently at about the same time by Atiyah and by Guillemin and Sternberg, following work of Kostant [38] for the case of coadjoint orbits.

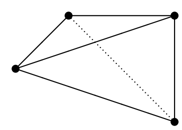
Theorem 1.3.9. ***(Atiyah [6], Guillemin-Sternberg [28])*** *Let (M, ω) be a compact connected symplectic manifold with a hamiltonian action of an m-torus, $\mathbb{T}^m$, and with moment map $\mu : M \to \mathbb{R}^m$. Then:*

(a) the levels of μ are connected;

(b) the image of μ is convex;

(c) the image of μ is the convex hull of a finite number of points, that are images of the fixed points of the action.

The image $\mu(M)$ of the moment map is called the **moment polytope**. A proof of Theorem 1.3.9 following Atiyah can be found in [46].



1.4 Symplectic Toric Manifolds

We start by analysing some crucial properties of the differential of a moment map. These are needed not only later in this section but also in Section 2.2.

Let (M, ω, G, μ) be a hamiltonian G-space. We denote by $\mathcal{O}$ the G-orbit through a point $p \in M$, by G_p the stabilizer[5] of p, and by $\mathfrak{g}_p$ the Lie algebra of G_p.

Lemma 1.4.1. *For the differential of the moment map $\mu : M \to \mathfrak{g}^*$ at p,*

$$d\mu_p : T_pM \longrightarrow \mathfrak{g}^* \ ,$$

where we identify a tangent space to the vector space $\mathfrak{g}^*$ with itself, we have that:

$$\textit{(I)}\ \ker d\mu_p = (T_p\mathcal{O})^\omega \qquad \textit{and} \qquad \textit{(II)}\ \operatorname{im} d\mu_p = (\mathfrak{g}_p)^0 \ ,$$

where $(T_p\mathcal{O})^\omega$ is the symplectic orthocomplement[6] of $T_p\mathcal{O}$ in the symplectic vector space (T_pM, ω_p), and $(\mathfrak{g}_p)^0$ is the annihilator[7] of $\mathfrak{g}_p$.

The proof of this lemma is contained in the next exercise.

Exercise 1.4.2. *Recall that, by definition of moment map (Definition 1.3.3), we have that*

$$\langle d\mu_p(v), X\rangle = \omega_p(v, X_p^\#) \qquad \textit{for all } X \in \mathfrak{g} \textit{ and } v \in T_pM .$$

[5]The *stabilizer (group)* (or *isotropy*) of a point p is $G_p := \{g \in G \mid g \cdot p = p\}$.

[6]If W is a subspace of a symplectic vector space (V, Ω), then the *symplectic orthocomplement* of W is the subspace $W^\Omega := \{v \in V \mid \Omega(v, w) = 0 \forall w \in W\}$.

[7]The *annihilator* of a linear subspace $W \subset V$ is the subset of V^* defined by $W^0 := \{\xi \in V^* \mid \xi(w) = 0 \forall w \in W\}$.

1. *Prove claim (I) in Lemma 1.4.1 by checking that*

$$d\mu_p(v) = 0 \quad \Longleftrightarrow \quad \omega_p(v, X_p^\#) = 0 \ , \ \forall X \in \mathfrak{g} \ .$$

   Note that the tangent space to the G-orbit through p is spanned by all the vectors $X_p^\#$.

2. *By counting dimensions, check that*

$$\begin{aligned} \dim(\ker d\mu_p) &= \dim M - \dim G + \dim G_p \\ \dim(\operatorname{im} d\mu_p) &= \dim G - \dim G_p \ . \end{aligned}$$

3. *Using the dimension count above, for checking claim (II) it is enough to show that*

$$\langle d\mu_p(v), X\rangle = 0 \qquad \forall X \in \mathfrak{g}_p, \forall v \in T_pM \ .$$

4. *Conclude from (II) that the stabilizer group of p is discrete if and only if $d\mu_p$ is surjective.*

5. *Conclude from (I) that the orbit through p is open if and only if $d\mu_p$ is injective.*

Effective hamiltonian tori actions

An action of a group G on a manifold M is called **effective** (or *faithful*) if it is injective as a map $G \to \mathrm{Diff}(M)$, i.e., each group element $g \neq e$ moves at least one point, that is, $\cap_{p\in M} G_p = \{e\}$.

The following two results use the crucial fact that any effective action $\mathbb{T}^m \to \mathrm{Diff}(M)$ has at least one orbit of dimension m; a proof may be found in [12, Ch.IV,§5].

Corollary 1.4.3. *Under the conditions of the convexity theorem (Theorem 1.3.9), if the $\mathbb{T}^m$-action is effective, then there must be at least $m+1$ fixed points.*

Proof. By Exercise 1.4.2, at any point p of an m-dimensional orbit, the stabilizer is discrete, so $d\mu_p$ is surjective. This means that a the moment map is a submersion, i.e., $(d\mu_1)_p, \ldots, (d\mu_m)_p$ are linearly independent. Hence, $\mu(p)$ is an interior point of $\mu(M)$, and $\mu(M)$ is a nondegenerate convex polytope. Any nondegenerate convex polytope in $\mathbb{R}^m$ must have at least $m+1$ vertices. The vertices of $\mu(M)$ are images of fixed points. □

Theorem 1.4.4. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then $\dim M \geqslant 2m$.*

Proof. Since the moment map is constant on an orbit $\mathcal{O}$, for $p \in \mathcal{O}$ the exterior derivative

$$d\mu_p : T_pM \longrightarrow \mathfrak{g}^*$$

maps $T_p\mathcal{O}$ to 0. Thus

$$T_p\mathcal{O} \subseteq \ker d\mu_p = (T_p\mathcal{O})^\omega ,$$

where $(T_p\mathcal{O})^\omega$ is the symplectic orthocomplement of $T_p\mathcal{O}$ (see Lemma 1.4.1). This shows that orbits $\mathcal{O}$ of a hamiltonian torus action are always isotropic submanifolds[8] of M. In particular, by symplectic linear algebra we have that $\dim \mathcal{O} \leqslant \frac{1}{2}\dim M$. Now consider an m-dimensional orbit. □

Definition of symplectic toric manifold

The so-called *symplectic toric manifolds* fit in the optimal case of effective hamiltonian tori actions:

Definition 1.4.5. *A* ***symplectic toric manifold*** *is a compact connected symplectic manifold (M,ω) equipped with an effective hamiltonian action of a standard torus $\mathbb{T}^n$ of dimension equal to half the dimension of the manifold,*

$$\dim \mathbb{T} = \frac{1}{2}\dim M ,$$

and with a choice of a corresponding moment map $\mu : M \to \mathbb{R}^n$.

In the examples below, we choose a scaling factor giving the *Fubini-Study form* on $\mathbb{CP}^n$ as

$$\omega_{\text{FS}} = \tfrac{i}{2}\partial\bar\partial \ln(1+|z|^2)$$

with respect to standard charts with n coordinates z_j, $0 \leqslant j \leqslant n$, $j \neq k$, on each open set

$$\mathcal{U}_k = \{[z_0 : \ldots : z_{k-1} : 1 : z_{k+1} : \ldots : z_n] \in \mathbb{CP}^n\} \quad \longrightarrow \quad \mathbb{C}^n .$$

In particular for $n=1$, we have that the sphere $\mathbb{CP}^1$ has $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{eucl}}$ and total area π with respect to ω_{FS}, whereas the euclidean area of a unit sphere in $\mathbb{R}^3$ is 4π.

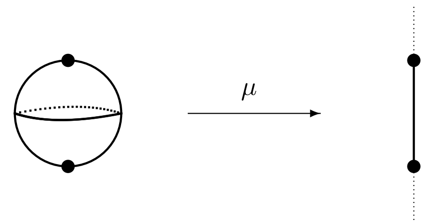
Examples of symplectic toric manifolds.

1. The circle S^1 acts on the 2-sphere $(S^2, \omega_{\text{eucl}} = d\theta \wedge dh)$ by rotations

$$e^{i\alpha}\cdot(\theta,h) = (\theta+\alpha,h)$$

with moment map $\mu = -h$ equal to minus the height function and moment polytope $[-1,1]$.

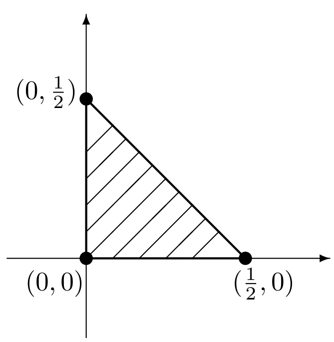
[8]A submanifold X of a symplectic manifold (M,ω) is **isotropic**, if the restriction of ω to X is trivial. This means that at each $p \in X$ the pairing of two tangent vectors to X by ω_p gives 0. We say that T_pX is an **isotropic subspace** of the *symplectic vector space* (T_pM,ω_p). When $\dim X = \frac{1}{2}\dim M$ we say that the isotropic submanifold X is **lagrangian**.



Equivalently, the circle S^1 acts on $\mathbb{CP}^1 = \mathbb{C}^2\backslash\{0\}/\sim$ with the Fubini-Study form $\omega_{\mathrm{FS}} = \frac{1}{4}\omega_{\mathrm{eucl}}$, by $e^{i\alpha}\cdot[z_0 : z_1] = [z_0 : e^{i\alpha}z_1]$. This is hamiltonian with moment map $\mu[z_0 : z_1] = \frac{1}{2}\cdot\frac{|z_1|^2}{|z_0|^2+|z_1|^2}$, and moment polytope $\left[0, \frac{1}{2}\right]$.

2. Let $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ be 2-(complex-)dimensional complex projective space equipped with the Fubini-Study form defined in Section 1.5. The $\mathbb{T}^2$-action on $\mathbb{CP}^2$ by $(e^{i\theta_1}, e^{i\theta_2})\cdot[z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1}z_1 : e^{i\theta_2}z_2]$ has moment map

$$\mu[z_0 : z_1 : z_2] = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}\right) .$$



The fixed points get mapped as

$$\begin{array}{ccc} [1:0:0] & \longmapsto & (0,0) \\ [0:1:0] & \longmapsto & \left(\frac{1}{2}, 0\right) \\ [0:0:1] & \longmapsto & \left(0, \frac{1}{2}\right) \end{array}$$

Notice that the stabilizer of a preimage of the edges is S^1, while the action is free at preimages of interior points of the moment polytope.

3. More generally, on $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ with diagonal action of $\mathbb{T}^n$ as $(e^{i\theta_1}, \ldots, e^{i\theta_n})\cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : e^{i\theta_1}z_1 : \ldots : e^{i\theta_n}z_n]$ we have as moment map

$$\mu[z_0 : z_1 : \ldots : z_n] = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+\ldots+|z_n|^2}, \ldots, \frac{|z_n|^2}{|z_0|^2+|z_1|^2+\ldots+|z_n|^2}\right) ,$$

whose image is an n-dimensional simplex.

Products of symplectic toric manifolds are naturally symplectic toric manifolds. For instance, on the product manifold $(\mathbb{CP}^1)^n$ with product symplectic structure given by the Fubini-Study form on each factor and with diagonal action of $\mathbb{T}^n$, we have as moment map

$$\mu(z_1, \ldots, z_n) = \frac{1}{2}\left(|z_1|^2, \ldots, |z_n|^2\right) ,$$

whose image is an n-dimensional cube. In particular, the moment polytope for the $\mathbb{T}^2$-action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1]) = ([z_0 : e^{i\theta} z_1], [w_0 : e^{i\eta} w_1])$$

is a square.

Equivalence between symplectic toric manifolds

The equivalence between symplectic toric manifolds is given by equivariant symplectomorphisms.

Definition 1.4.6. *Two symplectic toric manifolds, $(M_k, \omega_k, \mathbb{T}^n, \mu_k)$, $k = 1, 2$, are* ***isomorphic*** *if there exists an equivariant*[9] *symplectomorphism $\varphi : M_1 \to M_2$.*

Isomorphic symplectic toric manifolds are often undistinguished. Note that the torus is fixed and that the moment maps necessarily differ by a constant, in the sense that

$$\mu_1 = \mu_2 \circ \varphi + c \quad \text{for some } c \in \mathbb{R}^n.$$

(For general hamiltonian torus actions, moment maps are unique up to a constant).

Locally, there is an equivariant version of Darboux's theorem [58]. At a fixed point p for an action, there is an induced representation (i.e., a linear action) of the group on the tangent space T_pM given by differentiating the action. This is called the **isotropy representation** at the point p.

Theorem 1.4.7. ***(Equivariant Darboux)*** *Let $(M, \omega, \mathbb{T}^k, \mu)$ be a $2n$-dimensional hamiltonian torus space, and let p be a fixed point. Let $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^k$ be the weights of the isotropy representation of $\mathbb{T}^k$ on the tangent space T_pM.*

Then there is a $\mathbb{T}^k$-invariant neighborhood $\mathcal{U}$ of p in M and coordinate functions $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p with respect to which we have:

(a)

$$\omega|_{\mathcal{U}} = \sum_{j=1}^{n} dx_j \wedge dy_j = \tfrac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j ,$$

where $z_j = x_j + iy_j$, $\overline{z}_j = x_j - iy_j$,

[9] Equivariance means $\varphi([\theta] \cdot p) = [\theta] \cdot \varphi(p)$.

(b) *the action becomes the linear action of* $\mathbb{T}^k$ *with the given weights:*

$$(e^{i\theta_1}, \ldots, e^{i\theta_k}) \cdot (z_1, \ldots, z_n) = \left(e^{i \sum_j \lambda_j^{(1)} \theta_j} z_1, \ldots, e^{i \sum_j \lambda_j^{(n)} \theta_j} z_n \right) \qquad \textit{and}$$

(c) *the moment map becomes*

$$\mu|_{\mathcal{U}} = \mu(p) + \frac{1}{2} \sum_{j=1}^{n} \lambda^{(j)} (x_j^2 + y_j^2) = \mu(p) + \frac{1}{2} \sum_{j=1}^{n} \lambda^{(j)} |z_j|^2 \ .$$

The proof, which we omit, relies on:

- viewing the isotropy representation as a complex representation using a *compatible almost complex structure*; and
- adjusting Moser's argument for the standard Darboux theorem by using $\mathbb{T}^k$-invariant/equivariant data all along.

Exercise 1.4.8. *Suppose that* $\mathbb{T}^m$ *acts linearly on* $(\mathbb{C}^n, \omega_0)$ *as follows:*

$$(e^{i\theta_1}, \ldots, e^{i\theta_m}) \cdot (z_1, \ldots, z_n) = \left(e^{i\langle \lambda^{(1)}, \theta \rangle} z_1 \ , \ldots, e^{i\langle \lambda^{(n)}, \theta \rangle} z_n \right) \ ,$$

for some weights $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$. *In Exercise 1.3.8, we have seen that this action is hamiltonian with a moment map given by*

$$\mu(z_1, \ldots, z_n) = \tfrac{1}{2} \sum_{j=1}^{n} \lambda^{(j)} |z_j|^2 \ \textit{(+ constant)} \ .$$

(a) *Show that, if the action is effective, then* $m \leqslant n$ *and the weights* $\lambda^{(1)}, \ldots, \lambda^{(n)}$ $\mathbb{Z}$*-span* $\mathbb{Z}^m$.

(b) *Conclude that, if such a linear action of* $\mathbb{T}^n$ *on* $\mathbb{C}^n$ *is effective, then any moment map* μ *is a submersion, i.e., each differential* $d\mu_z : \mathbb{C}^n \to \mathbb{R}^n$ $(z \in \mathbb{C}^n)$ *is surjective.*

Exercise 1.4.9. *Show that for a symplectic toric manifold the weights of the isotropy representation at a fixed point,* $\lambda^{(1)}, \ldots, \lambda^{(n)}$, *form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$.

Hint: *Equivariant Darboux and previous exercise.*

1.5 Symplectic Reduction

Symplectic reduction is a fundamental construction of (new) symplectic manifolds starting from (old) symplectic manifolds with a hamiltonian group action by taking quotients in the symplectic sense.

Symplectic reduction is also the key for Delzant's proof of existence in his classification theorem, by providing the construction of a symplectic toric manifold out of the data encoded in an appropriate polytope.

First we recall *orbit spaces*. Let $\psi : G \to \mathrm{Diff}(M)$ be any action. The **orbit** of G through $p \in M$ is $\{\psi_g(p) \mid g \in G\}$.

Exercise 1.5.1. *If q is in the orbit of p, then their stabilizers G_q and G_p are conjugate subgroups. In particular, when G is abelian, all points in the same orbit have the same stabilizer.*

Definition 1.5.2. *We say that the action of G on M is:*

- ***transitive*** *if there is just one orbit,*
- ***free*** *if all stabilizers are trivial $\{e\}$,*
- ***locally free*** *if all stabilizers are discrete.*

Let $\sim$ be the orbit equivalence relation; for $p, q \in M$,

$$p \sim q \quad \Longleftrightarrow \quad p \text{ and } q \text{ are on the same orbit.}$$

The space of orbits $M/G := M/\sim$ is called the **orbit space**. Let

$$\begin{array}{rccl} \Pi : & M & \longrightarrow & M/G \\ & p & \longmapsto & \text{orbit through } p \end{array}$$

be the **point-orbit projection**.

We equip M/G with the weakest topology for which Π is continuous, i.e., $\mathcal{U} \subseteq M/G$ is open if and only if $\Pi^{-1}(\mathcal{U})$ is open in M. This is called the **quotient topology**. This topology can be *bad*. For instance:

Example. Let $G = \mathbb{C}\backslash\{0\}$ act on $M = \mathbb{C}^n$ by

$$\lambda \longmapsto \psi_\lambda = \text{multiplication by } \lambda \ .$$

The orbits are the punctured complex lines (through non-zero vectors $z \in \mathbb{C}^n$), plus one so-called *unstable* orbit through 0, which has a single point. The orbit space is

$$M/G = \mathbb{CP}^{n-1} \sqcup \{\text{point}\} \ .$$

The quotient topology restricts to the usual topology on $\mathbb{CP}^{n-1}$. The only open set containing $\{\text{point}\}$ in the quotient topology is the full space, hence the topology in M/G is *not* Hausdorff.

However, it suffices to remove 0 from $\mathbb{C}^n$ to obtain a Hausdorff orbit space:

$$\Big(\mathbb{C}^n\backslash\{0\}\Big)\Big/\Big(\mathbb{C}\backslash\{0\}\Big) = \mathbb{CP}^{n-1} \ .$$

$\diamondsuit$

We next address the previous example once again but from a compact and symplectic (yet not complex) viewpoint:

Example. Let $\omega = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k = \sum r_k dr_k \wedge d\theta_k$ be the standard symplectic form on $\mathbb{C}^n$. Consider the following S^1-action on $(\mathbb{C}^n, \omega)$:

$$\theta \in S^1 \longmapsto \psi_\theta = \text{multiplication by } \theta \ .$$

The vector field generated by this action is

$$X^{\#} = \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} + \cdots + \frac{\partial}{\partial \theta_n} \ .$$

This vector field is hamiltonian, i.e., the action ψ is hamiltonian with moment map

$$\begin{array}{rccl} \mu : & \mathbb{C}^n & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \frac{||z||^2}{2} + \text{constant} \end{array}$$

since

$$\imath_{X^{\#}}\omega = -\sum r_k dr_k = -\tfrac{1}{2}\sum d(r_k^2) = -d\mu \ .$$

If we conveniently choose the constant to be $-\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1} \ .$$

This description induces a symplectic form on $\mathbb{CP}^{n-1}$ as a particular instance of the following major theorem; see below. $\diamondsuit$

Meyer on one side and Marsden and Weinstein on the other proved independently the following mathematical formulation of the reduction process from physics. Later in this text, we will only be concerned with the case where the Lie group is a torus.

Theorem 1.5.3. ***(Marsden-Weinstein [44], Meyer [48])*** *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

(a) the orbit space $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold,

(b) $\Pi : \mu^{-1}(0) \to M_{\text{red}}$ is a principal G-bundle, and

(c) there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \Pi^*\omega_{\text{red}}$.

Note that this theorem does not assume that $\mu^{-1}(0)$ is a regular level, but this is a consequence of G acting freely on this level; see Exercise 1.4.2.

For a proof of Theorem 1.5.3, see for instance [15]. Here is just a sketch of the idea for the case $G = S^1$ and $\dim M = 4$ going back to Bott.

In this case the moment map is $\mu : M \to \mathbb{R}$. Let $p \in \mu^{-1}(0)$. Choose local coordinates:

- θ along the orbit through p,
- μ given by the moment map, and
- η_1, η_2 pullback of coordinates on $\mu^{-1}(0)/S^1$.

Then the symplectic form can be written

$$\omega = A\ d\theta \wedge d\mu + B_j\ d\theta \wedge d\eta_j + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2\ .$$

Since $d\mu = \imath\left(\frac{\partial}{\partial\theta}\right)\omega$, we must have $A = 1$, $B_j = 0$. Hence,

$$\omega = d\theta \wedge d\mu + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2\ .$$

Since ω is symplectic, we must have $D \neq 0$. Therefore, $i^*\omega = D\ d\eta_1 \wedge d\eta_2$ is the pullback of a symplectic form on M_{red}.

Definition 1.5.4. *The pair* $(M_{\text{red}}, \omega_{\text{red}})$ *is called the* ***symplectic reduction*** *of* (M, ω) *with respect to* G *and* μ *(or the* reduced space*, or the* symplectic quotient*, or the* Marsden-Weinstein-Meyer quotient*, etc.).*

Example. Consider the S^1-action on $(\mathbb{R}^{2n+2}, \omega_0)$ which, under the usual identification of $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, corresponds to multiplication by $e^{i\theta}$. This action is hamiltonian with a moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$\mu(z) = \tfrac{1}{2}||z||^2 - \tfrac{1}{2}\ .$$

Symplectic reduction yields complex projective space $\mu^{-1}(0)/S^1 = \mathbb{CP}^n$ equipped with the so-called **Fubini-Study symplectic form** $\omega_{\text{red}} = \omega_{\text{FS}}$. ♢

Exercise 1.5.5. *Recall that* $\mathbb{CP}^1 \simeq S^2$ *as real 2-dimensional manifolds. Check that*

$$\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{eucl}}\ ,$$

where $\omega_{\text{eucl}} = d\theta \wedge dh$ *is the euclidean area form on the unit sphere* S^2*.*

We consider here two basic extensions of the procedure of symplectic reduction. There is a further major extension to the case of *symplectic toric orbifolds*, which we briefly address in Chapter 3. Reduction for product groups (a.k.a. reduction in stages) will be needed in Chapter 2.

Reduction for product groups

Let G_1 and G_2 be compact connected Lie groups whose actions on a manifold M commute, and let $G = G_1 \times G_2$. Then $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}^* \simeq \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$. Suppose that (M, ω, G, ν) is a hamiltonian G-space with moment map

$$\nu : M \longrightarrow \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*\ .$$

Write $\nu = (\nu_1, \nu_2)$ where $\nu_k : M \to \mathfrak{g}_k^*$ for $k = 1, 2$. The fact that ν is equivariant implies that ν_1 is invariant under G_2 and ν_2 is invariant under G_1. Now reduce (M, ω) with respect to the G_1-action. Let

$$Z_1 = \nu_1^{-1}(0)\ .$$

Assume that G_1 acts freely on Z_1. Let $M_1 = Z_1/G_1$ be the reduced space and let ω_1 be the corresponding reduced symplectic form. The action of G_2 on Z_1 commutes with the G_1-action. Since G_2 preserves ω, it follows that G_2 acts symplectically on (M_1, ω_1). Since G_1 preserves ν_2, G_1 also preserves $\nu_2 \circ \iota_1 : Z_1 \to \mathfrak{g}_2^*$, where $\iota_1 : Z_1 \hookrightarrow M$ is inclusion. Thus $\nu_2 \circ \iota_1$ is constant on fibers of $Z_1 \xrightarrow{p_1} M_1$. We conclude that there exists a smooth map $\mu_2 : M_1 \to \mathfrak{g}_2^*$ such that $\mu_2 \circ p_1 = \nu_2 \circ \iota_1$.

Exercise 1.5.6. *Show that:*

(a) the map μ_2 is a moment map for the action of G_2 on (M_1, ω_1), and

(b) if G acts freely on $\nu^{-1}(0,0)$, then G_2 acts freely on $\mu_2^{-1}(0)$, and there is a natural symplectomorphism

$$\nu^{-1}(0,0)/G \ \simeq \ \mu_2^{-1}(0)/G_2 \ .$$

This technique of performing reduction with respect to one factor of a product group at a time is called **reduction in stages**. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H.

Example. Consider the hamiltonian S^1-action on $(\mathbb{C}^{n+1}, \omega_0)$ by multiplication by $e^{i\theta}$, for which symplectic reduction yields complex projective space $\mu^{-1}(0)/S^1 = \mathbb{CP}^n$ (see example above). Now $\mathbb{T}^{n+1}$ acts also on $(\mathbb{C}^{n+1}, \omega_0)$ by diagonal multiplication and this hamiltonian action commutes with the S^1-action. Hence, it descends to the reduced space $\mathbb{CP}^n$. The *reduced* moment map is given by

$$\begin{array}{ccc} \mathbb{CP}^n & \longrightarrow & \mathbb{R}^{n+1} \\ [z_0 : z_1 : \ldots : z_n] & \longmapsto & \frac{1}{2}\left(|z_0|^2, |z_1|^2, \ldots, |z_n|^2\right) \end{array}$$

where we choose $(z_0, z_1, \ldots, z_n) \in \mu^{-1}(0)$. $\diamondsuit$

Reduction at other levels

Suppose that a compact Lie group G acts on a symplectic manifold (M, ω) in a hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$. To reduce at the level ξ of μ, we need $\mu^{-1}(\xi)$ to be preserved by G, or else take the G-orbit of $\mu^{-1}(\xi)$, or equivalently take the inverse image $\mu^{-1}(\mathcal{O}_\xi)$ of the coadjoint orbit through ξ, or else take the quotient by the maximal subgroup of G which preserves $\mu^{-1}(\xi)$. Of course the level 0 is always preserved. Also, when G is a torus, any level is preserved and reduction at ξ for the moment map μ, is equivalent to reduction at 0 for a shifted moment map $\phi : M \to \mathfrak{g}^*$, $\phi(p) := \mu(p) - \xi$.

For the case of *torus* actions, are all levels equally easy, since the coadjoint action is trivial.

Example. Consider again the hamiltonian S^1-action on $(\mathbb{C}^{n+1}, \omega_0)$ by multiplication by $e^{i\theta}$ with moment map

$$\mu(z) = \frac{1}{2}||z||^2 - \frac{1}{2} \ ,$$

for which symplectic reduction at level 0 yields complex projective space

$$\mu^{-1}(0)/S^1 = \mathbb{CP}^n$$

equipped with the Fubini-Study symplectic form (see example above).

If we now reduce at another level $\xi > -\frac{1}{2}$, we obtain as reduced space the same smooth manifold

$$\mu^{-1}(\xi)/S^1 \simeq \mathbb{CP}^n \ ,$$

but the symplectic form will be scaled. $\diamondsuit$

Chapter 2

Delzant's Classification

Recall that a $2n$-dimensional symplectic toric manifold is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective hamiltonian action of a standard n-torus $\mathbb{T}^n$ and with a corresponding moment map. Two symplectic toric manifolds are called *isomorphic* – and thus considered *equivalent* – if they are equivariantly symplectomorphic (Section 1.4).

In this chapter, we state and prove the classification of equivalence classes of symplectic toric manifolds by their moment polytopes $\mu(M)$ up to translation. For the existence part we follow Delzant and for the uniqueness we follow Lerman. Moreover, we discuss first examples.

Although for the standard torus $\mathbb{T}^n$ both the Lie algebra and its dual are naturally identified with $\mathbb{R}^n$, we will distinguish $\mathbb{R}^n$ from $(\mathbb{R}^n)^*$ and write for the natural pairing $\langle \cdot, \cdot \rangle : (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$. In particular, a moment map will be denoted $\mu : M \to (\mathbb{R}^n)^*$.

2.1 Delzant's Theorem

A **polytope** in $\mathbb{R}^n$ is the convex hull[1] of a finite number of points in $\mathbb{R}^n$. A **convex polyhedron** is a subset of $\mathbb{R}^n$ that is the intersection of a finite number of affine half-spaces. It is a theorem, usually attributed to Weyl and Minkowski, that polytopes coincide with compact convex polyhedra.

Exercise 2.1.1. *($\star$)[2] Prove Weyl-Minkowski's theorem for $n = 2$. Although the claim is intuitive, its proof for higher n is involved.*

A **face** of a polytope Δ is a set of the form $F = \Delta \cap \{x \in \mathbb{R}^n \mid f(x) = c\}$ where $c \in \mathbb{R}$ and $f \in (\mathbb{R}^n)^*$ satisfies $f(x) \geqslant c$, $\forall x \in \Delta$. A **vertex** is a 0-dimensional face, whereas an **edge** is a 1-dimensional face. A **facet** of an n-dimensional polytope is an $(n-1)$-dimensional face.

[1] The *convex hull* of a given subset X of a vector space is the intersection of all convex sets containing X or, equivalently, the set of all convex combinations of points in X.

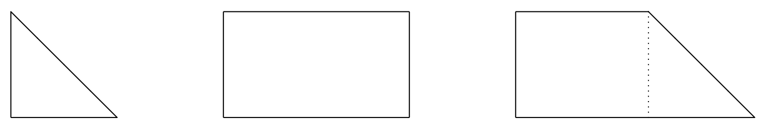
[2] Exercises marked with a star are either harder of less central to the exposition.

We now define the class of polytopes which arise in the classification of symplectic toric manifolds.

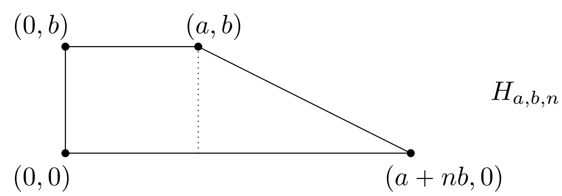
Definition 2.1.2. *A **Delzant polytope** Δ in $\mathbb{R}^n$ is a polytope satisfying:*

- ***simplicity**, i.e., there are n edges meeting at each vertex;*
- ***rationality**, i.e., the edges meeting at the vertex τ are rational in the sense that each edge is of the form $\tau + tu_k$, $t \geqslant 0$, where $u_k \in \mathbb{Z}^n$;*
- ***smoothness**, i.e., for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Examples of Delzant polytopes in $\mathbb{R}^2$:

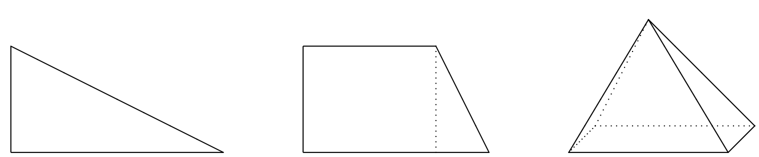


The dotted vertical line in the trapezoidal example is there just to stress that it is a picture of a rectangle plus an *isosceles* triangle. For "taller" triangles, smoothness would be violated. "Wider" triangles may still be Delzant as in the examples below. There is an integrality condition on the slope of the hypothenuse: $n = 0, 1, 2, \ldots$. We will call these examples **Hirzebruch trapezoids** and denote them $H_{a,b,n}$, where $a, b > 0$ and $n = 0, 1, 2, \ldots$. In particular, $H_{a,b,0}$ is just a rectangle.



$\diamondsuit$

Examples of polytopes that are not Delzant:



The picture on the left fails the smoothness condition on the upper vertex, whereas the one in the middle fails the smoothness condition on the two right vertices and the one on the right fails the simplicity condition. $\diamondsuit$

Exercise 2.1.3. *Show that, up to linear transformations in* $\mathrm{GL}(2;\mathbb{Z})$*, the Delzant polytopes in* $\mathbb{R}^2$ *with three vertices are just the isosceles right triangles.*

> ***Hint:*** *Once you use a transformation in* $\mathrm{GL}(2;\mathbb{Z})$ *to make one of the angles into a square angle, how are the lengths of the two edges forming that angle related?*

Conclude that the equivalence classes of Delzant polytopes in $\mathbb{R}^2$ *up to* $\mathrm{GL}(2;\mathbb{Z})$ *and translation by arbitrary vectors in* $\mathbb{R}^2$ *is the one-parameter family of triangles with vertices* $(0,0)$*,* $(a,0)$ *and* $(0,a)$*, where* $a > 0$.

Exercise 2.1.4. *Describe the class of Delzant polytopes in* $\mathbb{R}^2$ *with four vertices, up to linear transformations in* $\mathrm{GL}(2;\mathbb{Z})$.

> ***Hint:*** *Choose any vertex, translate it to zero and use* $\mathrm{GL}(2;\mathbb{Z})$ *to turn the corresponding edge vectors into the standard basis. Then the vertices will be of the form* $(0,0)$*,* $(c,0)$*,* $(0,b)$*,* (a,d) *with* $a,b,c,d > 0$*. By a reflection if necessary, assume that* $c > b$*. By convexity and the Delzant condition at* $(c,0)$*,* $(0,b)$*, the non-axial primitive edge vectors at those vertices must be of the form* $u_1 = (C,1)$ *and* $u_2 = (1,B)$ *with* $C, B \in \mathbb{Z}$*. Because those non-axial edges meet at* (a,d) *in a convex fashion, we must have that the determinant of the matrix with columns* u_1 *and* u_2 *must be negative, i.e.,* $BC < 1$*. Then the Delzant condition at* (a,d) *gives that that determinant must be* -1*, i.e.,* $BC = 0$*. Follow on, checking the cases* $B = 0$ *and* $C = 0$*.*

Delzant's theorem classifies (equivalence classes of) symplectic toric manifolds in terms of the combinatorial data encoded by a Delzant polytope (up to translation).

Theorem 2.1.5. ***(Delzant [19])*** *Symplectic toric manifolds are classified up to equivalence by Delzant polytopes up to translation. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\begin{array}{rcl} \{\textit{symplectic toric manifolds}\} & \stackrel{1-1}{\longrightarrow} & \{\textit{Delzant polytopes}\} \\ \textit{(mod equivalence)} & & \textit{(mod translation)} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M)\ . \end{array}$$

In order to prepare the construction of a symplectic toric manifold from a Delzant polytope, we will use the description of polytopes as convex polyhedra. For this passage, we use the following exercise, translating the Delzant condition into a similar condition in terms of normal vectors to the facets.

Exercise 2.1.6. *Consider one vertex of a Delzant polytope in* $\mathbb{R}^n$ *and a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$ *built up of the edge vectors meeting at that vertex,*

$$u_1, u_2, \ldots, u_n .$$

Show that then there are n *corresponding facets meeting at that vertex (each one containing all but one of the* u_k *vectors) and that the primitive inward-pointing normal vectors to these facets also form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$*.*

> ***Hint:*** *By a change of basis if necessary, you may assume that $u_1, u_2, \dots, u_n$ is the standard basis. Then the corresponding primitive inward-pointing normal vectors to the facets meeting at that vertex are also the vectors from the standard basis.*

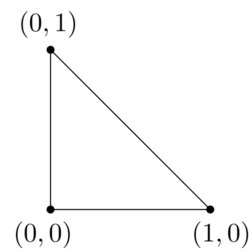
Let Δ be a Delzant polytope in $(\mathbb{R}^n)^*$ and with d facets.[3] Let $v_k \in \mathbb{Z}^n$, $k = 1, \dots, d$, be the primitive[4] inward-pointing normal vectors to the facets of Δ. Then we can describe Δ as an intersection of halfspaces

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle \geqslant \lambda_k,\ k = 1, \dots, d\} \quad \text{for some } \lambda_k \in \mathbb{R} ,$$

where $\langle \cdot, \cdot \rangle : (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$ is the natural pairing.

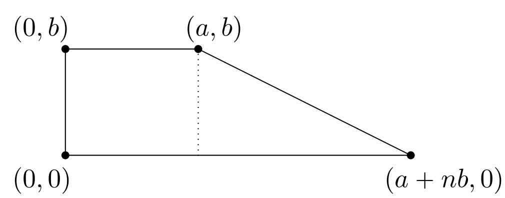
Example. For the picture below, we have

$$\begin{aligned} \Delta &= \{x \in (\mathbb{R}^2)^* \mid x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1 + x_2 \leqslant 1\} \\ &= \{x \in (\mathbb{R}^2)^* \mid \langle x, \underbrace{(1,0)}_{v_1} \rangle \geqslant 0 \ , \ \langle x, \underbrace{(0,1)}_{v_2} \rangle \geqslant 0 \ , \ \langle x, \underbrace{(-1,-1)}_{v_3} \rangle \geqslant -1\} \ . \end{aligned}$$



$\diamondsuit$

Exercise 2.1.7. *Describe the polytope $H_{a,b,n}$ as an intersection of four hyperplanes.*



Exercise 2.1.8. *($\star$) This is a generalization of Exercise 2.1.3. Show that, up to linear transformations in $\mathrm{GL}(n;\mathbb{Z})$, and translations, the Delzant polytopes in $\mathbb{R}^n$ with $n+1$ vertices are the simplices with vertices at the origin and at the points with all coordinates 0 except one equal to a ($a > 0$). In particular, in $\mathbb{R}^3$ that is the set of simplices with vertices $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, $(0,0,a)$.*

---

[3] It may be more clear for now to see Δ in $(\mathbb{R}^n)^*$. In particular, edge vectors will be in $(\mathbb{R}^n)^*$, whereas normal vectors to the facets will be regarded in $\mathbb{R}^n$.

[4] A lattice vector $v \in \mathbb{Z}^n$ is **primitive** if it cannot be written as $v = \ell u$ with $u \in \mathbb{Z}^n$, $\ell \in \mathbb{Z}$ and $|\ell| > 1$; for instance, $(1,1)$, $(4,3)$, $(1,0)$ are primitive, but $(2,2)$, $(4,6)$ are not.

Hint: *Choose any vertex, translate it to zero and use* $\mathrm{GL}(n;\mathbb{Z})$ *to turn the corresponding edge vectors into the standard basis. Now all vertices lie on the coordinate axes and let* $(a, 0, 0, \ldots, 0)$ *be one of the vertices. By another transformation in* $\mathrm{GL}(n;\mathbb{Z})$*, you may assume that this is a vertex closest to the origin. Then by rationality the other vertices will be of the form* $(0, c_2a, 0, \ldots, 0)$*,* $(0, 0, c_3a, 0, \ldots, 0)$*, etc with* $c_2, c_3, \ldots, c_n \in \mathbb{N}$*. Suppose that one of the* c_k'*s, say* c_2*, is not* 1*. Write the edge vectors at the vertices* $(a, 0, 0, \ldots, 0)$ *and* $(0, c_2a, 0, \ldots, 0)$ *and impose the Delzant condition.*

2.2 Proof of Existence

Following [19, 27], we prove the existence part (or surjectivity) in Delzant's theorem, by using symplectic reduction to associate to an n-dimensional Delzant polytope Δ a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ with $\mu_\Delta(M_\Delta) = \Delta$.

Let Δ be a Delzant polytope with d facets. Let $v_k \in \mathbb{Z}^n$, $k = 1, \ldots, d$, be the primitive inward-pointing normal vectors to the facets. For some $\lambda_k \in \mathbb{R}$, we can write

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k\rangle \geqslant \lambda_k,\ k = 1, \ldots, d\} \ .$$

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{array}{rccc} \Pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_k & \longmapsto & v_k \ . \end{array}$$

Then the map Π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$ since, for each vertex, the v_k's corresponding to the facets meeting at that vertex form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$; see Exercise 2.1.6.

Therefore, Π induces a surjective map, still called Π, between tori:

$$\begin{array}{ccccc} \mathbb{R}^d/(2\pi\mathbb{Z}^d) & \xrightarrow{\Pi} & \mathbb{R}^n/(2\pi\mathbb{Z}^n) & & \\ \| & & \| & & \\ \mathbb{T}^d & \longrightarrow & \mathbb{T}^n & \longrightarrow & \mathbb{1} \ . \end{array}$$

The kernel N of Π is a connected $(d-n)$-dimensional Lie subgroup of $\mathbb{T}^d$, hence a torus, with inclusion $i : N \hookrightarrow \mathbb{T}^d$. Let $\mathfrak{n}$ be the Lie algebra of N. The exact sequence of tori

$$\mathbb{1} \longrightarrow N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\Pi} \mathbb{T}^n \longrightarrow \mathbb{1}$$

induces an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\Pi} \mathbb{R}^n \longrightarrow 0$$

with dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\Pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0 \ .$$

Now consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k$, and standard hamiltonian action of $\mathbb{T}^d$ given by

$$(e^{i\theta_1}, \ldots, e^{i\theta_d}) \cdot (z_1, \ldots, z_d) = (e^{i\theta_1}z_1, \ldots, e^{i\theta_d}z_d) \ .$$

The moment map is $\phi : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$ defined by

$$\phi(z_1, \ldots, z_d) = \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + \text{ constant },$$

where we will choose the constant to be $(\lambda_1, \ldots, \lambda_d)$. By Exercise 1.3.5, the subtorus N acts on $\mathbb{C}^d$ in a hamiltonian way with moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^* .$$

Let $Z = (i^* \circ \phi)^{-1}(0)$ be the zero-level set. Note that Z is connected, because $(i^*)^{-1}(0)$ is a linear subspace of $\mathbb{R}^d$ and the fibers $\phi^{-1}(x)$ are path-connected.

Claim 1. The submanifold Z is compact and N acts freely on Z.

We postpone the proof of this claim until further down.

Now Z is the zero-level of a moment map for the action of the torus N on $\mathbb{C}^d$. Knowing that N acts freely on Z ensures that this is a *regular level*; this is a consequence of Exercise 1.4.2. Hence, Z is a *submanifold* of $\mathbb{C}^d$ of (real) dimension $2d - (d - n) = d + n$. We now use the following theorem from Lie theory:

Theorem 2.2.1. *If a compact Lie group N acts freely on a manifold Z, then the orbit space Z/N is a manifold and the point-orbit map $p : Z \to Z/N$ is a principal N-bundle.*

In our case, Z is a *compact* $(d + n)$-dimensional manifold, so the orbit space $M_\Delta = Z/N$ is a compact manifold of (real) dimension $\dim Z - \dim N = (d + n) - (d - n) = 2n$. The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ. Consider the diagram

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & \mathbb{C}^d \\ {\scriptstyle p}\downarrow & & \\ M_\Delta & & \end{array}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is inclusion. The Marsden-Weinstein-Meyer theorem (Theorem 1.5.3) guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^*\omega_\Delta = j^*\omega_0 .$$

Since Z is connected, the compact symplectic $2n$-dimensional manifold $(M_\Delta, \omega_\Delta)$ is also connected.

Proof of Claim 1. The set Z is clearly closed, hence in order to show that it is compact it suffices (by the Heine-Borel theorem) to show that Z is bounded. Let Δ' be the image of Δ by Π^*. We will show that $\phi(Z) = \Delta'$.

Lemma 2.2.2. *Let $y \in (\mathbb{R}^d)^*$. Then:*

$$y \in \Delta' \iff y \in \phi(Z) .$$

Proof. The value y is in the image of Z by ϕ if and only if both of the following conditions hold:

1. y is in the image of ϕ;
2. $i^*y = 0$.

Using the expression for ϕ and the dual exact sequence, we see that these conditions are equivalent to:

1. $\langle y, e_k\rangle \geqslant \lambda_k$ for $k = 1, \ldots, d$;
2. $y = \Pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$.

Suppose that the second condition holds, so that $y = \Pi^*(x)$. Then

$$\begin{aligned} \langle y, e_k\rangle \geqslant \lambda_k,\ \forall k &\iff \langle \Pi^*(x), e_k\rangle \geqslant \lambda_k,\ \forall k \\ &\iff \langle x, \Pi(e_k)\rangle \geqslant \lambda_k,\ \forall k \\ &\iff \langle x, v_k\rangle \geqslant \lambda_k,\ \forall k \\ &\iff x \in \Delta\ . \end{aligned}$$

Thus,

$$y \in \phi(Z) \iff y \in \Pi^*(\Delta) = \Delta'\ .$$

This concludes the proof of Lemma 2.2.2. □

Since we have that Δ' is compact, that ϕ is a proper map[5] and that $\phi(Z) = \Delta'$, we conclude that Z must be bounded, and hence compact.

It remains to show that N acts freely on Z.

Pick a vertex τ of Δ, and let $I = \{k_1, \ldots, k_n\}$ be the set of indices for the n facets meeting at τ. Pick $z \in Z$ such that $\phi(z) = \Pi^*(\tau)$. Then τ is characterized by n equations $\langle \tau, v_k\rangle = \lambda_k$ where k ranges in I:

$$\begin{aligned} \langle \tau, v_k\rangle = \lambda_k &\iff \langle \tau, \Pi(e_k)\rangle = \lambda_k \\ &\iff \langle \Pi^*(\tau), e_k\rangle = \lambda_k \\ &\iff \langle \phi(z), e_k\rangle = \lambda_k \\ &\iff i\text{-th coordinate of } \phi(z) \text{ is equal to } \lambda_k \\ &\iff \tfrac{1}{2}|z_k|^2 + \lambda_k = \lambda_k \\ &\iff z_k = 0\ . \end{aligned}$$

Hence, those z's are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we may assume that $I = \{1, \ldots, n\}$. The stabilizer of z is

$$(\mathbb{T}^d)_z = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}, 1, \ldots, 1) \in \mathbb{T}^d\}\ .$$

[5] A map between topological spaces is called *proper* if inverse images of compact subsets are compact.

As the restriction $\Pi : (\mathbb{R}^d)_z \to \mathbb{R}^n$ maps the vectors $e_1, \ldots, e_n$ to a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of $\mathbb{Z}^n$ respectively, at the level of groups the map $\Pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ must be bijective. Since $N = \ker(\Pi : \mathbb{T}^d \to \mathbb{T}^n)$, we conclude that $N \cap (\mathbb{T}^d)_z = \{\mathbb{1}\}$, i.e., $N_z = \{\mathbb{1}\}$. Hence, all N-stabilizers at points mapping to vertices are trivial. But this was the worst case, since other stabilizers $N_{z'}$ $(z' \in Z)$ are contained in stabilizers for points z which map to vertices. This concludes the proof of Claim 1. $\square$

Given a Delzant polytope Δ, we have constructed a symplectic manifold $(M_\Delta, \omega_\Delta)$ where $M_\Delta = Z/N$ is a compact $2n$-dimensional manifold and ω_Δ is the reduced symplectic form.

Claim 2. The manifold $(M_\Delta, \omega_\Delta)$ inherits a hamiltonian $\mathbb{T}^n$-action with a moment map μ_Δ having image $\mu_\Delta(M_\Delta) = \Delta$.

Proof of Claim 2. Let z be such that $\phi(z) = \Pi^*(\tau)$ where τ is a vertex of Δ, as in the proof of Claim 1. Let $\sigma : \mathbb{T}^n \to (\mathbb{T}^d)_z$ be the inverse for the earlier bijection $\Pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$. Since we have found a *section*, i.e., a right inverse for Π, in the exact sequence

$$\mathbb{1} \longrightarrow N \xrightarrow{i} \mathbb{T}^d \underset{\underset{\sigma}{\longleftarrow}}{\xrightarrow{\Pi}} \mathbb{T}^n \longrightarrow \mathbb{1} ,$$

the exact sequence *splits*, i.e., becomes like a sequence for a product, as we obtain an isomorphism

$$(i, \sigma) : N \times \mathbb{T}^n \xrightarrow{\simeq} \mathbb{T}^d .$$

The action of the $\mathbb{T}^n$ factor (or, more rigorously, $\sigma(\mathbb{T}^n) \subset \mathbb{T}^d$) descends to the quotient $M_\Delta = Z/N$.

It remains to show that the $\mathbb{T}^n$-action on M_Δ is hamiltonian with appropriate moment map.

Consider the diagram

$$\begin{array}{l} Z \overset{j}{\hookrightarrow} \mathbb{C}^d \xrightarrow{\phi} (\mathbb{R}^d)^* \simeq \mathfrak{n}^* \oplus (\mathbb{R}^n)^* \xrightarrow{\sigma^*} (\mathbb{R}^n)^* \\ p \downarrow \\ M_\Delta \end{array}$$

where the last horizontal map is simply projection onto the second factor. Since the composition of the horizontal maps is constant along N-orbits, it descends to a map

$$\mu_\Delta : M_\Delta \longrightarrow (\mathbb{R}^n)^*$$

which satisfies

$$\mu_\Delta \circ p = \sigma^* \circ \phi \circ j .$$

By Exercise 1.5.6 on reduction for product groups, this is a moment map for the action of $\mathbb{T}^n$ on $(M_\Delta, \omega_\Delta)$. Finally, the image of μ_Δ is:

$$\mu_\Delta(M_\Delta) = (\mu_\Delta \circ p)(Z) = (\sigma^* \circ \phi \circ j)(Z) = (\sigma^* \circ \Pi^*)(\Delta) = \Delta ,$$

because $\phi(Z) = \Pi^*(\Delta)$ and $\sigma^* \circ \Pi^* = (\Pi \circ \sigma)^* = \mathrm{id}$. $\square$

The above $\mathbb{T}^n$-action is effective because $\mathbb{T}^d$, and hence $\mathbb{T}^n$, acts freely on the open dense subset

$$\phi^{-1}\left(\Pi^*(\Delta^o)\right) \subset Z \ ,$$

where Δ^o denotes the interior of Δ.

We conclude that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ is the required toric manifold corresponding to Δ.

2.3 Discussion of Delzant's Correspondence

Delzant's theorem asserts that the map

$$\begin{array}{rcl} \{\text{symplectic toric manifolds}\} & \longrightarrow & \{\text{Delzant polytopes}\} \\ (\text{mod equivalence}) & & (\text{mod translation}) \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M) \ . \end{array}$$

is well-defined and bijective.

- In the previous section, we saw that it is indeed surjective.
- We will see now that it is well-defined.
- In the section after next, we will see that it is indeed injective.

Moreover, in this section, we later review the main idea behind Delzant's construction, check that the moment polytope is the orbit space of a symplectic toric manifold, and discuss concrete examples.

In Section 1.4, we had already observed that equivalent (i.e. isomorphic) symplectic toric manifolds have the same moment map up to a constant, hence have the same moment polytope up to translation. It remains to show that the moment polytope is *Delzant*.

Proposition 2.3.1. *Let $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold. Then the image Δ of μ is a Delzant polytope.*

Proof. By the Atiyah-Guillemin-Sternberg convexity theorem (Theorem 1.3.9) the image Δ is the convex hull of the images of the fixed points of the action.

Let τ be a vertex of Δ. Then there is $p \in M$ fixed by $\mathbb{T}^n$ and with $\mu(p) = \tau$.

By the equivariant Darboux theorem (Theorem 1.4.7), we can find a Darboux chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that:

- the neighborhood $\mathcal{U}$ is $\mathbb{T}^n$-invariant,
- the symplectic form becomes $\omega_{\mathcal{U}} = \sum_k dx_k \wedge dy_k = \frac{i}{2}\sum_k dz_k \wedge d\bar{z}_k$ where $z_k = x_k + iy_k$, $\bar{z}_k = x_k - iy_k$,

- in these coordinates the action of $\mathbb{T}^n$ is linear:

$$(e^{i\theta_1}, \ldots, e^{i\theta_k}) \cdot (z_1, \ldots, z_n) = \left(e^{i \sum_j \lambda_j^{(1)} \theta_j} z_1, \ldots, e^{i \sum_j \lambda_j^{(n)} \theta_j} z_n \right) ,$$

  where $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^n$ are the corresponding weights,

- thus the moment map has the form:

$$\mu_{\mathcal{U}}(z_1, \ldots, z_n) = \tau + \tfrac{1}{2} \sum_{k=1}^{n} \lambda^{(k)} |z_k|^2 .$$

Moreover, the weights $\lambda^{(1)}, \ldots, \lambda^{(n)}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ because the action is effective (cf. Exercise 1.4.8 (a)). This shows that the image of this neighborhood $\mathcal{U}$ by μ is of the form

$$\tau + \sum_{k=1}^{n} t_k \lambda^{(k)} \quad \text{with} \quad t_k \geqslant 0 ,$$

which by itself satisfies simplicity, rationality and smoothness.

Moreover, by the Atiyah-Guillemin-Sternberg theorem the levels of μ are connected, in particular, the level $\mu^{-1}(\tau)$ is connected. The above form shows that $\mu^{-1}(\tau) = \{p\}$, therefore the preimage of a neighborhood of τ is completely described by the model above and Delzant's conditions are globally satisfied. □

Exercise 2.3.2. *Use the previous proof to show that the fixed points of a symplectic toric manifold are isolated and the moment map of a symplectic toric manifold maps the fixed points of the action bijectively onto the vertices of the moment polytope. (This last fact will be generalized in Theorem 2.3.5.)*

Idea behind Delzant's construction:

The main idea of Delzant's construction is that the space $\mathbb{R}^d$ is *universal* in the sense that any n-dimensional (nondegenerate) polytope Δ with d facets can be obtained by intersecting the positive orthant $\mathbb{R}^d_+$ with an affine plane A. (We now identify $\mathbb{R}^n$ with its dual.) Given Δ, to construct A first write Δ as:

$$\Delta = \{x \in \mathbb{R}^n \mid \langle x, v_k \rangle \geqslant \lambda_k,\ k = 1, \ldots, d\} .$$

Define

$$\begin{array}{cccc} \Pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_k & \longmapsto & v_k \end{array} \qquad \text{with dual map} \qquad \Pi^* : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^d .$$

Then $\Pi^* - \lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ is an injective affine map, where $\lambda = (\lambda_1, \ldots, \lambda_d)$. Let A be the image of $\Pi^* - \lambda$. Then A is an n-dimensional affine space.

Lemma 2.3.3. *We have the equality* $(\Pi^* - \lambda)(\Delta) = \mathbb{R}^d_+ \cap A$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\begin{array}{rcl} (\Pi^* - \lambda)(x) \in \mathbb{R}^d_+ & \Longleftrightarrow & \langle \Pi^*(x) - \lambda, e_k \rangle \geqslant 0, \forall i \\ & \Longleftrightarrow & \langle x, \Pi(e_k) \rangle - \lambda_k \geqslant 0, \forall i \\ & \Longleftrightarrow & \langle x, v_k \rangle \geqslant \lambda_k, \forall i \\ & \Longleftrightarrow & x \in \Delta \ . \end{array}$$

$\square$

We conclude that $\Delta \simeq \mathbb{R}^d_+ \cap A$.

Now $\mathbb{R}^d_+$ is the image of the moment map for the standard hamiltonian action of $\mathbb{T}^d$ on $\mathbb{C}^d$

$$\begin{array}{rcl} \phi : \mathbb{C}^d & \longrightarrow & \mathbb{R}^d \\ (z_1, \ldots, z_d) & \longmapsto & \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \end{array}$$

and we assume that Δ is *Delzant*. Then the following facts hold:

- The set $\phi^{-1}(A) \subset \mathbb{C}^d$ is a compact submanifold. Let $i : \phi^{-1}(A) \hookrightarrow \mathbb{C}^d$ denote inclusion. Then $i^*\omega_0$ is a closed 2-form whose kernel is an integrable distribution. The corresponding foliation is called the **null foliation**.
- The null foliation of $i^*\omega_0$ is a principal fibration, so we take the quotient:

$$\begin{array}{ccccc} N & \circlearrowright & \phi^{-1}(A) & & \\ & & \downarrow p & & \\ & & M_\Delta & := & \phi^{-1}(A)/N \end{array}$$

  with induced (reduced) symplectic form ω_Δ satisfying $p^*\omega_\Delta = i^*\omega\omega_0$.
- The (non-effective) action of $\mathbb{T}^d = N \times \mathbb{T}^n$ on $\phi^{-1}(A)$ has a "moment map" with image $\phi(\phi^{-1}(A)) = \Delta$. By "moment map" we mean a map satisfying the usual definition even though the closed 2-form is not symplectic.

There is a remaining action of $\mathbb{T}^n$ on M_Δ which is hamiltonian with a moment map $\mu_\Delta : M_\Delta \to \mathbb{R}^n$ defined by the commutative diagram

$$\begin{array}{ccccc} \phi^{-1}(A) & \stackrel{j}{\hookrightarrow} & \mathbb{C}^d & \stackrel{\phi}{\longrightarrow} & \mathbb{R}^d \\ p \downarrow & & & & \downarrow \mathrm{pr}_2 \\ M_\Delta & & \stackrel{\mu_\Delta}{\dashrightarrow} & & \mathbb{R}^n \end{array}$$

where $\mathrm{pr}_2 : \mathbb{T}^d = N \times \mathbb{T}^n \to \mathbb{T}^n$, resp. $\mathrm{pr}_2 : \mathbb{R}^d = \mathfrak{n} \times \mathbb{R}^n \to \mathbb{R}^n$ is projection onto the second factor.

The moment polytope of a symplectic toric manifold is its orbit space:

Exercise 2.3.4. *As an experiment, consider the standard* $\mathbb{T}^3$*-action on* $(\mathbb{CP}^3, \omega_{_{FS}})$,

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] \ ,$$

with moment map

$$\mu[z_0 : z_1 : z_2 : z_3] = \frac{1}{2(|z_0|^2+|z_1|^2+|z_2|^2+|z_3|^2)}\left(|z_1|^2, |z_2|^2, |z_3|^2\right) \ .$$

Exhibit explicitly the subsets of $\mathbb{CP}^3$ for which the stabilizer under this action is $\{\mathbb{1}\}$, a circle, a 2-torus and $\mathbb{T}^3$. Show that the images of these subsets under the moment map are the interior, the facets, the edges and the vertices of $\Delta = \mu(\mathbb{CP}^3)$, respectively. Given $x \in \Delta$, how many $\mathbb{T}^3$-orbits is $\mu^{-1}(x)$?

Theorem 2.3.5. *For any $x \in \Delta$, we have that $\mu_\Delta^{-1}(x)$ is a single $\mathbb{T}^n$-orbit. Moreover, the dimension of that orbit is equal to the dimension of the smallest face to which x belongs.*

Proof. First consider the standard $\mathbb{T}^d$-action on $\mathbb{C}^d$ with moment map $\phi : \mathbb{C}^d \to \mathbb{R}^d$,

$$\phi(z_1, \ldots, z_d) = \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + (\lambda_1, \ldots, \lambda_d) \ .$$

Then $\phi^{-1}(y)$ is a single $\mathbb{T}^d$-orbit for any $y \in \phi(\mathbb{C}^d)$, its stabilizer (i.e., the stabilizer of any point on that orbit) is

$$\{(t_1, \ldots, t_d) \in \mathbb{T}^d \mid t_k = 1 \text{ whenever } \langle y, e_k\rangle > \lambda_k\}$$

and its dimension is equal to the number of indices k with $\langle y, e_k\rangle > \lambda_k$. The only fixed point is the origin mapping to the only vertex of the image.

Now let $x_0 \in \Delta$, take $y_0 = \Pi^*(x_0)$ and recall from Lemma 2.2.2 and the definition of Z that

$$y_0 \in \Delta' := \Pi^*(\Delta) \quad \Longleftrightarrow \quad y_0 \in \phi(Z) \quad \Longleftrightarrow \quad \phi^{-1}(y_0) \subseteq Z \ .$$

Then $\mu_\Delta^{-1}(x_0) = \phi^{-1}(y_0)/N$. But $\phi^{-1}(y_0)$ is a single $\mathbb{T}^d$-orbit where $\mathbb{T}^d = N \times \mathbb{T}^n$, hence $\mu_\Delta^{-1}(x_0)$ is a single $\mathbb{T}^n$-orbit.

Let F be the smallest face to which x_0 belongs and let m be the codimension of F. The face F is given as

$$F = \Delta \cap \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k\rangle = \lambda_k,\ k \in I_F\}$$

for some index subset $I_F \subset \{1, \ldots, d\}$ with cardinality $|I_F| = m$. Then y_0 belongs to the face of Δ' given by

$$\Delta' \cap \{y \in (\mathbb{R}^d)^* \mid \langle y, e_k\rangle = \lambda_k,\ k \in I_F\}$$

and thus has as stabilizer the m-dimensional subtorus

$$T_F := \{(t_1, \ldots, t_d) \in \mathbb{T}^d \mid t_k = 1 \text{ whenever } k \notin I_F\}$$

and the $\mathbb{T}^d$-orbit of y_0, namely $\phi^{-1}(y_0)$, is $(d - m)$-dimensional. It follows that the $\mathbb{T}^n$-orbit of x_0, namely $\mu_\Delta^{-1}(x_0)$, has stabilizer $\Pi(T_F)$. Since N acts freely on Z, we see that $\Pi(T_F)$ is also an m-dimensional torus and the orbit $\mu_\Delta^{-1}(x_0)$ has dimension

$$\dim\left(\mu_\Delta^{-1}(x_0)\right) = \underbrace{\dim\left(\phi^{-1}(y_0)\right)}_{d-m} - \underbrace{\dim N}_{d-n} = n - m \ .$$

$\square$

Therefore, in particular, for a symplectic toric manifold the moment polytope Δ is the orbit space.

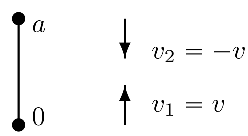
Concrete instances of Delzant's construction:

We will follow through the details of Delzant's construction for specific cases.

Example. We consider the case of $\Delta = [0, a] \subset \mathbb{R}^*$ $(n = 1, d = 2)$. Let $v = 1$ be the standard basis vector in $\mathbb{R}$. Then Δ is described by

$$\langle x, v \rangle \geqslant 0 \qquad \text{and} \qquad \langle x, -v \rangle \geqslant -a \ ,$$

so we have $v_1 = v$, $v_2 = -v$, $\lambda_1 = 0$ and $\lambda_2 = -a$.



The projection

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\Pi} & \mathbb{R} \\ e_1 & \longmapsto & v \\ e_2 & \longmapsto & -v \end{array}$$

has kernel equal to the span of $(e_1 + e_2)$, so that N is the diagonal subgroup of $\mathbb{T}^2 = S^1 \times S^1$. The exact sequences become

$$\begin{array}{ccccccccc} \mathbb{1} & \longrightarrow & N & \xrightarrow{i} & \mathbb{T}^2 & \xrightarrow{\Pi} & S^1 & \longrightarrow & \mathbb{1} \\ & & t & \longmapsto & (t,t) & & & & \\ & & & & (t_1, t_2) & \longmapsto & t_1 t_2^{-1} & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \xrightarrow{i} & \mathbb{R}^2 & \xrightarrow{\Pi} & \mathbb{R} & \longrightarrow & 0 \\ & & x & \longmapsto & (x,x) & & & & \\ & & & & (x_1, x_2) & \longmapsto & x_1 - x_2 & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^* & \xrightarrow{\Pi^*} & (\mathbb{R}^2)^* & \xrightarrow{i^*} & \mathfrak{n}^* & \longrightarrow & 0 \\ & & x & \longmapsto & (x,-x) & & & & \\ & & & & (x_1, x_2) & \longmapsto & x_1 + x_2 \ . & & \end{array}$$

We choose the moment map for the standard $\mathbb{T}^2$-action on $\mathbb{C}^2$:

$$\phi(z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2) + \underbrace{(0, -a)}_{(\lambda_1,\lambda_2)} \ .$$

The action of the diagonal subgroup $N = \{(e^{i\theta}, e^{i\theta}) \in S^1 \times S^1\}$ on $\mathbb{C}^2$,

$$(e^{i\theta}, e^{i\theta}) \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2) \ ,$$

has moment map

$$(i^* \circ \phi)(z_1, z_2) = \frac{1}{2}(|z_1|^2 + |z_2|^2) - a \ ,$$

with zero-level set

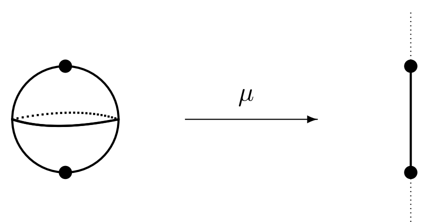
$$Z = (i^* \circ \phi)^{-1}(0) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 2a\} \ .$$

Hence, the reduced space is a projective space:

$$(i^* \circ \phi)^{-1}(0)/N = \mathbb{CP}^1 \ .$$

One can further check that the induced symplectic form is a multiple of the Fubini-Study form: $\omega_\Delta = 2a\omega_{\text{FS}}$; cf. Sections 1.4 and 1.5.

Here we see clearly the point-orbit correspondence given by the moment map. The boundary points of the moment polytope $\Delta = [0, a]$ correspond to the fixed points – North pole and South pole – whereas interior points correspond to free orbits – the other latitude circles.

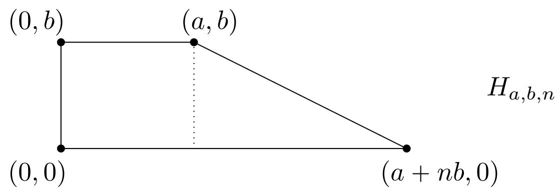


$\diamondsuit$

Exercise 2.3.6. *($\star$) Let Δ be the n-simplex in $\mathbb{R}^n$ spanned by the origin and the standard basis vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Show that the corresponding symplectic toric manifold is n-dimensional complex projective space, $M_\Delta = \mathbb{CP}^n$.*

Exercise 2.3.7. *($\star$) Which $2n$-dimensional toric manifolds have exactly $n + 1$ fixed points?*

Example. We consider Delzant's construction for the case of $\Delta = H_{a,b,n} \subset (\mathbb{R}^2)^*$. The manifolds we will obtain are known as **Hirzebruch surfaces**, $M_\Delta = \mathcal{H}_{a,b,n}$ [33].



The polytope $\Delta = H_{a,b,n}$ is described by

$$\begin{aligned}\Delta &= \{x \in (\mathbb{R}^2)^* \mid x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1 + nx_2 \leqslant a + nb,\ x_2 \leqslant b,\} \\ &= \{x \in (\mathbb{R}^2)^* \mid \langle x, \underbrace{(1,0)}_{v_1} \rangle \geqslant \underbrace{0}_{\lambda_1},\ \langle x, \underbrace{(0,1)}_{v_2} \rangle \geqslant \underbrace{0}_{\lambda_2}, \\ &\qquad \langle x, \underbrace{(-1,-n)}_{v_3} \rangle \geqslant \underbrace{-a-nb}_{\lambda_3},\ \langle x, \underbrace{(0,-1)}_{v_4} \rangle \geqslant \underbrace{-b}_{\lambda_4} \}\ .\end{aligned}$$

The projection

$$\begin{array}{ccl} \mathbb{R}^4 & \xrightarrow{\Pi} & \mathbb{R}^2 \\ e_1 & \longmapsto & v_1 = (1,0) \\ e_2 & \longmapsto & v_2 = (0,1) \\ e_3 & \longmapsto & v_3 = (-1,-n) \\ e_4 & \longmapsto & v_4 = (0,-1) \end{array}$$

has kernel equal to the span of $\{e_2 + e_4, e_1 + ne_2 + e_3\}$, so that

$$N := \{\left(e^{i\beta}, e^{i(\alpha+n\beta)}, e^{i\beta}, e^{i\alpha}\right)\} \subset \mathbb{T}^4,$$

the exact sequences are

$$\begin{array}{ccccccccc} \mathbb{1} & \longrightarrow & N & \xrightarrow{i} & \mathbb{T}^4 & \xrightarrow{\Pi} & \mathbb{T}^2 & \longrightarrow & \mathbb{1} \\ & & (a,b) & \longmapsto & (b, ab^n, b, a) & & & & \\ & & & & (t_1,t_2,t_3,t_4) & \longmapsto & (t_1 t_3^{-1}, t_2 t_3^{-n} t_4^{-1}) & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \xrightarrow{i} & \mathbb{R}^4 & \xrightarrow{\Pi} & \mathbb{R}^2 & \longrightarrow & 0 \\ & & (x,y) & \longmapsto & (y, x+ny, y, x) & & & & \\ & & & & (x_1,x_2,x_3,x_4) & \longmapsto & (x_1\text{-}x_3, x_2\text{-}nx_3\text{-}x_4) & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{R}^2)^* & \xrightarrow{\Pi^*} & (\mathbb{R}^4)^* & \xrightarrow{i^*} & \mathfrak{n}^* & \longrightarrow & 0 \\ & & (x,y) & \longmapsto & (x, y, \text{-}x\text{-}ny, \text{-}y) & & & & \\ & & & & (x_1,x_2,x_3,x_4) & \longmapsto & (x_2 + x_4, x_1 + nx_2 + x_3)\ . & & \end{array}$$

and the action of N has moment map

$$(i^* \circ \phi)(z_1,z_2,z_3,z_4) = \tfrac{1}{2}(|z_2|^2 + |z_4|^2, |z_1|^2 + n|z_2|^2 + |z_3|^2) + (-b, -a-nb)\ ,$$

with zero-level set

$$\begin{aligned} Z = (i^* \circ \phi)^{-1}(0) = &\{(z_1,z_2,z_3,z_4) \in \mathbb{C}^4 : \\ &\qquad |z_2|^2 + |z_4|^2 = 2b, |z_1|^2 + n|z_2|^2 + |z_3|^2 = 2(a+nb)\}\ . \end{aligned}$$

Hence, the reduced space is a so-called *Hirzebruch (complex) surface*

$$\mathcal{H}_{a,b,n} := Z/\sim\ ,$$

where the equivalence relation given by N is

$$(z_1, z_2, z_3, z_4) \sim \left(e^{i\beta} z_1, e^{i(\alpha+n\beta)} z_2, e^{i\beta} z_3, e^{i\alpha} z_4\right) .$$

$\diamondsuit$

Remark. One can see that each 4-dimensional manifold $\mathcal{H}_{a,b,n}$ is a sphere bundle over a sphere, by considering the projection $p : \mathcal{H}_{a,b,n} \to \mathbb{CP}^1$ induced by the map

$$\begin{array}{rccc} \widetilde{p}: & Z & \longrightarrow & \mathbb{CP}^1 \\ & (z_1, z_2, z_3, z_4) & \longmapsto & [z_1 : z_3] \ . \end{array}$$

The map p is well-defined because $\widetilde{p}$ is invariant by N. Moreover, one can check that the fibers of p are copies of $\mathbb{CP}^1$ and that it is locally trivial, hence actually a fibration. In particular, for $n = 0$ we get a product of spheres,

$$\mathcal{H}_{a,b,0} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$$

and all $\mathcal{H}_{a,b,n}$ for n even are diffeomorphic to this,[6] whereas all $\mathcal{H}_{a,b,n}$ for n odd are diffeomorphic to $\mathcal{H}_{a,b,1}$, the nontrivial S^2-bundle over S^2. Note that all these $\mathcal{H}_{a,b,n}$'s are distinct as complex manifolds, as well as symplectic manifolds (as well as symplectic toric manifolds). $\diamondsuit$

Exercise 2.3.8. *What are all the 4-dimensional symplectic toric manifolds that have exactly four fixed points?*

> ***Hint:*** *Exercise 2.1.4, observation after Theorem 2.3.5 and previous example.*

2.4 Lerman's Construction

In the 1990's Eugene Lerman gave an alternative version of Delzant's construction using his *symplectic cutting* trick. Whereas we will deal with the original symplectic cutting technique in Chapter 3, we will now follow the exposition in [47, Ch.7, §5] to do cutting w.r.t. a Delzant polytope by working with the cotangent bundle of the torus, $T^*(\mathbb{T}^n)$.

[6] Orientable sphere bundles over the sphere S^2 are trivializable over each half-sphere and hence obtained by gluing two trival bundles over a disk along the boundary by a map from the equator S^1 to the group of orientation preserving diffeomorphisms of S^2. Milnor showed that this group of diffeomorphisms retracts onto SO(3), and we have that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$. So there are only two diffeomorphism classes of such bundles: the class of the trivial bundle $S^2 \times S^2$ and the class of *the* nontrivial bundle.

Symplectic cutting w.r.t. a Delzant polytope:

Let Δ be a Delzant polytope given as an intersection of halfspaces as

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle \geqslant \lambda_k,\ k = 1, \ldots, d\} ,$$

where the $v_k \in \mathbb{Z}^n$, $k = 1, \ldots, d$, are the primitive inward-pointing normal vectors to the facets of Δ and where $\lambda_k \in \mathbb{R}$. We set $\lambda := (\lambda_1, \ldots, \lambda_d)$.

We trivialize the tangent bundle $T(\mathbb{T}^n)$ as $\mathbb{T}^n \times \mathfrak{t}^n$ by invariant vector fields and, correspondingly, the cotangent bundle $T^*(\mathbb{T}^n)$ as $\mathbb{T}^n \times (\mathfrak{t}^n)^*$. We equip $T^*(\mathbb{T}^n) \simeq \mathbb{T}^n \times (\mathfrak{t}^n)^*$ with the symplectic form given by

$$\sum_{k=1}^{n} d\xi_k \wedge d\theta_k$$

with respect to cotangent coordinates $(\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n)$, to which we refer as **action coordinates** ξ_k and **angle coordinates** θ_k. Note the sign convention for this symplectic form.

We call **standard action** to the action of $\mathbb{T}^n$ on its cotangent bundle $T^*\mathbb{T}^n$ by the lift of its multiplication action on itself.[7] W.r.t. the action-angle coordinates above, the element $(e^{i\theta'_1}, \ldots, e^{i\theta'_n}) \in \mathbb{T}^n$ acts by

$$(\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n) \longmapsto (\theta_1 + \theta'_1, \ldots, \theta_n + \theta'_n, \xi_1, \ldots, \xi_n) .$$

This action is hamiltonian with moment map given by projection onto the second factor $\mathrm{pr}_2 : \mathbb{T}^n \times (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$, $(\theta, \xi) \mapsto \xi$.

Now consider the Lie group homomorphism

$$\begin{array}{rcl} \rho_\Delta : \mathbb{T}^d & \longrightarrow & \mathbb{T}^n \\ (e^{i\alpha_1}, \ldots, e^{i\alpha_d}) & \longmapsto & \exp\left(\sum_{k=1}^{d} \alpha_k v_k\right) . \end{array}$$

Notice that the differential of this homomorphism is the linear map from Section 2.2

$$\begin{array}{rrcl} D\rho_\Delta \simeq \Pi : & \mathfrak{t}^d \simeq \mathbb{R}^d & \longrightarrow & \mathfrak{t}^n \simeq \mathbb{R}^n \\ & e_k & \longmapsto & v_k . \end{array}$$

The action of $\mathbb{T}^d$ on $T^*(\mathbb{T}^n)$ via the composition of ρ_Δ with the standard $\mathbb{T}^n$-action is hamiltonian with moment map ν_Δ given by the composition of the projection pr_2 with the adjoint of $D\rho_\Delta$ up to a constant. We fix the following moment map:

$$\begin{array}{rrcl} \nu_\Delta = (D\rho_\Delta)^* \circ \mathrm{pr}_2 : & \mathbb{T}^n \times (\mathfrak{t}^n)^* & \to & (\mathfrak{t}^d)^* \\ & (\theta, \xi) & \longmapsto & \sum_{k=1}^{d} \langle \xi, v_k \rangle e_k - \lambda , \end{array}$$

[7] If $\varphi : X \to X$ is a diffeomorphism, then its (cotangent) *lift* is $\varphi^\sharp : T^*X \to T^*X$, $(x, \xi) \mapsto \Big(\varphi(x), \big((d\varphi_x)^{-1}\big)^* \xi\Big)$. Then $\varphi^\sharp$ is a symplectomorphism for any (nonzero) multiple of the canonical symplectic form, $\sum dx_k \wedge d\xi_k$.

where we added the constant $-\lambda$ for later convenience.

Let $\mathbb{T}^d$ act on $(\mathbb{C}^d, \frac{i}{2}\sum dz_k \wedge d\bar{z}_k)$ by

$$(e^{i\alpha_1}, \dots, e^{i\alpha_d}) \cdot (z_1, \dots, z_d) = (e^{-i\alpha_1} z_1, \dots, e^{-i\alpha_d} z_d)$$

with moment map

$$(z_1, \dots, z_d) \longmapsto -\tfrac{1}{2}(|z_1|^2, \dots, |z_d|^2) \ .$$

Proposition 2.4.1. *Given a Delzant polytope Δ, consider the product manifold*

$$(T^*\mathbb{T}^n) \times \mathbb{C}^d$$

with:

- *product symplectic form*

$$\sum_{k=1}^{n} d\xi_k \wedge d\theta_k + \tfrac{i}{2} \sum_{k=1}^{d} dz_k \wedge d\bar{z}_k \ ,$$

- *product action of $\mathbb{T}^d$, where $\mathbb{T}^d$ acts on each factor as above, and*
- *moment map*

$$((\theta, \xi), z) \quad \longmapsto \quad \sum_{k=1}^{d} \langle \xi, v_k \rangle e_k - \lambda - \tfrac{1}{2}(|z_1|^2, \dots, |z_d|^2) \ .$$

Then the $\mathbb{T}^d$-action is free on the zero level set of the moment map, so the reduced space is a symplectic manifold. Moreover, this reduced space is naturally a $2n$-dimensional symplectic toric manifold with moment map image Δ.

We denote this reduced space by $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$ and call it **Lerman's symplectic toric manifold associated with Δ**.

Proof. Let $((\theta, \xi), z) \in T^*\mathbb{T}^n \times \mathbb{C}^d$ be a point in the zero level, i.e.,

$$\sum_{k=1}^{d} \langle \xi, v_k \rangle e_k = \lambda + \tfrac{1}{2}(|z_1|^2, \dots, |z_d|^2) \ .$$

If $z_k \neq 0$, then the kth factor of $\mathbb{T}^d$ acts freely on $((\theta, \xi), z)$. Thus we need only worry about the set I of indices k with $z_k = 0$. For such an index $k \in I$, we have that $\langle \xi, v_k \rangle = \lambda_k$. Let

$$T_I := \{(t_1, \dots, t_d) \in \mathbb{T}^d \mid t_k = 1 \text{ whenever } k \notin I\} \ .$$

By the Delzant condition (see the end of the proof of Claim 1 in Section 2.2), the restriction to T_I of the homomorphism ρ_Δ,

$$\rho_\Delta|_{T_I} : T_I \longrightarrow \mathbb{T}^n \ ,$$

is injective. Therefore, since $\mathbb{T}^n$ acts freely on $T^*(\mathbb{T}^n)$, so does T_I. This shows that the $\mathbb{T}^d$-action on the zero level set is free, so, by the Marsden-Weinstein-Meyer theorem (Theorem 1.5.3), the reduced space is a symplectic manifold, $(E^\Delta, \omega^\Delta)$.

This reduced space inherits a hamiltonian $\mathbb{T}^n$-action induced by the standard $\mathbb{T}^n$-action on $T^*(\mathbb{T}^n) = \mathbb{T}^n \times (\mathfrak{t}^n)^*$ with moment map $[(\theta, \xi), z] \mapsto \xi$ (this is well-defined since the $\mathbb{T}^d$-action preserves ξ). A point $\xi \in (\mathfrak{t}^n)^*$ is in the image by the moment map of some $((\theta, \xi), z) \in T^*\mathbb{T}^n \times \mathbb{C}^d$ from the zero level exactly when we can find z such that

$$\sum_{k=1}^{d} \langle \xi, v_k \rangle e_k = \lambda + \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \ ,$$

that is, when

$$\langle \xi, v_k \rangle \geqslant \lambda_k \ , \quad k = 1, \ldots, d \ ,$$

that is, when $\xi \in \Delta$. □

Interpretation of Lerman's construction:

We may view Δ as a *manifold with corners* in $(\mathfrak{t}^n)^*$. At every point x in the interior of a face F, the tangent space $T_x\Delta$ is the subspace of $(\mathfrak{t}^n)^* \simeq (\mathbb{R}^n)^*$ tangent to F.

The **interior** of Δ is the set of points given by strict inequalities:

$$\Delta^o := \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle > \lambda_k, \ k = 1, \ldots, d\}$$

and this is a manifold (just an open subset of euclidean space).

Essentially, what we do is take the product $\mathbb{T}^n \times \Delta$. Let x lie in the interior $\mathbb{T}^n \times \Delta^o$. The tangent space at x is $\mathfrak{t}^n \times (\mathfrak{t}^n)^* \simeq \mathbb{R}^n \times (\mathbb{R}^n)^*$. Define ω_x^o by:

$$\omega_x^o(v, \xi) = -\xi(v) = -\omega_x^o(\xi, v) \quad \text{and} \quad \omega_x^o(v, v') = \omega_x^o(\xi, \xi') = 0 \ ,$$

for all $v, v' \in \mathfrak{t}^n$ and $\xi, \xi' \in (\mathfrak{t}^n)^*$. Then ω^o is a closed nondegenerate 2-form on the interior of $\mathbb{T}^n \times \Delta$.

We will see that we can *close* the open subset $\mathbb{T}^n \times \Delta^o$ in a smooth and symplectic way.

At corners, there are tangent directions missing in $(\mathfrak{t}^n)^*$, so the extension of ω^o above would be a degenerate pairing. The missing directions at each corner point x are the normal directions to the facets of Δ meeting at that point. For all ξ in the tangent space to the kth facet, we have $\omega(v_k, \xi) := -\xi(v_k) = 0$, where v_k is the vector defining that facet, and v_k spans the annihilator of that tangent space. We fix the degeneracy by eliminating in the $\mathfrak{t}^n$ component of the tangent space the directions of the vectors v_k defining the facets that meet at the point x. To do this, we collapse the orbit of the subgroup of $\mathbb{T}^n$ generated by those v_k's. This is a blow-down process and the result is a smooth compact manifold. We thus simultaneously eliminate corners and singularities of ω .

Finally, $\mathbb{T}^n$ acts on $\mathbb{T}^n \times \Delta$ by multiplication on the $\mathbb{T}^n$ factor. The moment map for this action is projection onto the Δ factor. We thus obtain $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$.

Note that the interior $\mathbb{T}^n \times \Delta^o$ with symplectic form ω^o embeds symplectically into $(E^\Delta, \omega^\Delta)$ and here we have **action-angle coordinates**, namely the ξ_k's and the θ_k's, with respect to which the symplectic form is

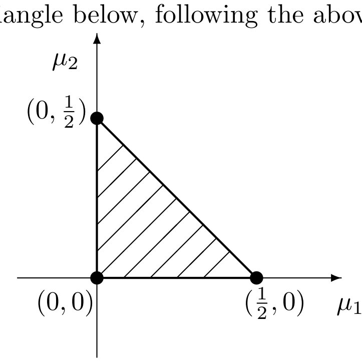
$$\omega^\Delta|_{\mathbb{T}^n \times \Delta^o} = \omega^o = \sum_{k=1}^n d\xi_k \wedge d\theta_k \ .$$

Example. Consider

$$(S^2, \omega = dh \wedge d\theta, S^1, \mu = h) \ ,$$

where S^1 acts on S^2 by rotation (with vector field $\frac{\partial}{\partial \theta}$). The image of μ is the line segment $I = [-1, 1]$. The product $S^1 \times I$ is an open-ended cylinder. By collapsing each circle end of the cylinder to a point, we recover the 2-sphere. Note that the notation for the symplectic form is only valid in the interior $(-1, 1) \times S^1$, so that is actually $\omega^o = dh \wedge d\theta$ presuming the extension. $\diamondsuit$

Example. We want to build $\mathbb{CP}^2$ from $\mathbb{T}^2 \times \Delta$ where Δ is the right-angled isosceles triangle below, following the above construction.



Consider, for instance, the edge of the triangle lying on the x-axis, whose tangent vectors ξ satisfy $\langle \xi, v_1 \rangle = 0$, where $v_1 = (0, 1) \in \mathfrak{t}^2$. For points of that edge, we collapse the subgroup of $\mathbb{T}^2$ generated by v_1, namely, the second circle factor. Similarly, for the edge of the triangle lying on the y-axis we collapse the first circle factor in $\mathbb{T}^2$, and for the hypothenuse we collapse the diagonal circle $\{(e^{i\theta}, e^{i\theta}) \in \mathbb{T}^2\}$. At the vertices (points lying in two facets), we collapse the whole $\mathbb{T}^2$.

All together, after the above collapses, the map

$$\begin{array}{ccc} \mathbb{T}^2 \times \Delta & \longrightarrow & \mathbb{CP}^2 \\ (e^{i\theta_1}, e^{i\theta_2}), (\mu_1, \mu_2) & \longmapsto & \left[\sqrt{1 - 2(\mu_1 + \mu_2)} : \sqrt{2\mu_1} e^{i\theta_1} : \sqrt{2\mu_2} e^{i\theta_2}\right] \ . \end{array}$$

provides an equivariant symplectomorphism. $\diamondsuit$

Exercise 2.4.2. *Build $\mathbb{CP}^n$ from $\mathbb{T}^n \times \Delta^n$ where Δ^n is simplex with vertices at the origin and at the points*

$$(\tfrac{1}{2}, 0, \ldots, 0), (0, \tfrac{1}{2}, 0, \ldots, 0), \ldots (0, \ldots, 0, \tfrac{1}{2}) \ .$$

Exercise 2.4.3. *($\star$) Build a Hirzebruch surface from $\mathbb{T}^2 \times \Delta$ where $\Delta = H_{a,b,n}$ is the polytope in Section 2.3.*

2.5 Proof of Uniqueness

For Delzant's theorem, it remains to prove:

Theorem 2.5.1. *Let $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold with moment polytope $\Delta := \mu(M)$. Then $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ is equivariantly symplectomorphic to Lerman's symplectic toric manifold associated with Δ, $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$, defined in Section 2.4*

The original proof due to Delzant uses a sheaf-theoretic argument. We will sketch here an alternative proof going back to ideas of Lerman [40] and Meinrenken [47].

Definition 2.5.2. *Let Δ be an n-dimensional Delzant polytope with d facets and let $(M, \omega, \mathbb{T}^n, \mu)$ be a hamiltonian $\mathbb{T}^n$-space. The* ***cut space of $(M, \omega, \mathbb{T}^n, \mu)$ w.r.t. Δ*** *is the hamiltonian $\mathbb{T}^n$-space obtained by symplectic reduction at level 0 of the the product manifold*

$$M \times \mathbb{C}^d$$

with:

- *product symplectic form*

$$\omega + \tfrac{i}{2} \sum_{k=1}^{d} dz_k \wedge d\bar{z}_k \ ,$$

- *diagonal action of $\mathbb{T}^d$, where $\mathbb{T}^d$ acts on M via the composition with the homomorphism $\rho_\Delta : \mathbb{T}^d \to \mathbb{T}^n$ from Section 2.4 and on $\mathbb{C}^d$ as in Proposition 2.4.1, and*

- *moment map*

$$(p, z) \quad \longmapsto \quad \sum_{k=1}^{d} \langle \mu(p), v_k \rangle e_k - \lambda - \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \ .$$

In particular, we saw in Section 2.4 that the cut space of

$$(T^*\mathbb{T}^n, \sum d\xi_k \wedge d\theta_k, \mathbb{T}^n, \mathrm{pr}_2)$$

w.r.t. Δ is Lerman's symplectic toric manifold associated with Δ, $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$.

Remark. If we extend the above construction to a half-line (instead of Δ), say $[\lambda, +\infty)$, we get the first instance of cut space defined by Lerman in [40]; cf. Section 3.3. $\diamondsuit$

Proof. (of Theorem 2.5.1)

Warning: This proof uses concepts such as *compatible almost complex structure*, *principal bundles* and *symplectic neighborhood*, which we have not yet discussed in these notes.

The idea is to present $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ as a cut space w.r.t. its moment polytope Δ of a hamiltonian torus space $(\widetilde{M}^{2n}, \widetilde{\omega}, \mathbb{T}^n, \widetilde{\mu})$ with *free* $\mathbb{T}^n$-action. Then the moment map $\widetilde{\mu} : \widetilde{M}^{2n} \to \mathbb{R}^n$ may be viewed as a lagrangian (torus) fibration over its image and we can introduce action-angle coordinates $(\xi_k,\, \theta_k)$, thus identifying $\widetilde{M}^{2n}$ (up to equivariant symplectomorphism) with an open subset of $T^*\mathbb{T}^n$. It then follows that $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ is the cut space w.r.t. Δ of

$$(T^*\mathbb{T}^n, \sum d\xi_k \wedge d\theta_k, \mathbb{T}^n, \mathrm{pr}_2) \ ,$$

i.e., is (equivariantly symplectomorphic to) Lerman's symplectic toric manifold $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$.

Here is a sketch of how to construct such a $(\widetilde{M}^{2n}, \widetilde{\omega}, \mathbb{T}^n, \widetilde{\mu})$.

Let

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k\rangle \geqslant \lambda_k,\ k = 1, \ldots, d\} \ ,$$

let $k_1 \in \{1, \ldots, d\}$, consider the corresponding facet

$$\Delta_1 := \Delta \cap \{x \in (\mathbb{R}^n)^* \mid \langle x, v_{k_1}\rangle = \lambda_{k_1}\}$$

and let $S = \mu^{-1}(\Delta_1) \subset M$ be the preimage of that facet. Then S is a connected component of the fixed point set of the subgroup

$$T := \{\exp(tv_{k_1}) \mid t \in \mathbb{R}\}$$

and is a symplectic submanifold of codimension 2. We denote by ω_S the symplectic form on S. We consider its *symplectic normal bundle*[8], TS^ω. We can equip TS^ω with the structure of an hermitian line bundle (this involves choosing a *compatible almost complex structure* on M) and thus extract its unit circle bundle, $\Pi_Q : Q \to S$, which is a $\mathbb{T}^n$-equivariant principal circle bundle and satisfies $TS^\omega = Q \times_{S^1} \mathbb{C}$. Choose a corresponding $\mathbb{T}^n$-invariant connection form α, i.e., a $\mathbb{T}^n$-invariant 1-form on Q satisfying, for the vertical vector field v generated by the circle action:

$$\imath_v \alpha = 1 \quad \text{ and } \quad \imath_v d\alpha = 0 \ .$$

[8]The **symplectic normal bundle** of a symplectic submanifold $S \subset M$ is the vector bundle over S whose fiber at each point s is given by the symplectic orthogonal of T_sS in (T_sM, ω_s).

Consider the closed 2-form on $Q \times \mathbb{C}$:

$$\omega_{Q\times\mathbb{C}} := \Pi_Q^*\omega_S + \omega_{\mathbb{C}} + \tfrac{1}{2}d\left(|z|^2\alpha\right) \ .$$

This 2-form is invariant for the circle action and vanishes on the vertical vector field,

$$\imath_{v-\frac{\partial}{\partial\theta}}\omega_{Q\times\mathbb{C}} = \imath_{-\frac{\partial}{\partial\theta}}\omega_{\mathbb{C}} + \imath_v r dr \wedge \alpha = rdr - rdr = 0 \ ,$$

so it descends to a closed 2-form ω_{TS^ω}. Moreover, ω_{TS^ω} is nondegenerate near its zero section, S_0. It follows that there exists an equivariant symplectomorphism between tubular neighborhoods of S in M and of S_0 in TS^ω.

Now the symplectic normal bundle TS^ω may be viewed as a *cut space* w.r.t. the interval $[0,+\infty)$ of the hamiltonian S^1-space $Q \times \mathbb{R}$ equipped with the symplectic form

$$\omega_{Q\times\mathbb{R}} := \Pi_Q^*\omega_S + d\,(t\alpha)$$

(where t is the coordinate function on $\mathbb{R}$). There is a natural $\mathbb{T}^n$-equivariant diffeomorphism between $Q \times \mathbb{R}^+$ and $TS^\omega \backslash S_0$ preserving the 2-forms. We can thus glue $M\backslash S$ with a small neighborhood of Q in $Q \times \mathbb{R}$, to obtain a new hamiltonian $\mathbb{T}^n$-space $(M_1, \omega_1, \mathbb{T}^n, \mu_1)$ with one orbit type stratum less. The original space is obtained from this M_1 by cutting with respect to the affine half-space

$$\mathcal{H}_1 := \{x \in (\mathbb{R}^n)^* \mid \langle x, v_{k_1}\rangle \geqslant \lambda_{k_1}\} \ .$$

Continuing in this fashion for each facet of Δ, we obtain a sequence of spaces $M_1, M_2, \ldots, M_d$ with the property that the final space M_d has a *free* action and each M_k is the cut space of M_{k-1} w.r.t. $\mathcal{H}_k$, setting $M_0 := M$. Hence, we have

$$M = (M_1)_{\mathcal{H}_1} = (M_2)_{\mathcal{H}_1\cap\mathcal{H}_2} = \ldots = (M_d)_\Delta$$

and we set $M_d = \widetilde{M}$. □

Exercise 2.5.3. *What would be the classification of symplectic toric manifolds if, instead of the equivalence relation defined in Section 1.4, one considered to be equivalent those* $(M_j, \omega_j, \mathbb{T}^n, \mu_j)$, $j = 1, 2$, *related by:*

(a) *a* $\mathbb{T}^n$*-equivariant symplectomorphism* φ *sucht that* $\mu_1 = \mu_2 \circ \varphi$*?*

(b) *an isomorphism* $\lambda : \mathbb{T}^n \to \mathbb{T}^n$ *and a* λ*-equivariant*[9] *symplectomorphism* $\varphi : M_1 \to M_2$*?*

Hint: *The* ***general affine group****,* $\mathrm{AGL}(n;\mathbb{Z}) := \mathbb{R}^n \rtimes \mathrm{GL}(n;\mathbb{Z})$*, is the group of all invertible affine integral transformations, whose elements are compositions of linear maps in* $\mathrm{GL}(n;\mathbb{Z})$ *and translations by arbitrary vectors in* $\mathbb{R}^n$*.*

[9] λ-equivariance means that $\varphi(t \cdot p) = \lambda(t) \cdot \varphi(p)$ for all $p \in M_1$ and $t \in \mathbb{T}^n$. An isomorphism of $\mathbb{T}^n$ is given by an element of $\mathrm{GL}(n;\mathbb{Z})$ (those are the linear maps $\mathbb{R}^n \to \mathbb{R}^n$ that are isomorphisms of the lattice $(2\pi\mathbb{Z})^n$).

Chapter 3

Further Topics

This chapter goes on exploring *how to understand a toric manifold from its polytope.*

3.1 Homology of Symplectic Toric Manifolds

After reviewing the basics of Morse theory following [50], we compute the homology of symplectic toric manifolds using Morse theory; an appropriate Morse function is provided by a moment map with respect to a suitable circle subgroup.

Review of Morse theory

Let M be an m-dimensional manifold and let $f : M \to \mathbb{R}$ be a smooth function.

A point $q \in M$ is a **critical point** of f if $df_q = 0$. A critical point is **nondegenerate** if the **hessian matrix**

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_q$$

is nonsingular, where the x_i's are local coordinates near q. (The condition that the hessian matrix is nonsingular is independent of the choice of coordinates.) The hessian matrix defines a symmetric bilinear function $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by inner product

$$(v, w) \longmapsto \langle v, \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_q w\rangle$$

and also called the **hessian** of f at q relative to the local coordinates x_i; the hessian is in fact the expression in coordinates of a natural bilinear form on the tangent space at q.

The **index** of a bilinear function $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is the maximal dimension of a subspace of $\mathbb{R}^m$ where H is negative definite. The **nullity** of H is the dimension of its nullspace, that is, the subspace consisting of all $v \in \mathbb{R}^m$ such that $H(v,w) = 0$ for all $w \in \mathbb{R}^m$. Hence, a critical point q of $f : M \to \mathbb{R}$ is nondegenerate if and only if the hessian $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ has nullity equal to zero.

Let q be a nondegenerate critical point for $f : M \to \mathbb{R}$. The **index of f at q** is the index of the hessian $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. This is well-defined, i.e., the index is independent of the choice of local coordinates. Moreover, the **Morse lemma** states that there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_m)$ centered at q such that

$$f|_{\mathcal{U}} = f(q) - (x_1)^2 - \ldots - (x_\lambda)^2 + (x_{\lambda+1})^2 + \ldots + (x_m)^2 \ ,$$

where λ is the index of f at q. In particular, nondegenerate critical points are necessarily isolated.

A smooth function $f : M \to \mathbb{R}$ is a **Morse function** on M if all of its *critical points* are *nondegenerate*.

For $a \in \mathbb{R}$ and a Morse function f on M, we define the corresponding **sublevel set** as

$$M^a = f^{-1}(-\infty, a] = \{p \in M \mid f(p) \leqslant a\} \ .$$

Theorem 3.1.1. ***(Morse [51], Milnor [50])***
Let f be a Morse function on M.

(a) Let $a < b$ and suppose that the set $f^{-1}[a,b]$, consisting of all $p \in M$ with $a \leqslant f(p) \leqslant b$, is compact, and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.

(b) Let q be a nondegenerate critical point with index λ and $f(q) = c$. Suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact, and contains no critical point of f other than q, for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.

(c) If each set M^a is compact, then the manifold M has the homotopy type of a CW-complex with one cell of dimension λ for each critical point of index λ.

A k-cell is simply a k-dimensional disk D^k, and it gets attached along its boundary S^{k-1}. Morse's original treatment did not include part (c) of Theorem 3.1.1. Instead, his main results were phrased in terms of inequalities. Let $b_k(M) := \dim H_k(M)$ be the ***k*-th Betti number of *M***. Let M be a compact manifold and f a Morse function on M. Let C_λ be the number of critical points of f with index λ.

Theorem 3.1.2. ***(Morse inequalities [51])***
In the conditions and notation of the above paragraph, we have:

(a)

$$b_\lambda(M) \leqslant C_\lambda \ ,$$

(b)

$$\sum_\lambda (-1)^\lambda b_\lambda(M) = \sum_\lambda (-1)^\lambda C_\lambda \ , \ \textit{and}$$

(c)

$$b_\lambda(M) - b_{\lambda-1}(M) + \ldots \pm b_0(M) \leqslant C_\lambda - C_{\lambda-1} + \ldots \pm C_0 \ .$$

A **perfect Morse function** is a Morse function for which the inequalities in the previous statement are equalities.

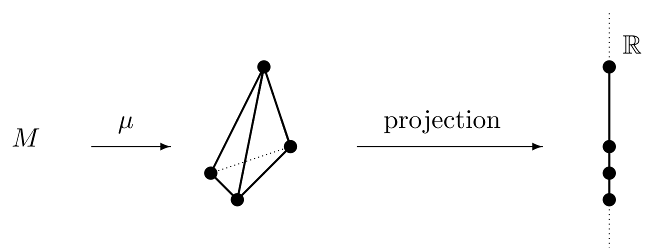
Corollary 3.1.3. *If all critical points of a Morse function f have even index, then f is a perfect Morse function.*

Homology of symplectic toric manifolds

Let $(M, \omega, \mathbb{T}^n, \mu)$ be a $2n$-dimensional symplectic toric manifold. Choose a suitably generic direction in $\mathbb{R}^n$ by picking a vector X whose components are independent over $\mathbb{Q}$. This condition ensures that:

- the one-dimensional subgroup, $T^X \subset \mathbb{T}^n$, generated by the vector X is dense in $\mathbb{T}^n$,
- X is not parallel to the facets of the moment polytope $\Delta := \mu(M)$, and
- the vertices of Δ have different projections along X.

Exercise 3.1.4. *Check that the fixed points for the $\mathbb{T}^n$-action are exactly the fixed points of the action restricted to $\mathbb{T}^X$, that is, are the zeros of the vector field, $X^\#$ on M corresponding to the $\mathbb{T}^X$-action.*



Let $\mu^X := \langle \mu, X \rangle : M \to \mathbb{R}$ be the projection of μ along X. By definition of moment map, μ^X is a hamiltonian function for the vector field $X^\#$ generated by X. We conclude from the previous exercise that the critical points of μ^X are precisely the fixed points of the $\mathbb{T}^n$-action.

By Theorem 1.4.7, if q is a fixed point for the $\mathbb{T}^n$-action, then there exists a chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at q and weights $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^n$ such that

$$\mu^X\big|_{\mathcal{U}} = \langle \mu, X \rangle|_{\mathcal{U}} = \mu^X(q) + \tfrac{1}{2} \sum_{k=1}^n \langle \lambda^{(k)}, X \rangle (x_k^2 + y_k^2) \ .$$

Since the components of X are independent over $\mathbb{Q}$, all coefficients $\langle\lambda^{(k)}, X\rangle$ are nonzero, so q is a *nondegenerate* critical point of μ^X. Moreover, the index of q is twice the number of labels k such that $\langle\lambda^{(k)}, X\rangle < 0$. But the $\lambda^{(k)}$'s are precisely the edge vectors u_i which satisfy Delzant's conditions. Therefore, geometrically, the index of q can be read from the moment polytope Δ, by taking twice the number of edges whose inward-pointing edge vectors at $\mu(q)$ *point down relative to* X, that is, whose inner product with X is negative. In particular, μ^X is a perfect Morse function. By applying Corollary 3.1.3 we conclude that:

Theorem 3.1.5. *Let $X \in \mathbb{R}^n$ have components independent over $\mathbb{Q}$. The degree-$2k$ homology group of the symplectic toric manifold $(M, \omega, \mathbb{T}^n, \mu)$ has dimension equal to the number of vertices of the moment polytope Δ where there are exactly k (primitive inward-pointing) edge vectors which point down relative to the projection along the X. All odd-degree homology groups of M are zero.*

By Poincaré duality (or by taking $-X$ instead of X), the words "point down" may be replaced by "point up".

Exercise 3.1.6. *Let $(M, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold. What is the Euler characteristic of M?*

3.2 Cutting Symplectic Toric Manifolds

Symplectic cutting is a construction proposed by Eugene Lerman [40] in the 1990's, which has since found many applications. It is basically the application of symplectic reduction to the product of a hamiltonian S^1-space with a standard $\mathbb{C}$, in a way that the reduced space for the original hamiltonian S^1-space embeds symplectically as a codimension 2 submanifold in the symplectic *cut* manifold.

Let (M, ω) be a symplectic manifold where S^1 acts in a hamiltonian way, $\rho : S^1 \to \mathrm{Diff}(M)$, with moment map $\mu : M \to \mathbb{R}$. Suppose that:

- M has a unique nondegenerate minimum at q where $\mu(q) = 0$, and
- for ε sufficiently small, S^1 acts freely on the level set $\mu^{-1}(\varepsilon)$.

Let $\mathbb{C}$ be equipped with the symplectic form $-\frac{i}{2} dz \wedge d\bar{z}$ (note the sign here for later convenience) and with standard circle action. Then the diagonal action of S^1 on the product $M \times \mathbb{C}$ is hamiltonian with moment map

$$\phi : M \times \mathbb{C} \longrightarrow \mathbb{R} , \qquad \phi(p, z) = \mu(p) - \tfrac{1}{2}|z|^2 .$$

Observe that S^1 acts freely on the ε-level of ϕ for ε small enough:

$$\begin{array}{rcl} \phi^{-1}(\varepsilon) & = & \{(p, z) \in M \times \mathbb{C} \mid \mu(p) - \frac{1}{2}|z|^2 = \varepsilon\} \\ & = & \{(p, 0) \in M \times \mathbb{C} \mid \mu(p) = \varepsilon\} \\ & & \quad \cup \quad \{(p, z) \in M \times \mathbb{C} \mid \frac{1}{2}|z|^2 = \mu(p) - \varepsilon > 0\} . \end{array}$$

The reduced space is hence

$$\phi^{-1}(\varepsilon)/S^1 \simeq \mu^{-1}(\varepsilon)/S^1 \sqcup \{p \in M \mid \mu(p) > \varepsilon\} \ .$$

One can check that the open submanifold of M given by $\{p \in M \mid \mu(p) > \varepsilon\}$ embeds as an open dense symplectic submanifold into $\phi^{-1}(\varepsilon)/S^1$, and the reduced space $\mu^{-1}(\varepsilon)/S^1$ embeds as a codimension 2 symplectic submanifold into $\phi^{-1}(\varepsilon)/S^1$.

As it is a local construction, the cutting operation may be more generally performed at a local minimum (or maximum) of the moment map μ.

There is a remaining S^1-action on the **ε-cut space**

$$M_{\text{cut}}^{\geqslant\varepsilon} := \phi^{-1}(\varepsilon)/S^1$$

induced by

$$\tau : S^1 \longrightarrow \text{Diff}(M \times \mathbb{C}) \ , \qquad \tau_t(p,z) = (\rho_t(p), z) \ .$$

In fact, τ is a hamiltonian S^1-action on $M \times \mathbb{C}$ which commutes with the diagonal action, thus descends to an action $\widetilde{\tau} : S^1 \to \text{Diff}(M_{\text{cut}}^{\geqslant\varepsilon})$.

Exercise 3.2.1. *Show that $\widetilde{\tau}$ is hamiltonian by describing a moment map.*

Loosely speaking, the cutting technique provides a hamiltonian way to close the open manifold $\{p \in M \mid \mu(p) > \varepsilon\}$, by using the reduced space at level ε, $\mu^{-1}(\varepsilon)/S^1$. We may similarly close $\{p \in M \mid \mu(p) < \varepsilon\}$. The resulting hamiltonian S^1-spaces are called **cut spaces**, and denoted $M_{\text{cut}}^{\geqslant\varepsilon}$ and $M_{\text{cut}}^{\leqslant\varepsilon}$.

If another group G acts on M in a hamiltonian way which commutes with the S^1-action, then the cut spaces are also hamiltonian G-spaces: Suppose that a compact Lie group G acts on a symplectic manifold (M, ω) in a hamiltonian way, and that $q \in M$ is a fixed point for the G-action. Then, by Theorem 1.4.7, there exists a Darboux chart $(\mathcal{U}, z_1, \dots, z_n)$ centered at q which is G-equivariant with respect to a linear action of G on $\mathbb{C}^n$. Consider an ε-cut space of M relative to this chart, for ε sufficiently small.

Exercise 3.2.2. *Check that G acts on the cut space in a hamiltonian way. Describe the moment map.*

Cutting symplectic toric manifolds

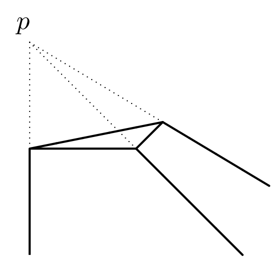
Let Δ be an n-dimensional Delzant polytope, and let $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ be the associated symplectic toric manifold. We consider the action of the diagonal circle $S^1 \subseteq \mathbb{T}^n$ on M_Δ. The corresponding ε-cut space of $(M_\Delta, \omega_\Delta)$ at a fixed point of the $\mathbb{T}^n$-action is a new symplectic toric manifold. What is the moment polytope Δ_ε corresponding to this new symplectic toric manifold?

Let q be a fixed point of the $\mathbb{T}^n$-action on $(M_\Delta, \omega_\Delta)$, and let $p = \mu_\Delta(q)$ be the corresponding vertex of Δ. (Cf. Exercise 2.3.2.) Let $u_1, \dots, u_n$ be the primitive (inward-pointing) edge vectors at p, so that the rays $p + tu_i$, $t \geqslant 0$, form the edges of Δ at p.

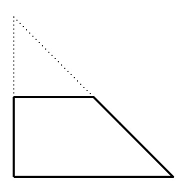
Theorem 3.2.3. *The ε-cut space of $(M_\Delta, \omega_\Delta)$ at a fixed point q is the symplectic toric manifold associated to the polytope Δ_ε obtained from Δ by replacing the vertex p by the n vertices*

$$p + \varepsilon u_i \ , \quad i = 1, \ldots, n \ .$$

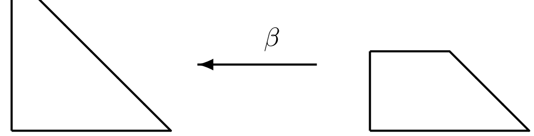
In other words, the moment polytope for a cut space of $(M_\Delta, \omega_\Delta)$ at q is obtained from Δ by chopping off the corner corresponding to q, thus substituting the original set of vertices by the same set with the vertex corresponding to q replaced by exactly n new vertices:



Proof. Exercise: Check that the new polytope is Delzant. We may view the ε-cut space of $(M_\Delta, \omega_\Delta)$ as being obtained from M_Δ by smoothly replacing q by $(\mathbb{CP}^{n-1}, \varepsilon\omega_{\text{FS}})$. Compute the restriction of the moment map to this set. Recall Exercise 2.3.6. $\square$

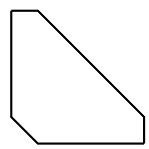


Example. The moment polytope for the standard $\mathbb{T}^2$-action on $(\mathbb{CP}^2, \omega_{\text{FS}})$ is a right isosceles triangle Δ. If we cut $\mathbb{CP}^2$ at $[0 : 0 : 1]$ we obtain a symplectic toric manifold associated to the trapezoid below. This manifold is a Hirzebruch surface, defined in Section 2.3.



$\diamondsuit$

Example. The following moment polytope corresponds to a toric manifold obtained by cutting $\mathbb{CP}^2$ at each of its three fixed points:



$\diamondsuit$

3.3 Blow-Up of Symplectic Toric Manifolds

There is a close connection between symplectic cutting and the classical blow-up construction in the category of symplectic toric manifolds.

Review of classic blow-up

Let L be the tautological line bundle over $\mathbb{CP}^{n-1}$, that is,

$$L = \{([p], z) \mid p \in \mathbb{C}^n \backslash \{0\} \ , \ z = \lambda p \text{ for some } \lambda \in \mathbb{C}\}$$

with projection to $\mathbb{CP}^{n-1}$ given by $([p], z) \mapsto [p]$. The fiber of L over the point $[p] \in \mathbb{CP}^{n-1}$ is the complex line in $\mathbb{C}^n$ represented by that point.

Definition 3.3.1. *The* ***blow-up of $\mathbb{C}^n$ at the origin*** *is the total space of the bundle L. The corresponding* ***blow-down map*** *is the map $\beta : L \to \mathbb{C}^n$ defined by $\beta([p], z) = z$.*

Notice that the total space of L may be decomposed as the disjoint union of two sets,

$$E := \{([p], 0) \mid p \in \mathbb{C}^n \backslash \{0\}\}$$

and

$$S := \{([p], z) \mid p \in \mathbb{C}^n \backslash \{0\} \ , \ z = \lambda p \text{ for some } \lambda \in \mathbb{C}^*\} \ .$$

The set E is called the **exceptional divisor**; it is diffeomorphic to $\mathbb{CP}^{n-1}$ and gets mapped to the origin by β. On the other hand, the restriction of β to the complementary set S is a diffeomorphism onto $\mathbb{C}^n \backslash \{0\}$. Hence, we may regard L as being obtained from $\mathbb{C}^n$ by smoothly replacing the origin by a copy of $\mathbb{CP}^{n-1}$.

There are actions of the unitary group $\mathrm{U}(n)$ on all of these sets induced by the standard linear action on $\mathbb{C}^n$, and the map β is $\mathrm{U}(n)$-equivariant.

Remark. Blow-up extends to an operation along a complex submanifold by considering the projectivization of the normal bundle to the submanifold. $\diamondsuit$

Symplectic blow-up

According to its first printed exposition in [45], the symplectic version of blow-up is due to Gromov.

Definition 3.3.2. *A* ***blow-up symplectic form*** *on* L *is a* $\mathrm{U}(n)$*-invariant symplectic form* ω *such that the difference* $\omega - \beta^*\omega_0$ *is compactly supported, where* $\omega_0 = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$ *is the standard symplectic form on* $\mathbb{C}^n$*.*

Two blow-up symplectic forms are called **equivalent** if one is the pullback of the other by a $\mathrm{U}(n)$-equivariant diffeomorphism of L. Guillemin and Sternberg [29] have shown that two blow-up symplectic forms are equivalent if and only if they have equal restrictions to the exceptional divisor $E \subset L$.

Let Ω^ε ($\varepsilon > 0$) be the set of all blow-up symplectic forms on L whose restriction to the exceptional divisor $E \simeq \mathbb{CP}^{n-1}$ is $\varepsilon\omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini-Study form on $\mathbb{CP}^{n-1}$ described in Section 1.4. An ***ε*-blow-up** of $\mathbb{C}^n$ at the origin is a pair (L, ω) with $\omega \in \Omega^\varepsilon$.

Let (M, ω) be a $2n$-dimensional symplectic manifold. It is a consequence of the Darboux theorem that, for each point $q \in M$, there exists a chart $(\mathcal{U}, z_1, \ldots, z_n)$ centered at q and with image in $\mathbb{C}^n$ where

$$\omega|_{\mathcal{U}} = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k \ .$$

It is shown in [29] that, for ε small enough, we can perform an ε-blow-up of M at q modeled on $\mathbb{C}^n$ at the origin, without changing the symplectic structure outside of a small neighborhood of q. The resulting manifold is then called an ***ε*-blow-up of *M* at *q***.

Example. Let $\mathbb{CP}(L \oplus \mathbb{C})$ be the $\mathbb{CP}^1$-bundle over $\mathbb{CP}^{n-1}$ obtained by projectivizing the direct sum of the tautological line bundle L with a trivial complex line bundle. Consider the map

$$\begin{array}{rrcl} \beta: & \mathbb{CP}(L \oplus \mathbb{C}) & \longrightarrow & \mathbb{CP}^n \\ & ([p], [\lambda p : w]) & \longmapsto & [\lambda p : w] \ , \end{array}$$

where $[\lambda p : w]$ on the right represents a line in $\mathbb{C}^{n+1}$, forgetting that, for each $[p] \in \mathbb{CP}^{n-1}$, that line sits in the 2-complex-dimensional subspace $L_{[p]} \oplus \mathbb{C} \subset \mathbb{C}^n \oplus \mathbb{C}$. Notice that β maps the *exceptional divisor*

$$E := \{([p], [0 : \ldots : 0 : 1]) \mid [p] \in \mathbb{CP}^{n-1}\} \simeq \mathbb{CP}^{n-1}$$

to the point $[0 : \ldots : 0 : 1] \in \mathbb{CP}^n$, whereas β is a diffeomorphism on the complement

$$S := \{([p], [\lambda p : w]) \mid [p] \in \mathbb{CP}^{n-1} \ , \ \lambda \in \mathbb{C}^* \ , \ w \in \mathbb{C}\} \simeq \mathbb{CP}^n \backslash \{[0 : \ldots : 0 : 1]\} \ .$$

Therefore, we may regard $\mathbb{CP}(L \oplus \mathbb{C})$ as being obtained from $\mathbb{CP}^n$ by smoothly replacing the point $[0 : \ldots : 0 : 1]$ by a copy of $\mathbb{CP}^{n-1}$. The space $\mathbb{CP}(L \oplus \mathbb{C})$ is the blow-up of $\mathbb{CP}^n$ at the point $[0 : \ldots : 0 : 1]$, and β is the corresponding blow-down map. The manifold $\mathbb{CP}(L \oplus \mathbb{C})$ for $n = 2$ is a Hirzebruch surface; cf. Section 2.3. Notice that it coincides with a cut-space of $\mathbb{CP}^n$ at the fixed point $[0 : \ldots : 0 : 1]$. ◊

Exercise 3.3.3. *($\star$) Write a definition for* blow-up of a symplectic manifold along a complex submanifold *by considering the projectivization of the normal bundle to the submanifold.*

3.4 Symplectic Toric Orbifolds

Orbifold singularities.

Roughly speaking, orbifolds (introduced by Satake in [54]) are singular manifolds where each singularity is locally modeled on $\mathbb{R}^m/\Gamma$, for some finite group $\Gamma \subset \mathrm{GL}(m;\mathbb{R})$. For the precise definition, let $|M|$ be a Hausdorff topological space satisfying the second axiom of countability.

Definition 3.4.1. *An* ***orbifold chart*** *on $|M|$ is a triple $(\mathcal{V},\Gamma,\varphi)$, where $\mathcal{V}$ is a connected open subset of some euclidean space $\mathbb{R}^m$, Γ is a finite group which acts linearly on $\mathcal{V}$ so that the set of points where the action is not free has codimension at least two, and $\varphi : \mathcal{V} \to |M|$ is a Γ-invariant map inducing a homeomorphism from $\mathcal{V}/\Gamma$ onto its image $\mathcal{U} \subset |M|$. An* ***orbifold atlas*** *$\mathcal{A}$ for $|M|$ is a collection of orbifold charts on $|M|$ such that: the collection of images $\mathcal{U}$ forms a basis of open sets in $|M|$, and the charts are compatible in the sense that, whenever two charts $(\mathcal{V}_1,\Gamma_1,\varphi_1)$ and $(\mathcal{V}_2,\Gamma_2,\varphi_2)$ satisfy $\mathcal{U}_1 \subseteq \mathcal{U}_2$, there exists an injective homomorphism $\lambda : \Gamma_1 \to \Gamma_2$ and a λ-equivariant open embedding $\psi : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\varphi_2 \circ \psi = \varphi_1$. Two orbifold atlases are* ***equivalent*** *if their union is still an atlas. An m-dimensional* ***orbifold*** *M is a Hausdorff topological space $|M|$ satisfying the second axiom of countability, plus an equivalence class of orbifold atlases on $|M|$.*

Notice that we do not require the action of each group Γ to be effective. Given a point p on an orbifold M, let $(\mathcal{V},\Gamma,\varphi)$ be an orbifold chart for a neighborhood $\mathcal{U}$ of p. The **orbifold structure group** of p, Γ_p, is (the isomorphism class of) the isotropy group of a pre-image of p under φ. We may always choose an orbifold chart $(\mathcal{V},\Gamma,\varphi)$ such that $\varphi^{-1}(p)$ is a single point (which is fixed by Γ). In this case $\Gamma \simeq \Gamma_p$, and we say that $(\mathcal{V},\Gamma,\varphi)$ is a **structure chart** for p.

An ordinary manifold is a special case of orbifold where each group Γ is the identity group. Quotients of manifolds by locally free actions of Lie groups are orbifolds. In fact, any orbifold M has a presentation of this form obtained as follows. Given a structure chart $(\mathcal{V},\Gamma,\varphi)$ for $p \in M$ with image $\mathcal{U}$, the **orbifold tangent space** at p is the quotient of the tangent space to $\mathcal{V}$ at $\varphi^{-1}(p)$ by the induced action of Γ:

$$T_pM := T_{\varphi^{-1}(p)}\mathcal{V}/\Gamma \ .$$

The collection of the orbifold tangent spaces at all p, builds up the **orbifold tangent bundle** TM, which has a natural structure of smooth manifold outside the zero section. The general linear group $\mathrm{GL}(m;\mathbb{R})$ acts locally freely on $TM\backslash\{0\}$, and $M \simeq (TM\backslash\{0\})/\mathrm{GL}(m;\mathbb{R})$. Choosing a riemannian metric and taking the orthonormal frame bundle, $O(TM)$, we present M as $O(TM)/\mathrm{O}(m)$.

Examples.

1. Let $G = \mathbb{T}^n$ be an n-torus acting on a symplectic manifold (M, ω) in a hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. For any $\xi \in \mathfrak{g}^*$, the level $\mu^{-1}(\xi)$ is preserved by the $\mathbb{T}^n$-action. Suppose that ξ is a regular value of μ.[1] Then $\mu^{-1}(\xi)$ is a submanifold of codimension n. Let G_p be the stabilizer of p, and $\mathfrak{g}_p$ its Lie algebra. Note that

$$\begin{aligned} \xi \text{ regular} \quad &\Longleftrightarrow \quad d\mu_p \text{ is surjective at all } p \in \mu^{-1}(\xi) \\ &\Longleftrightarrow \quad \mathfrak{g}_p = 0 \text{ for all } p \in \mu^{-1}(\xi) \\ &\Longleftrightarrow \quad \text{the stabilizers on } \mu^{-1}(\xi) \text{ are finite .} \end{aligned}$$

   By the slice theorem (see, for instance, [9, 15]), near $\mathcal{O}_p$ the orbit space $\mu^{-1}(\xi)/G$ is modeled by S/G_p, where S is a G_p-invariant disk in $\mu^{-1}(\xi)$ through p and transverse to $\mathcal{O}_p$. Hence, $\mu^{-1}(\xi)/G$ is an orbifold.

2. Consider the S^1-action on $\mathbb{C}^2$ by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\theta} z_2)$ for some fixed integer $k \geqslant 2$. This is hamiltonian with moment map

$$\begin{array}{rccc} \mu : & \mathbb{C}^2 & \longrightarrow & \mathbb{R} \\ & (z_1, z_2) & \longmapsto & \frac{1}{2}(k|z_1|^2 + |z_2|^2) \ . \end{array}$$

   Any $\xi > 0$ is a regular value and $\mu^{-1}(\xi)$ is a 3-dimensional ellipsoid. The stabilizer of $(z_1, z_2) \in \mu^{-1}(\xi)$ is $\{1\}$ if $z_2 \neq 0$, and is

$$\mathbb{Z}_k = \left\{ e^{i\frac{2\pi\ell}{k}} \mid \ell = 0, 1, \ldots, k-1 \right\}$$

   if $z_2 = 0$. The reduced space $\mu^{-1}(\xi)/S^1$ is called a **teardrop** orbifold or *conehead*; it has one **cone** (also known as a *dunce cap*) singularity with cone angle $\frac{2\pi}{k}$, that is, a point with orbifold structure group $\mathbb{Z}_k$.

3. Let S^1 act on $\mathbb{C}^2$ by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\ell\theta} z_2)$ for some integers $k, \ell \geqslant 2$. Suppose that k and ℓ are relatively prime. Then

$$\begin{array}{rll} (z_1, 0) & \text{has stabilizer } \mathbb{Z}_k & (\text{for } z_1 \neq 0) \ , \\ (0, z_2) & \text{has stabilizer } \mathbb{Z}_\ell & (\text{for } z_2 \neq 0) \ , \\ (z_1, z_2) & \text{has stabilizer } \{1\} & (\text{for } z_1, z_2 \neq 0) \ . \end{array}$$

   The quotient $\mu^{-1}(\xi)/S^1$ is called a **football** orbifold. It has two cone singularities, one with angle $\frac{2\pi}{k}$ and another with angle $\frac{2\pi}{\ell}$.

4. More generally, the reduced spaces of S^1 acting on $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{i\ell_1\theta} z_1, \ldots, e^{i\ell_n\theta} z_n) \ ,$$

   are called **weighted** (or *twisted*) **projective spaces**.

[1] By Sard's theorem, the singular values of μ form a set of measure zero.

◊

The differential-geometric notions of vector fields, differential forms, exterior differentiation, group actions, etc., extend naturally to orbifolds by gluing corresponding local Γ-invariant or Γ-equivariant objects. In particular, a **symplectic orbifold** is a pair (M, ω) where M is an orbifold and ω is a closed 2-form on M which is nondegenerate at every point of M.

Definition 3.4.2. *A **symplectic toric orbifold** is a compact connected symplectic orbifold (M, ω) equipped with an effective hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the orbifold,*

$$\dim \mathbb{T} = \frac{1}{2} \dim M \ ,$$

and with a choice of a corresponding moment map μ.

Symplectic toric orbifolds have been classified by Lerman and Tolman [43] in a theorem which generalizes Delzant's theorem: a symplectic toric orbifold is determined by its moment polytope plus a positive integer label attached to each of the polytope facets. The polytopes which occur in the Lerman-Tolman classification are more general than the Delzant polytopes in the sense that only simplicity and rationality are required; the edge vectors $u_1, \ldots, u_n$ need only form a rational basis of $\mathbb{Z}^n$. In the case where the integer labels are all equal to 1, the failure of the polytope smoothness accounts for all orbifold singularities.

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