Two-Point Gromov-Witten Formulas for Symplectic Toric Manifolds

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Abstract of the Dissertation

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We show that the standard generating functions for genus 0 two-point twisted Gromov-Witten invariants arising from concavex vector bundles over symplectic toric manifolds are explicit transforms of the corresponding one-point generating functions. The latter are, in turn, transforms of Givental's J-function. We obtain closed formulas for them and, in particular, for two-point Gromov-Witten invariants of non-negative toric complete intersections. Such two-point formulas should play a key role in the computation of genus 1 Gromov-Witten invariants (closed, open, and unoriented) of toric complete intersections as they indeed do in the case of the projective complete intersections.

Contents

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Chapter 1

Introduction

Torus actions on moduli spaces of stable maps into a smooth projective variety facilitate the computation of equivariant Gromov-Witten invariants [Gi1] via the Localization Theorem [ABo], [GraPa]. Equivariant formulas lead to other interesting consequences beyond the computation of non-equivariant Gromov-Witten invariants. In the case of the projective spaces, two-point equivariant Gromov-Witten formulas in [PoZ] lead to the confirmation of mirror symmetry predictions concerning open and unoriented genus 1 Gromov-Witten invariants in the same paper and to the computation of closed genus 1 Gromov-Witten invariants in [Po]. In this dissertation we obtain equivariant formulas expressing the standard two-point closed genus 0 generating function for certain twisted Gromov-Witten invariants of symplectic toric manifolds in terms of the corresponding one-point generating functions. We also obtain explicit formulas for the latter. In particular, we show that the standard generating function for these two-point invariants is a fairly simple transform of the well-known Givental's J-function. The formulas obtained in this dissertation compute, in particular, the twisted/un-twisted Gromov-Witten numbers (1.0.2)/(1.0.3) below.

For a smooth projective variety X and a class $A \in H_2(X;\mathbb{Z})$, $\overline{\mathfrak{M}}_{0,m}(X,A)$ denotes the moduli space of stable maps from genus 0 curves with m marked points into X representing A. Let

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{0,m}(X,A) \longrightarrow X$$

be the evaluation map at the i-th marked point; see [MirSym, Chapter 24]. All cohomology groups in this dissertation will be with rational coefficients unless otherwise specified. For each $i = 1,2,\ldots,m$, let $\psi_i \in H^2(\overline{\mathfrak{M}}_{0,m}(X,A))$ be the first Chern class of the universal cotangent line bundle for the i-th marked point. Let

$$\pi : \mathfrak{U} \longrightarrow \overline{\mathfrak{M}}_{0,m}(X,A)$$

be the universal curve and $\mathrm{ev} : \mathfrak{U} \longrightarrow X$ the natural evaluation map; see [MirSym, Section 24.3].

A holomorphic vector bundle $E \longrightarrow X$ is called concavex if

$$E = E^+ \oplus E^-, \qquad \text{with} \quad H^1\left(\mathbb{P}^1, f^*E^+\right) = 0, \quad H^0\left(\mathbb{P}^1, f^*E^-\right) = 0 \quad \forall\, f : \mathbb{P}^1 \longrightarrow X.$$

Such a vector bundle induces a vector orbi-bundle $\mathcal{V}_E$ over $\overline{\mathfrak{M}}_{0,m}(X,A)$:

$$\mathcal{V}_E \equiv \mathcal{V}_{E^+} \oplus \mathcal{V}_{E^-}, \quad \text{where} \quad \mathcal{V}_{E^+} \equiv \pi_*\mathrm{ev}^*E^+, \quad \mathcal{V}_{E^-} \equiv R^1\pi_*\mathrm{ev}^*E^-. \tag{1.0.1}$$

Given a class $A\in H_2(X;\mathbb{Z})$ and classes $\eta_1,\eta_2\in H^*(X)$, the corresponding genus 0 twisted two-point Gromov-Witten (GW) invariants of X are:

$$\left\langle \psi^{p_1}\eta_1,\psi^{p_2}\eta_2\right\rangle_{A,E}^X \equiv \int_{[\overline{\mathfrak{M}}_{0,2}(X,A)]^{vir}} \left(\psi_1^{p_1}\mathrm{ev}_1^*\eta_1\right)\left(\psi_2^{p_2}\mathrm{ev}_2^*\eta_2\right)e(\mathcal{V}_E)\in\mathbb{Q}. \tag{1.0.2}$$

In particular, if $E=E^+$, the twisted Gromov-Witten invariants (1.0.2) are the genus 0 two-point Gromov-Witten invariants of a complete intersection $Y\equiv s^{-1}(0)\hookrightarrow X$ defined by a generic holomorphic section $s\colon X\longrightarrow E^+$:

$$\left\langle \psi^{p_1}\eta_1,\psi^{p_2}\eta_2\right\rangle_{A,E^+}^X = \left\langle \psi^{p_1}\eta_1,\psi^{p_2}\eta_2\right\rangle_{A}^Y \equiv \left\langle \psi^{p_1}\eta_1,\psi^{p_2}\eta_2\right\rangle_{A,0}^Y \qquad \forall\,\eta_1,\eta_2\in H^*(Y); \tag{1.0.3}$$

the first equality follows from [El, Theorem 0.1.1, Remark 0.1.1].

The numbers (1.0.2) have been computed in the $X=\mathbb{P}^{n-1}$ case under various assumptions on E through various approaches. The case when E is a positive line bundle is solved in [BK] and [Z1] and extended to the case when E is a sum of positive line bundles in [PoZ]. The former led to the computation of the genus 1 Gromov-Witten invariants of Calabi-Yau hypersurfaces in [Z2], while the latter to the computation of the genus 1 Gromov-Witten invariants of Calabi-Yau complete intersections in [Po]. The case when E is a concavex vector bundle has been solved in [Ch] in the setting of [LLY1]. More recently, genus 0 formulas with any number of ψ classes have been obtained in [Z3]. In this dissertation we extend the approaches of [Z1] and [PoZ] to the case when X is an arbitrary compact symplectic toric manifold and E is a sum of non-negative and negative line bundles.

1.1 Some results

If n is a non-negative integer, we write

$$[n]\equiv\{1,2,\ldots,n\}.$$

Let $s\geqslant1$, $N_1,\ldots,N_s\geqslant2$ and for each $i\in[s]$ let

$$\mathrm{H}_i\equiv\mathrm{pr}_i^*\mathrm{H}\in H^2\left(\prod_{j=1}^s\mathbb{P}^{N_j-1}\right),$$

where $\mathrm{pr}_i\colon\prod\limits_{j=1}^s\mathbb{P}^{N_j-1}\longrightarrow\mathbb{P}^{N_i-1}$ is the projection onto the i-th component and $\mathrm{H}\in H^2(\mathbb{P}^{N_i-1})$ is the hyperplane class on $\mathbb{P}^{N_i-1}$.

Theorem 1.1.1. *Let $\mathbf{d}=(d_1,\ldots,d_s)\in(\mathbb{Z}^{>0})^s$. The degree $\mathbf{d}$ genus 0 two-point GW invariants (1.0.3) of $\prod\limits_{i=1}^s\mathbb{P}^{N_i-1}$ are given by the following identity in $\frac{\mathbb{Q}[\mathrm{A}_1,\ldots,\mathrm{A}_s,\mathrm{B}_1,\ldots,\mathrm{B}_s]}{\left(\mathrm{A}_i^{N_i},\mathrm{B}_i^{N_i}\quad\forall\, i\in[s]\right)}[[\hbar_1^{-1},\hbar_2^{-1}]]$:*

$$\sum_{\substack{a_1,\ldots,a_s\geqslant0\\ b_1,\ldots,b_s\geqslant0}}\mathrm{A}_1^{a_1}\ldots\mathrm{A}_s^{a_s}\mathrm{B}_1^{b_1}\ldots\mathrm{B}_s^{b_s}\left\langle\frac{\mathrm{H}_1^{N_1-1-a_1}\ldots\mathrm{H}_s^{N_s-1-a_s}}{\hbar_1-\psi},\frac{\mathrm{H}_1^{N_1-1-b_1}\ldots\mathrm{H}_s^{N_s-1-b_s}}{\hbar_2-\psi}\right\rangle_{\mathbf{d}}^{\prod_{i=1}^s\mathbb{P}^{N_i-1}}$$
$$=\frac{1}{\hbar_1+\hbar_2}\sum_{\substack{a_i,b_i,e_i,f_i\geqslant0\\ a_i+b_i=N_i-1\\ e_i+f_i=d_i}}\frac{(\mathrm{A}_1+e_1\hbar_1)^{a_1}\ldots(\mathrm{A}_s+e_s\hbar_1)^{a_s}(\mathrm{B}_1+f_1\hbar_2)^{b_1}\ldots(\mathrm{B}_s+f_s\hbar_2)^{b_s}}{\prod\limits_{i=1}^s\left(\prod\limits_{r=1}^{e_i}(\mathrm{A}_i+r\hbar_1)^{N_i}\prod\limits_{r=1}^{f_i}(\mathrm{B}_i+r\hbar_2)^{N_i}\right)}. \tag{1.1.1}$$

This follows from Corollary 3.2.4 in Section 3.2.

Remark 1.1.2. The sums on both sides of (1.1.1) are power series in $\hbar_1^{-1}$ and $\hbar_2^{-1}$ by expanding at $\hbar_1^{-1}=0$ and $\hbar_2^{-1}=0$; to see this on the right-hand side, divide both the numerator and denominator of each a_i,b_i,e_i,f_i-summand by $\hbar_1^{\sum_{i=1}^{s}N_ie_i}\hbar_2^{\sum_{i=1}^{s}N_if_i}$. Part of the statement of Theorem 1.1.1 is that the right-hand side sum in (1.1.1) is divisible by $\hbar_1+\hbar_2$ (i.e. it vanishes when evaluated at $(\hbar_1,\hbar_2)=(\hbar,-\hbar)$). Identity (1.1.1) should be interpreted by first dividing this sum by $\hbar_1+\hbar_2$ and then setting it equal to the left-hand side.

The results below concern the GW invariants of a compact symplectic toric manifold X_M^τ defined by (2.1.2) from a minimal toric pair (M,τ) as in Definition 2.1.1. We assume that the vector bundle E splits

$$E\equiv E^+\oplus E^-\longrightarrow X_M^\tau, \qquad \text{where} \quad E^+\equiv\bigoplus_{i=1}^{a}L_i^+, \quad E^-\equiv\bigoplus_{i=1}^{b}L_i^-, \tag{1.1.2}$$

L_i^+ are non-trivial, non-negative line bundles and L_i^- are negative line bundles.[1] Theorem 1.1.3 and Remark 1.1.4 below describe two-point twisted GW invariants in terms of one-point ones. As is usually done, the twisted GW invariants will be assembled into a generating function in the formal variables

$$Q=(Q_1,\ldots,Q_k)$$

with powers indexed by

$$\Lambda\equiv\left\{\mathbf{d}\in H_2(X_M^\tau;\mathbb{Z}):\left\langle\omega,\mathbf{d}\right\rangle\geqslant 0\quad\forall\,\omega\in\overline{\mathcal{K}}_M^\tau\right\}, \tag{1.1.3}$$

where $\overline{\mathcal{K}}_M^\tau$ is the closed Kähler cone of X_M^τ.[2]

A ring R and the monoid Λ induce an R-algebra denoted $R[[\Lambda]]$: to each $\mathbf{d}$ we associate a basis element denoted $Q^{\mathbf{d}}$ and set

$$R[[\Lambda]]\equiv\left\{\sum_{\mathbf{d}\in\Lambda}a_{\mathbf{d}}Q^{\mathbf{d}}:a_{\mathbf{d}}\in R\ \ \forall\,\mathbf{d}\in\Lambda\right\}.$$

[1] Recall that a line bundle $L\longrightarrow X_M^\tau$ is called positive (respectively negative) if $c_1(L)\in H^2(X_M^\tau;\mathbb{R})$ (respectively $-c_1(L)$) can be represented by a Kähler form on X_M^τ. A line bundle $L\longrightarrow X_M^\tau$ is called non-negative if $c_1(L)\in H^2(X_M^\tau;\mathbb{R})$ can be represented by a 2-form ω satisfying $\omega(v,Jv)\geqslant 0$ for all v. The assumptions that the line bundles L_i^+ are non-trivial and that L_i^- are negative (that is, $c_1(L_i^-)<0$ as opposed to just $c_1(L_i^-)\leqslant 0$) are only used in the theorems that rely on the one-point mirror theorem (5.1.2) of [LLY3], that is Theorems 3.2.1, Corollary 3.2.3, Corollary 3.2.4, and Theorem 4.2.3.

[2] By [Br, Theorem 4.5], a non-empty closed convex subset of $\mathbb{R}^d$ is the intersection of its supporting half-spaces. The supporting half-spaces of a closed convex cone C in $\mathbb{R}^d$ are all sets of the form $\{v\in\mathbb{R}^d:\left\langle v,w\right\rangle\geqslant 0\}$ for some $w\in\mathbb{R}^d$ such that $\left\langle v,w\right\rangle\geqslant 0$ for all $v\in C$. This implies that

$$\omega\in\overline{\mathcal{K}}_M^\tau \qquad\Longleftrightarrow\qquad \left\langle\omega,\mathbf{d}\right\rangle\geqslant 0\ \ \forall\,\mathbf{d}\in\Lambda.$$

Addition in $R[[\Lambda]]$ is defined naturally; multiplication is defined by

$$Q^{\mathbf{d}}\cdot Q^{\mathbf{d}'} \equiv Q^{\mathbf{d}+\mathbf{d}'} \qquad \forall\, \mathbf{d},\mathbf{d}'\in\Lambda$$

and extended by R-linearity.

For each $m\geqslant 1$ and each $\mathbf{d}\in\Lambda-\{0\}$, let $\sigma_i:\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})\longrightarrow\mathfrak{U}$ be the section of the universal curve given by the i-th marked point,

$$\begin{aligned}\dot{\mathcal{V}}_E&\equiv R^0\pi_*\left(\mathrm{ev}^*E^+(-\sigma_1)\right)\oplus R^1\pi_*\left(\mathrm{ev}^*E^-(-\sigma_1)\right)\longrightarrow\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d}), \quad \text{and}\\ \ddot{\mathcal{V}}_E&\equiv R^0\pi_*\left(\mathrm{ev}^*E^+(-\sigma_2)\right)\oplus R^1\pi_*\left(\mathrm{ev}^*E^-(-\sigma_2)\right)\longrightarrow\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d}) \ \text{ whenever } m\geqslant 2.\end{aligned} \tag{1.1.4}$$

If $m\geqslant 3$ and $\mathbf{d}=0$, $\dot{\mathcal{V}}_E$ and $\ddot{\mathcal{V}}_E$ are well-defined as well and they are 0. We next define the genus 0 two-point generating function $\dot{Z}$:

$$\dot{Z}\left(\hbar_1,\hbar_2,Q\right)\equiv\frac{\hbar_1\hbar_2}{\hbar_1+\hbar_2}\sum_{\mathbf{d}\in\Lambda}Q^{\mathbf{d}}\left(\mathrm{ev}_1\times\mathrm{ev}_2\right)_*\left[\frac{e(\dot{\mathcal{V}}_E)}{\left(\hbar_1-\psi_1\right)\left(\hbar_2-\psi_2\right)}\right], \tag{1.1.5}$$

where $\mathrm{ev}_1,\mathrm{ev}_2:\overline{\mathfrak{M}}_{0,3}(X_M^\tau,\mathbf{d})\longrightarrow X_M^\tau$ are the evaluation maps at the first two marked points. This is used - in the case of the projective spaces - for the computation of the genus 1 GW invariants of Calabi-Yau complete intersections.

With $\mathrm{ev}_1,\mathrm{ev}_2:\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d})\longrightarrow X_M^\tau$ denoting the evaluation maps at the two marked points and for all $\eta\in H^2(X_M^\tau)$, let

$$\begin{aligned}\dot{Z}_\eta\left(\hbar,Q\right)&\equiv\eta+\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}\mathrm{ev}_{1*}\left[\frac{e(\dot{\mathcal{V}}_E)\mathrm{ev}_2^*\eta}{\hbar-\psi_1}\right]\in H^*(X_M^\tau)[\hbar^{-1}][[\Lambda]],\\ \ddot{Z}_\eta\left(\hbar,Q\right)&\equiv\eta+\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}\mathrm{ev}_{1*}\left[\frac{e(\ddot{\mathcal{V}}_E)\mathrm{ev}_2^*\eta}{\hbar-\psi_1}\right]\in H^*(X_M^\tau)[\hbar^{-1}][[\Lambda]].\end{aligned} \tag{1.1.6}$$

Theorem 1.1.3. *Let* $\mathrm{pr}_i:X_M^\tau\times X_M^\tau\longrightarrow X_M^\tau$ *denote the projection onto the i-th component and let* $\eta_j,\check{\eta}_j\in H^*(X_M^\tau)$ *be such that*

$$\sum_{j=1}^{s}\mathrm{pr}_1^*\eta_j\mathrm{pr}_2^*\check{\eta}_j\in H^{2(N-k)}(X_M^\tau\times X_M^\tau)$$

is the Poincaré dual to the diagonal class, where $N-k$ *is the complex dimension of* X_M^τ. *Then,*

$$\dot{Z}\left(\hbar_1,\hbar_2,Q\right)=\frac{1}{\hbar_1+\hbar_2}\sum_{j=1}^{s}\mathrm{pr}_1^*\dot{Z}_{\eta_j}\left(\hbar_1,Q\right)\mathrm{pr}_2^*\ddot{Z}_{\check{\eta}_j}\left(\hbar_2,Q\right).$$

This follows from Theorem 4.2.1 below, which is an equivariant version of Theorem 1.1.3.

Remark 1.1.4. The genus 0 two-point twisted GW invariants (1.0.2) are assembled into

$$Z^*\left(\hbar_1,\hbar_2,Q\right)\equiv\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}\left(\mathrm{ev}_1\times\mathrm{ev}_2\right)_*\left[\frac{e(\mathcal{V}_E)}{\left(\hbar_1-\psi_1\right)\left(\hbar_2-\psi_2\right)}\right], \tag{1.1.7}$$

where $\mathrm{ev}_1, \mathrm{ev}_2 : \overline{\mathfrak{M}}_{0,2}(X_M^\tau, \mathbf{d}) \longrightarrow X_M^\tau$. By the string relation [MirSym, Section 26.3],

$$Z^*(\hbar_1, \hbar_2, Q) = \frac{\hbar_1\hbar_2}{\hbar_1+\hbar_2} \sum_{\mathbf{d}\in\Lambda-0} (\mathrm{ev}_1\times\mathrm{ev}_2)_* \left[\frac{e(\mathcal{V}_E)}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\right] \in H^*(X_M^\tau \times X_M^\tau)\,[\hbar_1^{-1}, \hbar_2^{-1}][[\Lambda]],$$

where $\mathrm{ev}_1, \mathrm{ev}_2 : \overline{\mathfrak{M}}_{0,3}(X_M^\tau, \mathbf{d}) \longrightarrow X_M^\tau$. By (1.1.4) and (1.0.1),

$$e(\dot{\mathcal{V}}_E)\mathrm{ev}_1^* e(E^+) = e(\mathcal{V}_E)\mathrm{ev}_1^* e(E^-).$$

The last two equations imply that

$$\dot{Z}^*(\hbar_1, \hbar_2, Q)\mathrm{pr}_1^* e(E^+) = Z^*(\hbar_1, \hbar_2, Q)\mathrm{pr}_1^* e(E^-),$$

where $\dot{Z}^*$ is obtained from $\dot{Z}$ by disregarding the $Q^{\mathbf{0}}$ term and $\mathrm{pr}_1 : X_M^\tau \times X_M^\tau \longrightarrow X_M^\tau$ is the projection onto the first component. This together with Theorem 1.1.3 expresses Z^* in terms of $\dot{Z}_\eta$, $\ddot{Z}_\eta$ in the $E = E^+$ case. In all other cases, Z^* can be expressed in terms of one-point GW generating functions which can be computed under one additional assumption; see Remark 3.2.5.

Remark 1.1.5. If $E = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ and $\mathrm{H} \in H^2(\mathbb{P}^2)$ is the hyperplane class, then

$$\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^2,d)} e(\mathcal{V}_E)\mathrm{ev}_1^*\mathrm{H}^2\mathrm{ev}_2^*\mathrm{H} = \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^2,d)} e(\mathcal{V}_E)\mathrm{ev}_1^*\mathrm{H}\mathrm{ev}_2^*\mathrm{H}^2 = (-1)^d \frac{(2d)!}{2d(d!)^2} \qquad \forall\, d \geqslant 1.$$

If $E = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathrm{H} \in H^2(\mathbb{P}^2)$ is the hyperplane class, then

$$\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^2,d)} e(\mathcal{V}_E)\mathrm{ev}_1^*\mathrm{H}^2\mathrm{ev}_2^*\mathrm{H}^2 = \frac{(-1)^{d+1}}{d} \qquad \forall\, d \geqslant 1.$$

If $E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathrm{H} \in H^2(\mathbb{P}^1)$ is the hyperplane class, then

$$\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^1,d)} e(\mathcal{V}_E)\mathrm{ev}_1^*\mathrm{H}\,\mathrm{ev}_2^*\mathrm{H} = \frac{1}{d} \qquad \forall\, d \geqslant 1.$$

These follow from (3.2.12) in Section 3.2 which relies on Theorem 4.2.1, the equivariant version of Theorem 1.1.3 above. The first of these equations implies the first statement in [KlPa, Proposition 2] by the divisor relation of [MirSym, Section 26.3], the second recovers the first statement in [PaZ, Lemma 3.1], and the third implies the Aspinwall-Morrison formula.

1.2 Outline of the dissertation

Chapter 2 presents the facts about symplectic toric manifolds needed for the Gromov-Witten theory parts of the dissertation. This chapter is inspired by the view in [Gi2] of a symplectic toric manifold as given by a matrix and the choice of a certain regular value together with the holomorphic charts of [Ba]. It contains proofs of all statements or references to the ones that are omitted. The reader interested only in the Gromov-Witten theory part may want to skip all proofs in Chapter 2.

Section 3.2 gives formulas for the one-point GW generating functions $\dot{Z}_\eta$, $\ddot{Z}_\eta$ of (1.1.6) under an additional assumption in terms of explicit formal power series constructed in Section 3.1. It begins with a short setup.

The explicit GW formulas of Section 3.2 and Theorem 1.1.3 above follow from the equivariant statements of Section 4.2. In particular, equivariant versions of $\dot{Z}_\eta$ and $\ddot{Z}_\eta$ are expressed in terms of explicit power series constructed in Section 4.1. Chapter 4 also begins with a short setup.

An outline of the proofs of the equivariant theorems of Section 4.2 is given in Section 5.1. The remaining sections of Chapter 5 provide the details.

Chapter 2

Overview of symplectic toric manifolds

This chapter reviews the basics of symplectic toric manifolds and sets up notation that will be used throughout the rest of the dissertation. It combines the perspectives of [Au, Chapter VII], [McDSa, Section 11.3], [Ba, Section 2], [CK, Section 3.3.4], [Gi2], [Gi3], and [Sp, Sections 5,6].

Sections 2.1-2.2 give the definition and describe the basic properties of a compact symplectic toric manifold. Section 2.3 is a preparation for localization computations in a toric setting; it describes the fixed points and curves and the equivariant cohomology.

2.1 Definition, charts, and Kähler classes

Throughout this dissertation, k and N denote fixed positive integers such that $k \leqslant N$ and

$$[N] \equiv \{1, 2, \dots, N\}.$$

If $v \in \mathbb{R}^k$ (or $v \in \mathbb{C}^N$) and $j \in [k]$ (or $j \in [N]$), let $v_j \in \mathbb{R}$ (or $v_j \in \mathbb{C}$) denote the j-th component of v and define

$$\mathrm{supp}(v) \equiv \{j : \; v_j \neq 0\}.$$

If $J \subseteq [N]$, let

$$\mathbb{R}^J \equiv \{v \in \mathbb{R}^N : \mathrm{supp}(v) \subseteq J\} \cong \mathbb{R}^{|J|}, \qquad \mathbb{C}^J \equiv \{z \in \mathbb{C}^N : \mathrm{supp}(z) \subseteq J\} \cong \mathbb{C}^{|J|}.$$

If $A = (a_{ij})_{i \in [k], j \in [N]}$ is a $k \times N$ matrix and $J \subseteq [N]$, denote by A_J the $k \times |J|$ submatrix of A consisting of the columns indexed by the elements of J. Let

$$\omega_{\mathrm{std}} \equiv \frac{\mathrm{i}}{2} \sum_{j=1}^{N} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$$

be the standard symplectic form on $\mathbb{C}^N$. Let

$$\mu_{\mathrm{std}} : \mathbb{C}^N \longrightarrow \mathbb{R}^N, \quad \mu_{\mathrm{std}}(z_1, \dots, z_N) \equiv \big(|z_1|^2, \dots, |z_N|^2\big)$$

be the moment map for the restriction of the standard action of $\mathbb{T}^N \equiv (\mathbb{C}^*)^N$ on $(\mathbb{C}^N, -2\,\omega_{\mathrm{std}})$,

$$(t_1,\ldots,t_N)\cdot(z_1,\ldots,z_N) = (t_1z_1,\ldots,t_Nz_N),$$

to $(S^1)^N \subset \mathbb{T}^N$.

An integer $k\times N$ matrix $M=(m_{ij})_{i\in[k],j\in[N]}$ induces an action of $\mathbb{T}^k\equiv(\mathbb{C}^*)^k$ on $(\mathbb{C}^N,-2\omega_{\mathrm{std}})$,

$$(t_1,\ldots,t_k)\cdot(z_1,\ldots,z_N) = (t_1^{m_{11}}t_2^{m_{21}}\ldots t_k^{m_{k1}}z_1,\ldots,t_1^{m_{1N}}t_2^{m_{2N}}\ldots t_k^{m_{kN}}z_N); \qquad (2.1.1)$$

the moment map of its restriction to $(S^1)^k\subset\mathbb{T}^k$ is

$$\mu_M \equiv M\circ\mu_{\mathrm{std}}\colon \mathbb{C}^N \longrightarrow \mathbb{R}^k.$$

If in addition $\tau\in\mathbb{R}^k$, let

$$\begin{gathered} P_M^\tau \equiv M^{-1}(\tau)\cap(\mathbb{R}^{\geqslant 0})^N,\\ \widetilde{X}_M^\tau \equiv \mathbb{C}^N - \bigcup_{\substack{J\subseteq[N]\\ \mathbb{C}^J\cap\mu_M^{-1}(\tau)=\varnothing}} \mathbb{C}^J = \big\{z\in\mathbb{C}^N\colon\ \mathbb{C}^{\mathrm{supp}(z)}\cap\mu_M^{-1}(\tau)\neq\varnothing\big\}, \qquad X_M^\tau\equiv\widetilde{X}_M^\tau/\mathbb{T}^k; \end{gathered} \qquad (2.1.2)$$

see diagram (2.1.3). By Proposition 2.1.2 below, X_M^τ is a compact projective manifold if the pair (M,τ) is toric in the sense of Definition 2.1.1. In this case, $\mu_{\mathrm{std}}^{-1}(P_M^\tau)/(S^1)^k$ has a unique smooth structure making the projection

$$\mu_{\mathrm{std}}^{-1}(P_M^\tau) \longrightarrow \mu_{\mathrm{std}}^{-1}(P_M^\tau)/(S^1)^k$$

a submersion. With this smooth structure, $\mu_{\mathrm{std}}^{-1}(P_M^\tau)/(S^1)^k$ is diffeomorphic to X_M^τ via a diffeomorphism induced by the inclusion $\mu_{\mathrm{std}}^{-1}(P_M^\tau)\hookrightarrow\widetilde{X}_M^\tau$. We summarize this setup in a diagram:

$$\begin{array}{ccccccccc}
 & & & & & & P_M^\tau\equiv M^{-1}(\tau)\cap(\mathbb{R}^{\geqslant 0})^N & & \\
 & & & & & & \big\downarrow & & \\
\mu_M^{-1}(\tau)\equiv\mu_{\mathrm{std}}^{-1}(P_M^\tau) & \hookrightarrow & \widetilde{X}_M^\tau & \hookrightarrow & \mathbb{C}^N & \xrightarrow{\ \mu_{\mathrm{std}}\ } & (\mathbb{R}^{\geqslant 0})^N & \hookrightarrow & \mathbb{R}^N \\
\big\downarrow{\scriptstyle\text{projection}} & & \big\downarrow{\scriptstyle\text{projection}} & & & \searrow^{\mu_M} & & & \big\downarrow{\scriptstyle M} \\
\frac{\mu_{\mathrm{std}}^{-1}(P_M^\tau)}{(S^1)^k} & \xrightarrow{\ \text{diffeo}\ } & X_M^\tau & & & & & & \mathbb{R}^k\ni\tau
\end{array} \qquad (2.1.3)$$

Given a pair (M,τ) consisting of an integer $k\times N$ matrix M and a vector $\tau\in\mathbb{R}^k$, we define

$$\begin{aligned} \mathscr{V}_M^\tau &\equiv \Big\{J\subseteq[N] : |J|=k,\, P_M^\tau\cap\mathbb{R}^J\neq\varnothing\Big\}\\ &\equiv \Big\{J\subseteq[N] : |J|=k,\, \exists\, v\in M^{-1}(\tau)\cap(\mathbb{R}^{\geqslant 0})^N \ \text{ s.t. } \ \mathrm{supp}(v)\subseteq J\Big\}. \end{aligned} \qquad (2.1.4)$$

Definition 2.1.1. *A pair (M,τ) consisting of an integer $k\times N$ matrix M and a vector $\tau\in\mathbb{R}^k$ is toric if* *(i)* τ *is a regular value of* μ_M *and* $P_M^\tau \neq \varnothing$*;*

(ii) $\det M_J \in \{\pm 1\}$ *for all* $J \in \mathscr{V}_M^\tau$*;*

(iii) $P_M^0 = \{0\}$ *(*$\Longleftrightarrow P_M^\tau$ *is bounded).*

A toric pair (M,τ) *is* **minimal** *if*

(iv) $P_M^\tau \cap \mathbb{R}^{[N]-\{j\}} \neq \varnothing$ *for all* $j \in [N]$*.*

If a pair (M,τ) satisfies (ii) in Definition 2.1.1 above, then

$$z \in \mathbb{C}^N,\ \mathrm{supp}(z) \supseteq J \quad \text{for some} \quad J \in \mathscr{V}_M^\tau \quad \Longrightarrow \quad \exists\, t \in \mathbb{T}^k \quad \text{such that} \quad (t\cdot z)_j = 1 \qquad \forall\, j \in J.$$

If (M,τ) is a toric pair, then a point $z \in \mathbb{C}^N$ lies in $\widetilde{X}_M^\tau$ if and only if $\mathrm{supp}(z) \supseteq J$ for some $J \in \mathscr{V}_M^\tau$ and the $\mathbb{T}^N$-fixed points of X_M^τ are indexed by $\mathscr{V}_M^\tau$; see Lemma 2.1.4(i) and Corollary 2.3.2(a).

Proposition 2.1.2. *If* (M,τ) *is a toric pair, then* X_M^τ *is a connected compact projective manifold of complex dimension* $N-k$ *endowed with a* $\mathbb{T}^N$*-action induced from the standard action of* $\mathbb{T}^N$ *on* $\mathbb{C}^N$*.*

Proof of Proposition 2.1.2. By Lemmas 2.1.5(a), (g), and (h) below, X_M^τ is a connected, compact complex manifold. It admits a positive line bundle by Lemmas 2.2.1, 2.1.7(b), and 2.1.9 below. By the Kodaira Embedding Theorem [GriH, p181], X_M^τ is then projective. □

Remark 2.1.3. If X is a compact symplectic toric manifold in the sense of [Ca, Definition 1.6.1], then the image of its moment map is a Delzant polytope P (a polytope with certain properties [Ca, Definition 2.1.1]); see [At, Theorem 1] or [GuS, Theorem 5.2]. This polytope P determines a fan Σ_P, which in turn determines a compact complex manifold X_{Σ_P}; see [Au, Section VII.1.ac]. This complex manifold X_{Σ_P} is endowed with a symplectic form, a torus action, and a moment map with image P making it into a symplectic toric manifold; see [Au, Theorem VII.2.1]. Moreover, this symplectic form is Kähler with respect to the complex structure, as stated in [Gi2, Section 3] and can be deduced from [Au, Chapter VII]. Since X and X_{Σ_P} have the same moment polytope (i.e. image of the moment map), they are isomorphic as symplectic toric manifolds by Delzant's uniqueness theorem [De, Theorem 2.1]. On the other hand, $X_{\Sigma_P} = X_M^\tau$ for some minimal toric pair (M,τ) by the proof of [Au, Theorem VII.2.1]. Thus, a compact symplectic toric manifold $(X^{2n}, \omega, (S^1)^n, \mu)$ in the sense of [Ca, Definition 1.6.1] admits a complex structure $\mathcal{J}$ so that $(X,\omega,\mathcal{J})$ is Kähler and $(X,\mathcal{J})$ is isomorphic to X_M^τ for some minimal toric pair (M,τ).

Lemma 2.1.5 relies on parts (i) and (j) of Lemma 2.1.4 below which in turn rely on the other parts of Lemma 2.1.4. Lemma 2.1.9 is based on Lemma 2.1.8 and Lemma 2.1.7(d). Lemma 2.1.7(b) follows from Lemma 2.1.7(a), while the proof of Lemma 2.1.7(d) uses Lemma 2.1.7(c).

For $t = (t_1, t_2, \ldots, t_k) \in \mathbb{T}^k$ and $\mathbf{p} = (p_1, p_2, \ldots, p_k) \in \mathbb{Z}^k$, let

$$t^{\mathbf{p}} \equiv t_1^{p_1} t_2^{p_2} \ldots t_k^{p_k}.$$

Lemma 2.1.4. *Let (M,τ) be a toric pair.*

(a) *The subset $P_M^\tau \subset (\mathbb{R}^{\geqslant 0})^N$ is a polytope (i.e. the convex hull of a finite set of points).*

(b) *Let $\eta \in \mathbb{R}^k$ be any regular value of μ_M. If $w \in P_M^\eta$, then*

$$M\colon \{v \in \mathbb{R}^N\colon \operatorname{supp}(v) \subseteq \operatorname{supp}(w)\} \longrightarrow \mathbb{R}^k$$

is onto. In particular, if $w \in P_M^\eta$, then $|\operatorname{supp}(w)| \geqslant k$.

(c) *If $J \in \mathscr{V}_M^\tau$, then $J = \operatorname{supp}(y)$ for some $y \in \mu_M^{-1}(\tau)$.*

(d) *If $J \subseteq [N]$ and $\operatorname{supp}(v) \subseteq J$ for some $v \in P_M^\tau$, then $\operatorname{supp}(w) = J$ for some $w \in P_M^\tau$.*

(e) *The polytope P_M^τ has dimension $N-k$.*

(f) *If v is a vertex of P_M^τ, then $\operatorname{supp}(v) \in \mathscr{V}_M^\tau$.*

(g) *If $\mathrm{Vertices}_M^\tau$ is the set of vertices of the polytope P_M^τ, the map*

$$\operatorname{supp}: \mathrm{Vertices}_M^\tau \longrightarrow \mathscr{V}_M^\tau, \quad v \longrightarrow \operatorname{supp}(v),$$

is a bijection.

(h) *If $y \in \mu_M^{-1}(\tau)$, then $\operatorname{supp}(y) \supseteq J$ for some $J \in \mathscr{V}_M^\tau$.*

(i) *Let $z \in \mathbb{C}^N$. Then, $z \in \widetilde{X}_M^\tau$ if and only if $\operatorname{supp}(z) \supseteq J$ for some $J \in \mathscr{V}_M^\tau$.*

(j) *Let $I, J \in \mathscr{V}_M^\tau$ and $t^{(n)} \in \mathbb{T}^k$. If $|t^{(n)}| \longrightarrow \infty$ and there exists $\delta > 0$ such that $|t_i^{(n)}| \geqslant \delta$ for all $i \in [k]$, then $|(t^{(n)})^{M_I^{-1} M_j}|$ is unbounded for some $j \in J$.*

Proof. (*a*) By [Zi, Theorem 1.1], a subset of $\mathbb{R}^N$ is a polytope if and only if it is a bounded intersection of half-spaces. Thus, the claim follows from (iii) in Definition 2.1.1.
(*b*) This is immediate from the surjectivity of $\mathrm{d}_w \mu_M$.
(*c*) This follows from the second statement in (*b*).
(*d*) Assume that $\operatorname{supp}(v) \subseteq I \subsetneq J$ and that there exists $v' \in P_M^\tau$ with $\operatorname{supp}(v') = I$. Let $I_1 \supset I$ with $I_1 \subseteq J$ and $|I_1| = |I|+1$. We show that there exists $w \in P_M^\tau$ with $\operatorname{supp}(w) = I_1$. By the first statement in (*b*), there exists $w' \in M^{-1}(\tau) \subset \mathbb{R}^N$ with $\operatorname{supp}(w') = I_1$. Let $w = (1-\lambda)v' + \lambda w'$ with $\lambda \in \mathbb{R}$ satisfying

$$\lambda w'_j > 0 \quad \text{if} \quad j \in I_1 - I \quad \text{and} \quad \lambda\left(1 - \frac{w'_j}{v'_j}\right) < 1 \quad \forall\, j \in I.$$

(*e*) By (*d*) together with the second condition in (i) in Definition 2.1.1, $\operatorname{supp}(w) = [N]$ for some $w \in P_M^\tau$ and thus $\dim P_M^\tau = N-k$, since M has rank k by (*b*).
(*f*) By (*e*), $|\operatorname{supp}(v)| \leqslant k$; the opposite inequality follows from the second statement in (*b*).
(*g*) By (*f*), $\operatorname{supp}(v) \in \mathscr{V}_M^\tau$ for every vertex v of P_M^τ. The map supp is injective by (ii) in Definition 2.1.1 and surjective by (*c*) and (ii) in Definition 2.1.1.

(h) By [Zi, Proposition 2.2], every polytope is the convex hull of its vertices; since $\mu_{\mathrm{std}}(y)\in P_M^\tau$ and P_M^τ is a polytope by (a),

$$\mu_{\mathrm{std}}(y) = \sum_{s=1}^{r} \lambda_s v_s$$

for some vertices $v_1, v_2, \ldots, v_r \in P_M^\tau$ and $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{R}^{>0}$. Then, $\mathrm{supp}(y) \supseteq \mathrm{supp}(v_1)$ and $\mathrm{supp}(v_1) \in \mathscr{V}_M^\tau$ by (f).
(i) If $z\in \widetilde{X}_M^\tau$, there exists $y\in\mathbb{C}^{\mathrm{supp}(z)}\cap\mu_M^{-1}(\tau)$. By (h), there exists $J\in\mathscr{V}_M^\tau$ with $J\subseteq\mathrm{supp}(y)$. Since $\mathrm{supp}(y)\subseteq\mathrm{supp}(z)$, it follows that $J\subseteq\mathrm{supp}(z)$. The converse follows from (c).
(j) By (c), there exist $v, w\in(\mathbb{R}^{>0})^k$ such that $M_I v=\tau=M_J w$. By (ii) in Definition 2.1.1, it follows that there exists $\mathfrak{a}\in(\mathbb{Z}^{>0})^k$ such that $M_I^{-1}M_J\mathfrak{a}\in(\mathbb{Z}^{>0})^k$.

Assume by contradiction that $|(t^{(n)})^{M_I^{-1}M_j}|$ is a bounded sequence for all $j\in J$. By passing to subsequences, we may assume that $|(t^{(n)})^{M_I^{-1}M_j}|$ is convergent for all $j\in J$. It follows that

$$\prod_{j\in J} \left|(t^{(n)})^{M_I^{-1}M_j}\right|^{\mathfrak{a}_j} = \left|(t^{(n)})^{M_I^{-1}M_J\mathfrak{a}}\right| \tag{2.1.5}$$

is also convergent. On the other hand, by passing to some subsequences, we may assume that for each $i\in[k]$, $|t_i^{(n)}|$ has a limit (possibly ∞). Since at least one of these limits is ∞ and none is 0, the right-hand side of (2.1.5) diverges leading to a contradiction. $\square$

For $z\in\mathbb{C}^N$ and $J=\{j_1<j_2<\ldots<j_n\}\subseteq[N]$, let

$$z_J \equiv (z_{j_1}, z_{j_2}, \ldots, z_{j_n}).$$

For $z\in\widetilde{X}_M^\tau$, let $[z]\in X_M^\tau$ denote the corresponding class.

Lemma 2.1.5. *Let (M,τ) be a toric pair.*

(a) *The space $\widetilde{X}_M^\tau$ is path-connected.*

(b) *The torus $\mathbb{T}^k$ acts freely on $\widetilde{X}_M^\tau$.*

(c) *The subset $\mathbb{T}^k\cdot\mu_M^{-1}(\tau)$ of $\mathbb{C}^N$ is open.*

(d) *The subset $\mathbb{T}^k\cdot\mu_M^{-1}(\tau)$ of $\widetilde{X}_M^\tau$ is closed.*

(e) *There is a unique map*

$$\rho_M^\tau\colon \widetilde{X}_M^\tau \longrightarrow (\mathbb{R}^{>0})^k \subset \mathbb{T}^k \qquad \text{s.t.} \quad \rho_M^\tau(z)\cdot z \in \mu_M^{-1}(\tau) \quad \forall\, z\in\widetilde{X}_M^\tau\,.$$

Furthermore, this map is smooth.

(f) *The quotient $\mu_M^{-1}(\tau)/(S^1)^k$ is a compact and Hausdorff.*

(g) *The inclusion $\mu_M^{-1}(\tau)\hookrightarrow\widetilde{X}_M^\tau$ induces a homeomorphism*

$$\mu_M^{-1}(\tau)/(S^1)^k \longrightarrow X_M^\tau. \tag{2.1.6}$$

In particular, X_M^τ is compact and Hausdorff.

(h) The space X_M^τ is a complex manifold of complex dimension $N-k$.

Proof. (a) This holds since $\widetilde{X}_M^\tau$ is the complement of coordinate subspaces in $\mathbb{C}^N$.
(b) Let $t\in\mathbb{T}^k$ and $z\in\widetilde{X}_M^\tau$ be such that $t\cdot z=z$. By Lemma 2.1.4(i), there exists $J\in\mathscr{V}_M^\tau$ such that

$$J\equiv\{j_1<\ldots<j_k\}\subseteq\mathrm{supp}(z).$$

By (ii) in Definition 2.1.1, the group homomorphism

$$\mathbb{T}^k\longrightarrow\mathbb{T}^k,\quad t\longrightarrow(t^{M_{j_1}},\ldots,t^{M_{j_k}}),$$

is injective and so $t=(1,1,\ldots,1)$.
(c) For each $z\in\mathbb{C}^N$, let

$$M_z\equiv M\begin{pmatrix}|z_1| & & 0\\ & \ddots & \\ 0 & & |z_N|\end{pmatrix}.$$

If $z\in\mu_M^{-1}(\tau)$, $\mathrm{supp}(z)\supseteq J$ for some $J\in\mathscr{V}_M^\tau$ by Lemma 2.1.4(h). Since M_J is invertible by (ii) in Definition 2.1.1, so are $(M_z)_J$ and $M_z(M_z)^{\mathrm{tr}}$. Since the differential of the map

$$\left(\mathbb{R}^{>0}\right)^k\longrightarrow\mathbb{R}^k,\qquad t\longrightarrow\mu_M(t\cdot z),$$

at $t=(1,\ldots,1)\in(\mathbb{R}^{>0})^k\subset\mathbb{T}^k$ is $2M_z(M_z)^{\mathrm{tr}}$, the differential of the map

$$\mathbb{T}^k\times\mu_M^{-1}(\tau)\longrightarrow\mathbb{R}^k,\qquad(t,z)\longrightarrow\mu_M(t\cdot z),$$

is surjective at $(1,z)$ for all $z\in\mu_M^{-1}(\tau)$. Since the restriction of this differential to the second component vanishes, the differential of the map

$$\mathbb{T}^k\times\mu_M^{-1}(\tau)\longrightarrow\mathbb{C}^N,\qquad(t,z)\longrightarrow t\cdot z,\tag{2.1.7}$$

is surjective at $(1,z)$ for all $z\in\mu_M^{-1}(\tau)$ and so, by the Inverse Function Theorem, the image of (2.1.7) contains an open neighborhood of $\mu_M^{-1}(\tau)$ in $\mathbb{C}^N$.
(d) Let $z^{(n)}\in\widetilde{X}_M^\tau$ and $t^{(n)}\in\mathbb{T}^k$ be sequences such that

$$\lim_{n\longrightarrow\infty}z^{(n)}=z\in\widetilde{X}_M^\tau\qquad\text{and}\qquad y^{(n)}\equiv t^{(n)}\cdot z^{(n)}\in\mu_M^{-1}(\tau).$$

By (iii) in Definition 2.1.1, we can assume that $y^{(n)}\longrightarrow y\in\mu_M^{-1}(\tau)$. By Lemma 2.1.4$(i)$, there exist $J(y),J(z)\in\mathscr{V}_M^\tau$ such that

$$J(y)\equiv\{j_1<\ldots<j_k\}\subseteq\mathrm{supp}(y)\qquad\text{and}\qquad J(z)\subseteq\mathrm{supp}(z);$$

we can assume that $J(y),J(z)\subseteq\mathrm{supp}(y^{(n)})=\mathrm{supp}(z^{(n)})$ for all n. By (ii) in Definition 2.1.1, $M_{J(y)}$ is invertible and so

$$t_i^{(n)}=\left(\tilde{t}^{(n)}\right)^{\left(M_{J(y)}^{-1}\right)_i},\qquad\text{where}\quad\left(\tilde{t}^{(n)}\right)_i=\frac{(y^{(n)})_{j_i}}{(z^{(n)})_{j_i}}\qquad\forall\,i=1,\ldots,k.$$

Since $(y^{(n)})_j \longrightarrow y_j \neq 0$ for all $j \in J(y)$ and $(z^{(n)})_j \longrightarrow z_j$, $|(\tilde{t}^{(n)})_i| \geqslant \delta$ for some $\delta \in \mathbb{R}^{>0}$ and for all n and i. If $|(\tilde{t}^{(n)})|$ is not bounded above, after passing to a subsequence we can assume that $|\tilde{t}^{(n)}| \longrightarrow \infty$. By Lemma 2.1.4($j$), there exists $j \in J(z)$ such that, after passing to a subsequence,

$$\left|(t^{(n)})^{M_j}\right| = \left|(\tilde{t}^{(n)})^{M_{J(y)}^{-1}M_j}\right| \longrightarrow \infty.$$

Since $t^{(n)} \cdot z^{(n)} \longrightarrow y$, it follows that $(z^{(n)})_j \longrightarrow 0$ and so $j \notin \mathrm{supp}(z)$, contrary to the assumption. Thus, $\{\tilde{t}^{(n)}\}$ is a compact subset of $\mathbb{T}^k$. After passing to a subsequence, we can thus assume that $t^{(n)} \longrightarrow t \in \mathbb{T}^k$. It follows that

$$t \cdot z = \lim_{n \longrightarrow \infty} t^{(n)} \cdot \lim_{n \longrightarrow \infty} z^{(n)} = \lim_{n \longrightarrow \infty} t^{(n)} \cdot z^{(n)} = \lim_{n \longrightarrow \infty} y^{(n)} = y.$$

Thus, $z \in \mathbb{T}^k \cdot \mu_M^{-1}(\tau)$.

(e) By the proof of (c), τ is a regular value of the smooth map

$$\Phi\colon \left(\mathbb{R}^{>0}\right)^k \times \widetilde{X}_M^\tau \longrightarrow \mathbb{R}^k, \qquad (t,z) \longrightarrow \mu_M(t \cdot z),$$

and the projection map $\pi_2 : \Phi^{-1}(\tau) \longrightarrow \widetilde{X}_M^\tau$ is a submersion. By (a), (c), and (d), this map is surjective. We show that it is also injective; by (a), (c), and (d), this is equivalent to showing that

$$(r_1,\ldots,r_k) \in \mathbb{R}^k,\ z, (\mathrm{e}^{r_1},\ldots,\mathrm{e}^{r_k}) \cdot z \in \mu_M^{-1}(\tau) \qquad \Longrightarrow \qquad r_i = 0 \qquad \forall\, i = 1,\ldots,k,$$

where the action of $(\mathrm{e}^{r_1},\ldots,\mathrm{e}^{r_k}) \in \mathbb{T}^k$ on z is defined by (2.1.1) as above. We present the argument in the proof of [Ki, 7.2 Lemma]. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(u) \equiv \Big\langle \mu_M\left[(\mathrm{e}^{ur_1},\ldots,\mathrm{e}^{ur_k}) \cdot z\right], (r_1,\ldots,r_k) \Big\rangle \qquad \forall\, u \in \mathbb{R}.$$

Since $f(0) = f(1)$, there exists $u_0 \in (0,1)$ such that $f'(u_0) = 0$. Since

$$f'(u_0) = 2\sum_{j=1}^{N} \mathrm{e}^{2u_0\langle (r_1,\ldots,r_k), M_j\rangle} \Big\langle (r_1,\ldots,r_k), M_j \Big\rangle^2 |z_j|^2,$$

$f'(u_0) = 0$ implies that $\langle (r_1,\ldots,r_k), M_j\rangle z_j = 0$ for all $j \in [N]$. By Lemma 2.1.4(i), there exists $J \in \mathscr{V}_M^\tau$ such that $J \subseteq \mathrm{supp}(z)$ and so $\langle (r_1,\ldots,r_k), M_j\rangle = 0$ for all $j \in J$. By (ii) in Definition 2.1.1, this implies that $r_i = 0$ for all $i \in [k]$. The map ρ_M^τ is π_2^{-1} composed with the projection $(\mathbb{R}^{>0})^k \times X_M^\tau \longrightarrow (\mathbb{R}^{>0})^k$.

(f) Since $\mu_M^{-1}(\tau)$ is compact by (iii) in Definition 2.1.1, so is the quotient space $\mu_M^{-1}(\tau)/(S^1)^k$. If p is the quotient projection map and $A \subset \mu_M^{-1}(\tau)$ is a closed subset,

$$p^{-1}\big(p(A)\big) = (S^1)^k \cdot A \equiv \big\{t \cdot z \colon z \in A,\ t \in (S^1)^k\big\}$$

is the image of the compact subset $(S^1)^k \times A$ in $\mu_M^{-1}(\tau)$ under the continuous multiplication map

$$(S^1)^k \times \mu_M^{-1}(\tau) \longrightarrow \mu_M^{-1}(\tau)$$

and thus compact. Since $\mu_M^{-1}(\tau)$ is Hausdorff, it follows that $p^{-1}(p(A))$ is a closed subset of $\mu_M^{-1}(\tau)$. We conclude the quotient map p is a closed map. Since $\mu_M^{-1}(\tau)$ is a normal

topological space, by [Mu, Lemma 73.3] so is $\mu_M^{-1}(\tau)/(S^1)^k$.
(g) The map (2.1.6) is well-defined, since the inclusion $\mu_M^{-1}(\tau)\hookrightarrow\widetilde{X}_M^\tau$ is equivariant under the inclusion $(S^1)^k\hookrightarrow\mathbb{T}^k$, and is continuous by the defining property of the quotient topology. The map

$$\widetilde{X}_M^\tau \longrightarrow \mu_M^{-1}(\tau), \qquad z\longrightarrow \rho_M^\tau(z)\cdot z,$$

is equivariant with respect to the natural projection $\mathbb{T}^k\longrightarrow(S^1)^k$ by the uniqueness property in (e) and thus induces a continuous map in the opposite direction to (2.1.6). Since $\rho_M^\tau|_{(\mu_M^{-1}(\tau))}=(1,\ldots,1)$, the two maps are easily seen to be mutual inverses.
(h) We cover X_M^τ by holomorphic charts as in [Ba, Propositions 2.17, 2.18]. For each $J\in\mathscr{V}_M^\tau$, let

$$[N]-J\equiv\{i_1<i_2<\ldots<i_{N-k}\},\quad \widetilde{U}_J\equiv\left\{z\in\mathbb{C}^N\colon \mathrm{supp}(z)\supseteq J\right\},\quad U_J\equiv\widetilde{U}_J/\mathbb{T}^k,$$
$$h_J:U_J\longrightarrow\mathbb{C}^{N-k},\qquad h_J[z]\equiv\left(\frac{z_{i_1}}{z_J^{M_J^{-1}M_{i_1}}},\frac{z_{i_2}}{z_J^{M_J^{-1}M_{i_2}}},\ldots,\frac{z_{i_{N-k}}}{z_J^{M_J^{-1}M_{i_{N-k}}}}\right). \tag{2.1.8}$$

By Lemma 2.1.4(i), the collections $\{\widetilde{U}_J: J\in\mathscr{V}_M^\tau\}$ and $\{U_J: J\in\mathscr{V}_M^\tau\}$ cover $\widetilde{X}_M^\tau$ and X_M^τ, respectively. The map h_J is well-defined. First, M_J^{-1} exists and is an integer matrix by (ii) in Definition 2.1.1. Second, if $t\in\mathbb{T}^k$, $z\in\widetilde{U}_J$, and $J\equiv\{j_1<j_2<\ldots<j_k\}$, then

$$\begin{aligned}(t\cdot z)_J^{-M_J^{-1}M_{i_s}}(t\cdot z)_{i_s}&=\left(\left(t^{M_{j_1}}z_{j_1}\right)^{-\left(M_J^{-1}M_{i_s}\right)_1}\ldots\left(t^{M_{j_k}}z_{j_k}\right)^{-\left(M_J^{-1}M_{i_s}\right)_k}\right)t^{M_{i_s}}z_{i_s}\\&=t^{-M_J\left(M_J^{-1}M_{i_s}\right)+M_{i_s}}z_J^{-M_J^{-1}M_{i_s}}z_{i_s}=z_J^{-M_J^{-1}M_{i_s}}z_{i_s},\qquad\forall\, s\in[N-k].\end{aligned}$$

The map h_J^{-1} is the composition of the continuous maps

$$\mathbb{C}^{N-k}\xrightarrow{\widetilde{h_J^{-1}}}\widetilde{U}_J\xrightarrow{\text{projection}}U_J,\qquad\left(\widetilde{h_J^{-1}}(z)\right)_i=\begin{cases}z_s, & \text{if } i=i_s,\\ 1, & \text{if } i\in J,\end{cases}\qquad\forall\, i\in[N].$$

The composition $\mathbb{C}^{N-k}\longrightarrow U_J\xrightarrow{h_J}\mathbb{C}^{N-k}$ is obviously the identity. The other relevant composition is given by $U_J\ni[z]\longrightarrow[y]\in U_J$, where

$$y_i=\begin{cases}z_J^{-M_J^{-1}M_{i_s}}\cdot z_{i_s}, & \text{if } i=i_s,\\ 1, & \text{if } i\in J,\end{cases}\qquad\forall\, j\in[N].$$

Let $t_r\equiv z_J^{-(M_J^{-1})_r}$ for all $r\in[k]$; it follows that $t\cdot z=y$.

If in addition $J'\in\mathscr{V}_M^\tau$, the domain and image of the overlap map $h_J\circ h_{J'}^{-1}$ are complements of some the coordinate subspaces in $\mathbb{C}^{N-k}$, and every component of this map is a ratio of monomials in the complex coordinates. In particular, this map is holomorphic. □

Remark 2.1.6. Let (M,τ) be a toric pair. The projection $\pi\colon\widetilde{X}_M^\tau\longrightarrow X_M^\tau$ is a holomorphic submersion; this can be seen using the charts (2.1.8).

Let K_M^τ be the connected component of τ inside the regular value locus of μ_M.

Lemma 2.1.7. *Let (M,τ) be a toric pair.*

*(*a*) Let $\eta\in\mathbb{R}^k$. Then, η is a regular value of μ_M if and only if $\eta\notin M_J(\mathbb{R}^{\geqslant 0})^{|J|}$ for every $J\subset[N]$ with $|J|\leqslant k-1$.*

*(*b*) The subset K_M^τ of $\mathbb{R}^k$ is an open cone (i.e. an open subset of $\mathbb{R}^k$ such that $\lambda\eta\in K_M^\tau$ whenever $\lambda>0$ and $\eta\in K_M^\tau$).*

*(*c*) For every $\eta\in K_M^\tau$, $\mathscr{V}_M^\eta=\mathscr{V}_M^\tau$.*

*(*d*) For every $\eta\in K_M^\tau$, (M,η) is a toric pair and $X_M^\eta=X_M^\tau$.*

Proof. (a) If η is a regular value of μ_M, $\eta\notin M_J(\mathbb{R}^{\geqslant 0})^{|J|}$ for every $J\subset[N]$ with $|J|\leqslant k-1$ by the second statement in Lemma 2.1.4(b). Suppose $\eta\notin M_J(\mathbb{R}^{\geqslant 0})^{|J|}$ for every $J\subset[N]$ with $|J|\leqslant k-1$. We prove that for every $v\in P_M^\eta$ there exists $J\subseteq\text{supp}(v)$ such that $|J|=k$ and $\det M_J\neq 0$. Suppose not, i.e. $\det M_J=0$ for all $J\subseteq\text{supp}(v)$ with $|J|=k$. We show that there exists $v'\in P_M^\eta$ with $|\text{supp}(v')|<k$; this contradicts the assumption on η. If $|\text{supp}(v)|\geqslant k$, there exists $w\in M^{-1}(0)\subset\mathbb{R}^N$ such that $\text{supp}(w)\subseteq\text{supp}(v)$ and $w_{j_0}>0$ for some $j_0\in\text{supp}(v)$. Let

$$\lambda\equiv\min\left\{\frac{v_j}{w_j}:j\in\text{supp}(v)\text{ such that }w_j>0\right\}.$$

It follows that $v-\lambda w\in P_M^\eta$ and $\text{supp}(v-\lambda w)\subsetneqq\text{supp}(v)$. Continuing in this way, we obtain $v'\in P_M^\eta$ with $|\text{supp}(v')|<k$.
(b) This follows immediately from (a).
(c) We show that the set $\{\eta\in K_M^\tau:\mathscr{V}_M^\eta=\mathscr{V}_M^\tau\}$ is open and closed in K_M^τ and thus equals K_M^τ. It suffices to show that for any $\mathscr{P}\subseteq\{J\subseteq[N]:|J|=k\}$ the set

$$\{\eta\in K_M^\tau:\mathscr{V}_M^\eta=\mathscr{P}\}=\bigcap_{J\in\mathscr{P}}\{\eta\in K_M^\tau:P_M^\eta\cap\mathbb{R}^J\neq\varnothing\}\cap\bigcap_{\substack{J\subseteq[N],|J|=k\\ J\notin\mathscr{P}}}\{\eta\in K_M^\tau:P_M^\eta\cap\mathbb{R}^J=\varnothing\}$$

is open. We show that the set

$$\left\{\eta\in K_M^\tau:\ P_M^\eta\cap\mathbb{R}^J\neq\varnothing\right\}$$

with $J\subseteq[N]$ and $|J|=k$ is open. Let η' be any of its elements and let $w\in P_M^{\eta'}\cap\mathbb{R}^J$. By the surjectivity of $\text{d}_w\mu_M$, $\text{supp}(w)=J$ and $\det M_J\neq 0$; this shows that $M_J(\mathbb{R}^{>0})^k$ is open and

$$\eta'\in M_J\left(\mathbb{R}^{>0}\right)^k\cap K_M^\tau\subseteq\{\eta\in K_M^\tau:P_M^\eta\cap\mathbb{R}^J\neq\varnothing\}.$$

The set

$$\left\{\eta\in K_M^\tau:\ P_M^\eta\cap\mathbb{R}^J=\varnothing\right\}=K_M^\tau-M_J(\mathbb{R}^{\geqslant 0})^{|J|}$$

with $J\subseteq[N]$ and $|J|=k$ is open as well.
(d) Since $P_M^\tau\neq\varnothing$, $\mu_M^{-1}(\tau)\neq\varnothing$ and so $\mathscr{V}_M^\tau\neq\varnothing$ by Lemma 2.1.4(h). Since $\mathscr{V}_M^\tau\neq\varnothing$, $\mathscr{V}_M^\eta\neq\varnothing$ by (c) and so $P_M^\eta\neq\varnothing$. Since (M,τ) satisfies (ii) in Definition 2.1.1, by (c) so does (M,η). Thus, (M,η) is toric. The equality $X_M^\eta=X_M^\tau$ follows from (c) together with Lemma 2.1.4(i). □

Lemma 2.1.8. *Let (M,τ) be a toric pair.*

(a) *The quotient $\mu_M^{-1}(\tau)/(S^1)^k$ admits a unique smooth structure such that the projection*

$$\pi_\tau : \mu_M^{-1}(\tau) \longrightarrow \mu_M^{-1}(\tau)/(S^1)^k \tag{2.1.9}$$

is a submersion.

(b) *There exists a unique symplectic form ω_τ on $\mu_M^{-1}(\tau)/(S^1)^k$ such that*

$$\pi_\tau^*\omega_\tau = \omega_{\mathrm{std}}\Big|_{\mu_M^{-1}(\tau)},$$

where π_τ is the projection (2.1.9).

(c) *The map (2.1.6) is a diffeomorphism.*

Proof. (a) By [tD, Proposition 5.2], if G is a compact Lie group acting freely and smoothly on a manifold M, then the quotient M/G carries a unique differentiable structure such that the projection

$$M \longrightarrow M/G$$

is a submersion. Thus, the claim follows from (i) in Definition 2.1.1 and Lemma 2.1.5(b).
(b) This follows from the Marsden-Weinstein symplectic reduction theorem [MW, Theorem 1].
(c) By (a) and Lemma 2.1.5(g), it is enough to show that the restriction

$$\pi\Big|_{\mu_M^{-1}(\tau)} : \mu_M^{-1}(\tau) \longrightarrow X_M^\tau$$

of the projection $\pi : \widetilde{X}_M^\tau \longrightarrow X_M^\tau$ is a submersion. This follows from the fact that the map

$$\mathbb{T}^k \times \mu_M^{-1}(\tau) \longrightarrow X_M^\tau, \quad (t,z) \longrightarrow [z],$$

is a submersion whose differential at (t,z) vanishes on $T_t\mathbb{T}^k \times 0$. This map is a submersion because it is the composition of two submersions,

$$\mathbb{T}^k \times \mu_M^{-1}(\tau) \longrightarrow \widetilde{X}_M^\tau, \ \ (t,z) \longrightarrow t\cdot z \quad \text{and} \quad \pi : \widetilde{X}_M^\tau \longrightarrow X_M^\tau.$$

The former map is a submersion by the proof of Lemma 2.1.5(c), while π is a submersion by Remark 2.1.6. □

If (M,τ) is a toric pair, we abuse notation and denote by ω_τ not only the form on $\mu_M^{-1}(\tau)/(S^1)^k$ defined by Lemma 2.1.8(b), but also the form it induces on X_M^τ via the diffeomorphism (2.1.6) of Lemma 2.1.8(c). In this case, by Lemma 2.1.7(d) and Lemma 2.1.8(b), for every $\eta \in K_M^\tau$, ω_η is the unique symplectic form on X_M^τ satisfying

$$\pi^*\omega_\eta\Big|_{\mu_M^{-1}(\eta)} = \omega_{\mathrm{std}}\Big|_{\mu_M^{-1}(\eta)}, \quad \text{where} \quad \pi : \widetilde{X}_M^\tau \longrightarrow X_M^\tau \tag{2.1.10}$$

is the projection; see also diagram (2.1.3).

Lemma 2.1.9. *Let (M,τ) be a toric pair. For every $\eta \in K_M^\tau$, ω_η is Kähler with respect to the complex structure on X_M^τ.*

Proof. The form ω_η is positive with respect to the complex structure on X_M^τ by (2.1.10) together with the equality $\mathbb{T}^k \cdot \mu_M^{-1}(\eta) = \widetilde{X}_M^\tau$ (justified by Lemmas 2.1.5(g) and 2.1.7(d)), Remark 2.1.6, and the positivity of ω_{std}. □

Remark 2.1.10. If (M,τ) is a toric pair and $J \subseteq [N]$, the pair (M_J,τ) is toric if and only if $P_{M_J}^\tau \neq \varnothing$. In this case, $X_{M_J}^\tau$ is a connected compact projective manifold of complex dimension $|J|-k$ by Proposition 2.1.2. It is biholomorphic to

$$X_M^\tau(J) \equiv \{[z] \in X_M^\tau : \mathrm{supp}(z) \subseteq J\}$$

via the map

$$X_{M_J}^\tau \ni [z] \longrightarrow [\iota_J(z)] \in X_M^\tau(J), \qquad \text{where} \quad \big(\iota_J(z)\big)_j \equiv \begin{cases} z_r, & \text{if } j = j_r, \\ 0, & \text{if } j \notin J, \end{cases} \tag{2.1.11}$$

if $J = \{j_1 < j_2 < \ldots < j_r\}$. In particular, if (M,τ) is a minimal toric pair and $M_{\hat{j}}$ is the matrix obtained from M by deleting the j-th column, then $X_{M_{\hat{j}}}^\tau$ is a connected compact projective manifold of complex dimension $N-1$. The map (2.1.11) identifies $X_{M_{\hat{j}}}^\tau$ with the hypersurface

$$X_M^\tau([N]-\{j\}) \equiv D_j \equiv \{[z] \in X_M^\tau : z_j = 0\}\,. \tag{2.1.12}$$

If $J \in \mathscr{V}_M^\tau$ with $\mathscr{V}_M^\tau$ defined by (2.1.4), then $X_M^\tau(J)$ is the point

$$[J] \equiv [z_1, \ldots, z_N], \qquad \text{where} \qquad z_j \equiv \begin{cases} 1, & \text{if } \ j \in J; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.13}$$

This follows from Lemma 2.1.4(i) and (ii) in Definition 2.1.1.

If $J \subseteq [N]$ is such that $P_M^\tau \cap \mathbb{R}^J \neq \varnothing$ and $|J| = k+1$, then $X_M^\tau(J)$ is a one-dimensional complex manifold and there exist exactly 2 multi-indices $I \in \mathscr{V}_M^\tau$ with $I \subset J$. The latter follows since multi-indices $I \in \mathscr{V}_M^\tau$ with $I \subset J$ correspond bijectively via ι_J to elements of $\mathscr{V}_{M_J}^\tau$, which in turn correspond to the vertices of $P_{M_J}^\tau$ by Lemma 2.1.4(g); $P_{M_J}^\tau$ has dimension 1 by Lemma 2.1.4(e).

Remark 2.1.11. If (M,τ) is a toric pair with M a $k \times N$ matrix, then $(VM, V\tau)$ is a toric pair whenever $V \in \mathrm{GL}_k(\mathbb{Z})$. In this case, $\mathscr{V}_M^\tau = \mathscr{V}_{VM}^{V\tau}$ and X_M^τ is biholomorphic to $X_{VM}^{V\tau}$. The pair $(VM, V\tau)$ satisfies the first condition of (i) in Definition 2.1.1, since V is an isomorphism. Since $P_M^\tau = P_{VM}^{V\tau}$, $\mathscr{V}_M^\tau = \mathscr{V}_{VM}^{V\tau}$ and so $(VM, V\tau)$ satisfies the second condition of (i), (ii), and (iii) in Definition 2.1.1 as well.

Remark 2.1.12. If (M,τ) is a toric pair with M a $k \times (k+1)$ matrix, then X_M^τ is biholomorphic to $\mathbb{P}^1$. In order to see this, note first that $|\mathscr{V}_M^\tau| = 2$ by Lemma 2.1.4(g) and Lemma 2.1.4(e). By Remark 2.1.11, we can assume that $M_J = \mathrm{Id}_k$ for some $J \in \mathscr{V}_M^\tau$. The claim now follows from (2.1.8): X_M^τ is a compact manifold covered by two charts

$$h_J : U_J \xrightarrow{\sim} \mathbb{C}, \qquad h_I : U_I \xrightarrow{\sim} \mathbb{C}$$

satisfying $h_J(U_J \cap U_I) = h_I(U_J \cap U_I) = \mathbb{C}^*$ since $I \cup J = [k+1]$ and $h_I \circ h_J^{-1}(z) = z^{\pm 1}$ by (ii) in Definition 2.1.1.

2.2 Cohomology, Kähler cone, and Picard group

Throughout the remaining part of this dissertation, (M,τ) is a toric pair. In order to complete the proof in Section 2.1 that X_M^τ is projective, we describe some holomorphic line bundles over it. For each $\mathbf{p}\in\mathbb{Z}^k$, let

$$L_{\mathbf{p}} \equiv \widetilde{X}_M^\tau \times_{\mathbf{p}} \mathbb{C} \equiv \widetilde{X}_M^\tau \times \mathbb{C}/\sim, \qquad \text{where} \quad (z,c)\sim(t^{-1}\cdot z, t^{\mathbf{p}}c), \ \forall\, t\in\mathbb{T}^k. \tag{2.2.1}$$

Since $\pi: \widetilde{X}_M^\tau \longrightarrow X_M^\tau$ with $\pi(z)\equiv[z]$ is a $\mathbb{T}^k$-principal bundle by Lemma 2.1.5(b) and Remark 2.1.6,

$$L_{\mathbf{p}} \longrightarrow X_M^\tau, \qquad [z,c]\longrightarrow[z],$$

is a holomorphic line bundle. Furthermore,

$$L_0 = \mathcal{O}_{X_M^\tau}, \qquad L_{\mathbf{p}}^* = L_{-\mathbf{p}}, \qquad L_{\mathbf{p}}\otimes L_{\mathbf{r}} = L_{\mathbf{p}+\mathbf{r}}.$$

The line bundle L_{-M_j} admits a holomorphic section

$$s_j : X_M^\tau \longrightarrow L_{-M_j}, \qquad [z]\longrightarrow[z,z_j]. \tag{2.2.2}$$

Since s_j is transverse to the zero set by (2.1.8) and $s_j^{-1}(0)=D_j$ by (2.1.12), $c_1(L_{-M_j}) = \mathrm{PD}(D_j)$.

For all $j\in[N]$ and $i\in[k]$, let

$$\mathrm{U}_j \equiv c_1(L_{-M_j}), \qquad \gamma_i \equiv L_{e_i}, \qquad \mathrm{H}_i \equiv c_1(\gamma_i^*), \tag{2.2.3}$$

where $\{e_i : i\in[k]\}\subset\mathbb{Z}^k$ is the standard basis. Thus,

$$L_{-M_j} = \gamma_1^{*\otimes m_{1j}}\otimes\gamma_2^{*\otimes m_{2j}}\otimes\ldots\otimes\gamma_k^{*\otimes m_{kj}} \quad\Longrightarrow\quad \mathrm{U}_j = \sum_{i=1}^k m_{ij}\mathrm{H}_i \qquad \forall\, j\in[N]. \tag{2.2.4}$$

Lemma 2.2.1 below is used in the proof of Proposition 2.1.2 in Section 2.1 and to describe the Kähler cone of X_M^τ in Proposition 2.2.4 below.

Lemma 2.2.1. *For every $\eta\in\mathbb{Z}^k\cap K_M^\tau$,*

$$c_1(L_{-\eta}) = \frac{1}{\pi}[\omega_\eta],$$

where ω_η is the Kähler form defined by (2.1.10).

Proof. We follow closely the proof of [Au, Proposition VII.3.1]. Let

$$L_{-\eta}^{\mathbb{R}} \longrightarrow \mu_M^{-1}(\eta)/(S^1)^k$$

be the pull-back of $L_{-\eta}$ via the diffeomorphism (2.1.6) of Lemma 2.1.8(c) and

$$L_{-\eta}^{S^1} \equiv \mu_M^{-1}(\eta)\times_{-\eta} S^1 \xrightarrow{p} \frac{\mu_M^{-1}(\eta)}{(S^1)^k}$$

be its sphere bundle. Let

$$\begin{array}{ccc} \mu_M^{-1}(\eta)\times S^1 & \xrightarrow{\widetilde{p}} & \mu_M^{-1}(\eta) \\ \Big\downarrow q & & \Big\downarrow \pi_\eta \\ L_{-\eta}^{S^1} & \xrightarrow{p} & \frac{\mu_M^{-1}(\eta)}{(S^1)^k} \end{array}$$

be the natural projections.

Let $e_i^\#$ be the fundamental vector field on $\mu_M^{-1}(\eta)\times S^1$ corresponding to $e_i\in\mathbb{R}^k$ for the $\mathbb{T}^k$-action given by (2.2.1) with $\mathbf{p}=-\eta$. Thus,

$$\begin{aligned} e_i^\# &\equiv \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp(\mathrm{i}te_i)\cdot(x_1+\mathrm{i}y_1,\ldots,x_N+\mathrm{i}y_N,x+\mathrm{i}y)) \\ &= \sum_{j=1}^N m_{ij}\left(y_j\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial y_j}\right)+\eta_i\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right), \end{aligned}$$

where x_j,y_j,x,y are the standard coordinates on $\mathbb{C}^N\equiv(\mathbb{R}^2)^N$ and $\mathbb{C}\equiv\mathbb{R}^2$, respectively. Let

$$\alpha\equiv\sum_{j=1}^N(-x_j\mathrm{d}y_j+y_j\mathrm{d}x_j)\in\Omega^1\left(\mu_M^{-1}(\eta)\right) \qquad \text{and} \qquad \sigma\equiv x\mathrm{d}y-y\mathrm{d}x\in\Omega^1(S^1).$$

Since $\iota_{e_i^\#}(\alpha\oplus\sigma)=0$ on $\mu_M^{-1}(\eta)\times S^1$ for all $i\in[k]$, $\alpha\oplus\sigma$ descends to a 1-form $(\alpha\oplus\sigma)_{S^1}$ on $L_{-\eta}^{S^1}$. This form is a connection 1-form for the principal S^1-bundle $L_{-\eta}^{S^1}$ because it satisfies

$$\mathcal{L}_{X^\#}(\alpha\oplus\sigma)_{S^1}=0, \qquad \iota_{X^\#}(\alpha\oplus\sigma)_{S^1}=1,$$

where

$$X^\#=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$

is the fundamental vector field for the S^1-action on $L_{-\eta}^{S^1}$ as a principal S^1-bundle; see [Au, Exercises V.4,V.5]. Let β denote the curvature form associated to $(\alpha\oplus\sigma)_{S^1}$. By [Au, Section V.4.c], it is uniquely determined by

$$p^*\beta=\mathrm{d}(\alpha\oplus\sigma)_{S^1}.$$

Since $q^*\mathrm{d}((\alpha\oplus\sigma)_{S^1})=-2\widetilde{p}^*\omega_{\mathrm{std}}$, $\beta=-2\omega_\eta$ by the uniqueness of reduced symplectic form ω_η of Lemma 2.1.8(b). Thus, by [Au, Proposition VI.1.18] and [Au, Section VI.5.b],

$$c_1(L_{-\eta}^{\mathbb{R}})=\frac{-1}{2\pi}[\beta]=\frac{1}{\pi}[\omega_\eta]\in H^2_{\mathrm{deR}}(\mu_M^{-1}(\eta)/(S^1)^k),$$

as claimed. □

We define

$$\mathcal{E}_M^\tau\equiv\left\{J\subseteq[N]:\bigcap_{j\in J}D_j=\varnothing\right\}=\left\{J\subseteq[N]:M_{J^c}^{-1}(\tau)\cap(\mathbb{R}^{>0})^{|J^c|}=\varnothing\right\}; \tag{2.2.5}$$

the second equality follows from Lemma 2.1.4(i)(c)(d) and (2.1.12).

Proposition 2.2.2. *If* (M,τ)*is a toric pair,*

$$H^*\left(X_M^\tau\right) \cong \frac{\mathbb{Q}\left[\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_k, \mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_N\right]}{\left(\mathrm{U}_j - \sum\limits_{i=1}^{k} m_{ij}\mathrm{H}_i,\ 1\leqslant j\leqslant [N]\right) + \left(\prod\limits_{j\in J}\mathrm{U}_j : J\in\mathcal{E}_M^\tau\right)}.$$

If, in addition (M,τ) *is minimal,* $H^2\left(X_M^\tau;\mathbb{Z}\right)$ *is free with basis* $\{\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_k\}$.

Proof. This follows from [McDSa, Section 11.3] together with Lemma 2.1.8(c). □

Remark 2.2.3. By Proposition 2.2.2, $H^*(X_M^\tau)$ is generated as a $\mathbb{Q}$-algebra by $\{\mathrm{H}_1, \ldots, \mathrm{H}_k\}$. Along with (ii) in Definition 2.1.1, this implies that $H^*(X_M^\tau)$ is generated as a $\mathbb{Q}$-algebra by $\{\mathrm{U}_1, \ldots, \mathrm{U}_N\}$.

Proposition 2.2.4. *If* (M,τ) *is a minimal toric pair, there is a basis* $\{c_1(L_{-\eta_i}) : i\in[k]\}$ *for* $H^2(X_M^\tau)$ *formed by the first Chern classes of ample line bundles, with* $L_{-\eta_i}$ *as in (2.2.1). In particular, the Kähler cone* $\mathcal{K}_M^\tau$ *of* X_M^τ *has dimension* k.

Proof. By Lemmas 2.2.1, 2.1.9, and 2.1.7(b)(d), there exists a subset $\{\eta_1, \ldots, \eta_k\} \subseteq \mathbb{Z}^k$, linearly independent over $\mathbb{Q}$, such that the line bundles $L_{-\eta_j}$ are positive. The first Chern classes of these line bundles form a $\mathbb{Q}$-basis of $H^2(X_M^\tau)$ by the last statement in Proposition 2.2.2. □

Proposition 2.2.5. *If* (M,τ) *is a minimal toric pair, the Picard group of* X_M^τ *is free of rank* k *and has a* $\mathbb{Z}$*-basis given by* $\gamma_1, \ldots, \gamma_k$ *defined by (2.2.3).*

Proof. The first Chern class homomorphism is an isomorphism because $h^{0,1}(X_M^\tau) = h^{0,2}(X_M^\tau) = 0$ which in turn follows from Proposition 2.2.2. □

Remark 2.2.6. If (M,τ) is a toric pair, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{X_M^\tau}^{\oplus k} \xrightarrow{F} \bigoplus_{j=1}^{N} L_{-M_j} \xrightarrow{G} TX_M^\tau \longrightarrow 0. \tag{2.2.6}$$

Specifically, we can take

$$F([z], e_i) \equiv \left[z, m_{i1}z_1, m_{i2}z_2, \ldots, m_{iN}z_N\right] \qquad \forall\, i\in[k],\ [z]\in X_M^\tau,$$
$$G(z, y_1, \ldots, y_N) \equiv \sum_{j=1}^{N} y_j \mathrm{d}_z\pi\left(\frac{\partial}{\partial z_j}\Big|_z\right), \qquad \forall\, z \in \widetilde{X}_M^\tau,\ y_1, \ldots, y_N\in\mathbb{C},$$

where $\{e_i : i\in[k]\}$ is the standard basis for $\mathbb{C}^k$ and $\pi : \widetilde{X}_M^\tau \longrightarrow X_M^\tau$ is the projection. Thus,

$$c_1\left(TX_M^\tau\right) = \sum_{j=1}^{N} \mathrm{U}_j. \tag{2.2.7}$$

2.3 Torus action and equivariant notation

The equivariant cohomology of a topological space X endowed with a continuous $\mathbb{T}^N$-action is

$$H^*_{\mathbb{T}^N}(X) \equiv H^*\left(E\mathbb{T}^N \times_{\mathbb{T}^N} X\right),$$

where $E\mathbb{T}^N \equiv (\mathbb{C}^\infty - 0)^N$ is the classifying space for $\mathbb{T}^N$. In particular, the equivariant cohomology of a point is

$$H^*_{\mathbb{T}^N}(point) \cong H^*_{\mathbb{T}^N} \equiv H^*\big((\mathbb{P}^\infty)^N\big) = \mathbb{Q}[\alpha_1,\dots,\alpha_N] \equiv \mathbb{Q}[\alpha],$$

where $\alpha_j \equiv c_1(\pi_j^*\mathcal{O}_{\mathbb{P}^\infty}(1))$, $\pi_j : (\mathbb{P}^\infty)^N \longrightarrow \mathbb{P}^\infty$ is the projection onto the j-th component and $\mathcal{O}_{\mathbb{P}^\infty}(1)$ is dual to the tautological line bundle over $\mathbb{P}^\infty$. The equivariant Euler class of an oriented vector bundle $V \longrightarrow X$ endowed with a lift of the $\mathbb{T}^N$-action on X is

$$\mathbf{e}(V) \equiv e(E\mathbb{T}^N \times_{\mathbb{T}^N} V) \in H^*_{\mathbb{T}^N}(X).$$

A $\mathbb{T}^N$-equivariant map $f : X \longrightarrow Y$ between compact oriented manifolds induces a push-forward map

$$f_* : H^s_{\mathbb{T}^N}(X) \longrightarrow H^{s+\dim Y-\dim X}_{\mathbb{T}^N}(Y)$$

characterized by

$$\int_X (f^*\eta)\eta' = \int_Y \eta(f_*\eta') \qquad \forall\, \eta \in H^*_{\mathbb{T}^N}(Y), \eta' \in H^*_{\mathbb{T}^N}(X). \tag{2.3.1}$$

If Y is a point, f_* is the integration along the fiber homomorphism $\int_X : H^s_{\mathbb{T}^N}(X) \longrightarrow H^{s-\dim X}_{\mathbb{T}^N}$. The push-forward map f_* extends to a homomorphism between the modules of fractions with denominators in $\mathbb{Q}[\alpha]$; in particular, the integration along the fiber homomorphism extends to

$$\int_X : H^*_{\mathbb{T}^N}(X) \otimes_{\mathbb{Q}[\alpha]} \mathbb{Q}(\alpha) \longrightarrow \mathbb{Q}(\alpha), \qquad \text{where} \qquad \mathbb{Q}(\alpha) \equiv \mathbb{Q}(\alpha_1,\dots,\alpha_N)$$

is the field of fractions of $\mathbb{Q}[\alpha]$. If X is a compact oriented manifold on which $\mathbb{T}^N$ acts smoothly, then, by the classical Localization Theorem [ABo]

$$\mathbb{Q}[\alpha] \ni \int_X \eta = \sum_{F \subset X^{\mathbb{T}^N}} \int_F \frac{\eta}{\mathbf{e}(N_{F/X})} \in \mathbb{Q}(\alpha), \qquad \forall\, \eta \in H^*_{\mathbb{T}^N}(X), \tag{2.3.2}$$

where the sum runs over the components of the $\mathbb{T}^N$ pointwise fixed locus $X^{\mathbb{T}^N}$ of X.

Lemma 2.3.1. *If (M,τ) is a toric pair, $(\mathbb{T}^N \cdot z/\mathbb{T}^k)$ is diffeomorphic to $\mathbb{T}^{|\mathrm{supp}(z)|-k}$ for every $z \in \widetilde{X}^\tau_M$.*

Proof. By Lemma 2.1.4(i) and (ii) in Definition 2.1.1, there exists $J \subseteq \mathrm{supp}(z)$ with $|J| = k$ and $\det M_J \in \{\pm 1\}$. The map

$$\frac{\mathbb{T}^N \cdot z}{\mathbb{T}^k} \ni [y_1,\dots,y_N] \longrightarrow \Big(y_J^{-M_J^{-1}M_s} y_s\Big)_{s \in \mathrm{supp}(z)-J} \in \mathbb{T}^{|\mathrm{supp}(z)|-k}$$

is a diffeomorphism with inverse

$$\mathbb{T}^{|\mathrm{supp}(z)|-k} \ni \lambda \longrightarrow [t_1 z_1, \ldots, t_N z_N] \in \frac{\mathbb{T}^N \cdot z}{\mathbb{T}^k}, \qquad \text{where} \qquad t_j \equiv \begin{cases} 1, & \text{if } j \notin \mathrm{supp}(z), \\ \frac{\lambda_s}{z_{j_s}}, & \text{if } j = j_s, \\ \frac{1}{z_j}, & \text{if } j \in J, \end{cases}$$

and $\mathrm{supp}(z) - J \equiv \{j_1 < \ldots < j_{|\mathrm{supp}(z)|-k}\}$; see the proof of Lemma 2.1.5(h) in Section 2.1. □

Corollary 2.3.2. *(*a*) The $\mathbb{T}^N$-fixed points in X_M^τ are the points $[J]$ of (2.1.13).*

*(*b*) The closed $\mathbb{T}^N$-fixed curves in X_M^τ are the submanifolds $X_M^\tau(J)$ of Remark 2.1.10 with $|J| = k+1$; all such tuples J are of the form $J = I_1 \cup I_2$ with with $I_1, I_2 \in \mathscr{V}_M^\tau$ and $|I_1 \cap I_2| = k-1$. These curves are biholomorphic to $\mathbb{P}^1$.*

Proof. The first two statements follow from Lemma 2.3.1. The third follows from the last part of Remark 2.1.10. The last follows from Remarks 2.1.10 and 2.1.12. □

We next consider lifts of the standard action of $\mathbb{T}^N$ on X_M^τ to the line bundles $L_{\mathbf{p}}$ of (2.2.1) which will be used in describing the equivariant cohomology of X_M^τ. One such lift is the canonical one

$$(t_1, \ldots, t_N) \cdot [z_1, \ldots, z_N, c] \equiv [t_1 z_1, \ldots, t_N z_N, c] \tag{2.3.3}$$

for all $(t_1, \ldots, t_N) \in \mathbb{T}^N$, $(z_1, \ldots, z_N) \in \widetilde{X}_M^\tau$, and $c \in \mathbb{C}$. We denote by

$$E\mathbb{T}^N \times_{\mathrm{triv}} L_{\mathbf{p}} \longrightarrow E\mathbb{T}^N \times_{\mathbb{T}^N} X_M^\tau$$

the induced line bundle. Another lift is given by

$$(t_1, \ldots, t_N) \cdot [z_1, \ldots, z_N, c] \equiv [t_1 z_1, \ldots, t_N z_N, t_j c] \tag{2.3.4}$$

for all $(t_1, \ldots, t_N) \in \mathbb{T}^N$, $(z_1, \ldots, z_N) \in \widetilde{X}_M^\tau$, and $c \in \mathbb{C}$. We denote by

$$E\mathbb{T}^N \times_j L_{\mathbf{p}} \longrightarrow E\mathbb{T}^N \times_{\mathbb{T}^N} X_M^\tau$$

the induced line bundle. These line bundles are related by isomorphisms

$$(E\mathbb{T}^N \times_{\mathrm{triv}} L_{\mathbf{p}}) \otimes (E\mathbb{T}^N \times_j L_0) \cong E\mathbb{T}^N \times_j L_{\mathbf{p}}, \tag{2.3.5}$$

$$E\mathbb{T}^N \times_j L_0 \cong \mathrm{pr}_1^* \pi_j^* \mathcal{O}_{\mathbb{P}^\infty}(-1), \tag{2.3.6}$$

where $\mathrm{pr}_1 : E\mathbb{T}^N \times_{\mathbb{T}^N} X_M^\tau \longrightarrow (\mathbb{P}^\infty)^N$ denotes the natural projection. The first of these follows by considering the isomorphism

$$L_{\mathbf{p}} \otimes L_0 \longrightarrow L_{\mathbf{p}}, \quad [z, c_1] \otimes [z, c_2] \longrightarrow [z, c_1 c_2] \qquad \forall\, z \in \widetilde{X}_M^\tau, c_1, c_2 \in \mathbb{C}$$

which is $\mathbb{T}^N$-equivariant with respect to the $\mathbb{T}^N$ action on $L_{\mathbf{p}} \otimes L_0$ obtained by tensoring (2.3.3) with (2.3.4) and the action (2.3.4) on $L_{\mathbf{p}}$. The second is given by

$$E\mathbb{T}^N \times_j L_0 \ni (e, z, c) \longrightarrow (e, z, c e_j) \in \mathrm{pr}_1^* \pi_j^* \mathcal{O}_{\mathbb{P}^\infty}(-1) \quad \forall\, e = (e_1, \ldots, e_N) \in E\mathbb{T}^N, z \in \widetilde{X}_M^\tau, c \in \mathbb{C}.$$

For all $j\in[N], i\in[k]$ and with γ_i defined by (2.2.3), let

$$[\mathbf{D_j}]\equiv E\mathbb{T}^N\times_j L_{-M_j}, \quad \boldsymbol{\gamma}_i\equiv E\mathbb{T}^N\times_{\mathrm{triv}}\gamma_i; \qquad u_j\equiv c_1([\mathbf{D_j}]), \quad x_i\equiv c_1(\boldsymbol{\gamma}_i^*)\in H^*_{\mathbb{T}^N}(X_M^\tau). \tag{2.3.7}$$

For each $J\in\mathscr{V}_M^\tau$, the inclusion $[J]\hookrightarrow X_M^\tau$ induces a restriction map

$$\cdot\ (J), \quad \cdot\Big|_J : H^*_{\mathbb{T}^N}(X_M^\tau)\longrightarrow H^*_{\mathbb{T}^N}([J])\cong H^*_{\mathbb{T}^N}. \tag{2.3.8}$$

By (2.2.4), (2.3.5) and (2.3.6),

$$u_j=\sum_{i=1}^k m_{ij}x_i-\alpha_j \qquad \forall\, j\in[N]. \tag{2.3.9}$$

For each $j\in[N]$, the section (2.2.2) of $L_{-M_j}\longrightarrow X_M^\tau$ is $\mathbb{T}^N$-equivariant with respect to the action (2.3.4) and thus induces a section $\mathbf{s}_j$ of $[\mathbf{D}_j]$ over $E\mathbb{T}^N\times_{\mathbb{T}^N}X_M^\tau$. If $J\in\mathscr{V}_M^\tau$ and $j\in J$, $\mathbf{s}_j$ does not vanish on $E\mathbb{T}^N\times_{\mathbb{T}^N}[J]$ and thus

$$J\in\mathscr{V}_M^\tau \qquad \Longrightarrow \qquad u_j(J)=0 \ \ \forall\, j\in J. \tag{2.3.10}$$

On the other hand, if $J\in\mathscr{E}_M^\tau$, with $\mathscr{E}_M^\tau$ defined by (2.2.5), then $\bigoplus_{j\in J}\mathbf{s}_j$ is a nowhere zero section of $\bigoplus_{j\in J}[\mathbf{D}_j]$ and thus

$$J\in\mathscr{E}_M^\tau \qquad \Longrightarrow \qquad \prod_{j\in J}u_j=0\in H^*_{\mathbb{T}^N}(X_M^\tau). \tag{2.3.11}$$

Proposition 2.3.3. *Let (M,τ) be a toric pair.*

(a) If $J=(j_1<\ldots<j_k)\in\mathscr{V}_M^\tau$,

$$\Big(x_1(J)\ x_2(J)\ \ldots\ x_k(J)\Big)=\Big(\alpha_{j_1}\ \alpha_{j_2}\ \ldots\ \alpha_{j_k}\Big)M_J^{-1}.$$

(b) With x_i and u_j defined by (2.3.7),

$$H^*_{\mathbb{T}^N}(X_M^\tau)=\frac{\mathbb{Q}[\alpha]\left[x_1,x_2,\ldots,x_k,u_1,u_2,\ldots,u_N\right]}{\Big(u_j-\sum\limits_{i=1}^k m_{ij}x_i+\alpha_j,\quad 1\leqslant j\leqslant N\Big)+\Big(\prod\limits_{j\in J}u_j\colon J\in\mathscr{E}_M^\tau\Big)}. \tag{2.3.12}$$

If in addition $P\in H^*_{\mathbb{T}^N}(X_M^\tau)$, then $P=0$ if and only if $P(J)=0$ for all $J\in\mathscr{V}_M^\tau$.

Proof. (*a*) This follows from (2.3.9) and (2.3.10).
(*b*) By Remark 2.2.3, there exists $B\subset(\mathbb{Z}^{\geqslant 0})^k$ such that $\{\mathrm{H}^{\mathbf{p}}:\mathbf{p}\in B\}$ is a $\mathbb{Q}$-basis for $H^*(X_M^\tau)$. The map

$$H^*(X_M^\tau)\ni\mathrm{H}^{\mathbf{p}}\longrightarrow x^{\mathbf{p}}\in H^*_{\mathbb{T}^N}(X_M^\tau) \qquad \forall\,\mathbf{p}\in B$$

defines a cohomology extension of the fiber for the fiber bundle $E\mathbb{T}^N\times_{\mathbb{T}^N}X_M^\tau\longrightarrow(\mathbb{P}^\infty)^N$. Thus, by the Leray-Hirsch Theorem [Spa, Chapter 5], the map

$$H^*_{\mathbb{T}^N}\otimes H^*(X_M^\tau)\ni P\otimes\mathrm{H}^{\mathbf{p}}\longrightarrow Px^{\mathbf{p}}\in H^*_{\mathbb{T}^N}(X_M^\tau) \qquad \forall\,\mathbf{p}\in B$$

is an isomorphism of vector spaces. The relations in (2.3.12) hold by (2.3.9) and (2.3.11). We show below that there are no other relations and simultaneously verify the last claim.

Suppose $P\in H^*_{\mathbb{T}^N}(X^\tau_M)$, $P(J)=0$ for all $J\in\mathscr{V}^\tau_M$. By (ii) in Definition 2.1.1, any element P of $H^*_{\mathbb{T}^N}(X^\tau_M)$ is a polynomial in $u_1,\ldots,u_N$ with coefficients in $\mathbb{Q}[\alpha]$. If $J\in\mathscr{V}^\tau_M$ and $j\in[N]-J$, then

$$u_j(J)\Big|_{\alpha_i=0\,\forall\, i\in J}=-\alpha_j \tag{2.3.13}$$

by (2.3.9) and (a). By (2.3.10) and (2.3.13), whenever $u_{i_1}^{a_1}\ldots u_{i_s}^{a_s}$ is a monomial appearing in P and $J\in\mathscr{V}^\tau_M$, $\{i_1,\ldots,i_s\}\cap J\neq\varnothing$. This shows that

$$P\in H^*_{\mathbb{T}^N}(X^\tau_M), P(J)=0 \;\;\forall\, J\in\mathscr{V}^\tau_M \qquad\Longrightarrow\qquad P\in\mathbf{H}',$$

where $\mathbf{H}'$ is the ideal

$$\mathbf{H}'\equiv\Big(u_{i_1}\ldots u_{i_s}:\{i_1,\ldots,i_s\}\cap J\neq\varnothing\;\forall\, J\in\mathscr{V}^\tau_M\Big)\subset\mathbb{Q}[\alpha][u_1,\ldots,u_N].$$

Since $\mathbf{H}'\subseteq(\prod_{j\in J}u_j : J\in\mathscr{E}^\tau_M)$ by Lemma 2.1.4(i),

$$P\in H^*_{\mathbb{T}^N}(X^\tau_M), P(J)=0 \quad\forall\, J\in\mathscr{V}^\tau_M \qquad\Longrightarrow\qquad P\in\left(\prod_{j\in J}u_j : J\in\mathscr{E}^\tau_M\right).$$

By (2.3.11), this implies that $P=0\in H^*_{\mathbb{T}^N}(X^\tau_M)$ if $P(J)=0$ for all $J\in\mathscr{V}^\tau_M$. □

For every $J\in\mathscr{V}^\tau_M$, let

$$\phi_J\equiv\prod_{j\in[N]-J}u_j.$$

By (2.2.6) and (2.3.10),

$$\phi_J(J)=\mathbf{e}\left(T_{[J]}X^\tau_M\right),\qquad \phi_J(I)=0\quad\forall\, I\in\mathscr{V}^\tau_M-\{J\}. \tag{2.3.14}$$

Thus, by the Localization Theorem (2.3.2),

$$\int_{X^\tau_M}P\phi_J=P(J)\qquad\forall\, P\in H^*_{\mathbb{T}^N}(X^\tau_M), J\in\mathscr{V}^\tau_M, \tag{2.3.15}$$

i.e. ϕ_J is the equivariant Poincaré dual of the point $[J]\in X^\tau_M$.

2.4 Examples

Example 2.4.1 (*the complex projective space $\mathbb{P}^{N-1}$ with the standard action of $\mathbb{T}^N$*)**.** If

$$M\equiv(1,\ldots,1)\in\mathbb{R}^N\qquad\text{and}\qquad\tau\in\mathbb{R}^{>0},$$

then

$$\mu_M:\mathbb{C}^N\longrightarrow\mathbb{R},\quad \mu_M(z)=|z_1|^2+\ldots+|z_N|^2,\qquad P^\tau_M=\left\{v\in\left(\mathbb{R}^{\geqslant 0}\right)^N : v_1+\ldots+v_N=\tau\right\},$$

(M,τ) is a minimal toric pair, $\widetilde{X}^\tau_M=\mathbb{C}^N-0$,

$$X^\tau_M=\mathbb{P}^{N-1}\cong(S^{2n-1}(\sqrt{\tau}))/S^1,\qquad H^*_{\mathbb{T}^N}(\mathbb{P}^{N-1})\cong\mathbb{Q}[\alpha_1,\ldots,\alpha_N][x]\Big/\prod_{k=1}^N(x-\alpha_k).$$

Example 2.4.2 (*the Hirzebruch surfaces* $\mathbb{F}_k \equiv \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$)**.** If $k \geqslant 0$,

$$M \equiv \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & k \end{pmatrix}, \qquad \tau \equiv \begin{pmatrix} 1 \\ k+1 \end{pmatrix},$$

then

$$\begin{aligned}(t_1,t_2)\cdot(z_1,z_2,z_3,z_4) &\equiv \big(t_2z_1, t_2z_2, t_1z_3, t_1t_2^kz_4\big)\,,\\ \mu_M:\mathbb{C}^4 \longrightarrow \mathbb{R}^2, \quad \mu_M(z) &\equiv \begin{pmatrix} |z_3|^2+|z_4|^2 \\ |z_1|^2+|z_2|^2+k|z_4|^2 \end{pmatrix},\\ P_M^\tau &= \Big\{ v \in \big(\mathbb{R}^{\geqslant 0}\big)^4 : v_3+v_4=1,\ v_1+v_2+kv_4=k+1 \Big\}\,,\end{aligned}$$

(M,τ) is a minimal toric pair,

$$\widetilde{X}_M^\tau = \mathbb{C}^4 - \Big(\mathbb{C}^2 \times 0 \cup 0 \times \mathbb{C}^2\Big), \qquad X_M^\tau = \widetilde{X}_M^\tau/\mathbb{T}^2.$$

The map

$$X_M^\tau \xrightarrow{\sim} \mathbb{F}_k, \qquad [z_1,z_2,z_3,z_4] \longrightarrow \Big[[z_1,z_2], z_3, \big((z_1,z_2)^{\otimes k} \longrightarrow z_4\big)\Big]\,, \tag{2.4.1}$$

is a $\mathbb{T}^4$-equivariant biholomorphism with respect to the action of $\mathbb{T}^4$ on $\mathbb{F}_k$ given by

$$(t_1,t_2,t_3,t_4)\cdot[[z_1,z_2],z_3,\varphi] \equiv \Big[[t_1z_1,t_2z_2], t_3z_3, (t_1y_1,t_2y_2)^{\otimes k} \longrightarrow t_4\varphi\Big((y_1,y_2)^{\otimes k}\Big)\Big]\,,$$
$$\forall [z_1,z_2] \in \mathbb{P}^1, z_3 \in \mathbb{C}, \varphi \in \mathcal{O}_{\mathbb{P}^1}(k)\big|_{[z_1,z_2]}.$$

By Proposition 2.2.2,

$$\begin{aligned} H^*(\mathbb{F}_k) &= \frac{\mathbb{Q}[\mathrm{H}_1,\mathrm{H}_2,\mathrm{U}_1,\mathrm{U}_2,\mathrm{U}_3,\mathrm{U}_4]}{(\mathrm{U}_1-\mathrm{H}_2,\mathrm{U}_2-\mathrm{H}_2,\mathrm{U}_3-\mathrm{H}_1,\mathrm{U}_4-\mathrm{H}_1-k\mathrm{H}_2)+(\mathrm{U}_1\mathrm{U}_2,\mathrm{U}_3\mathrm{U}_4)}\\ &\cong \frac{\mathbb{Q}[\mathrm{H}_1,\mathrm{H}_2]}{\big(\mathrm{H}_2^2, \mathrm{H}_1(\mathrm{H}_1+k\mathrm{H}_2)\big)}.\end{aligned}$$

Since the toric hypersurfaces D_2 and D_3 defined by (2.1.12) intersect at one point,

$$\mathrm{H}_1\mathrm{H}_2 = \mathrm{U}_2\mathrm{U}_3 = 1, \qquad \mathrm{H}_1^2 = -k\mathrm{H}_1\mathrm{H}_2 = -k.$$

The isomorphism (2.4.1) maps D_4 onto E_0 and D_3 onto E_∞, where

$$\begin{aligned} E_0 &\equiv \text{ image of the section } (1,0) \text{ in } \mathbb{F}_k,\\ E_\infty &\equiv \text{ closure of the image of } (0,\sigma) \text{ in } \mathbb{F}_k,\end{aligned}$$

where σ is any non-zero holomorphic section of $\mathcal{O}_{\mathbb{P}^1}(k)$.

Since $\mathscr{V}_M^\tau = \{(1,3),(1,4),(2,3),(2,4)\}$, by Corollary 2.3.2, the $\mathbb{T}^4$-fixed points in X_M^τ are

$$[1,0,1,0], [1,0,0,1], [0,1,1,0], \quad \text{and} \quad [0,1,0,1],$$

while the closed $\mathbb{T}^4$-fixed curves are all 4 toric hypersurfaces D_1, D_2, D_3, and D_4. By Proposition 2.3.3(b),

$$H_{\mathbb{T}^4}^*(\mathbb{F}_k) \equiv \frac{\mathbb{Q}[\alpha_1,\alpha_2,\alpha_3,\alpha_4][x_1,x_2]}{\Big((x_2-\alpha_1)(x_2-\alpha_2), (x_1-\alpha_3)(x_1+kx_2-\alpha_4)\Big)}.$$

Example 2.4.3 (*products*)**.** Let (M_1,τ_1) and (M_2,τ_2) be (minimal) toric pairs, where M_j is a $k_j\times N_j$ matrix. Define

$$M_1\oplus M_2\equiv\begin{pmatrix}M_1 & 0\\ 0 & M_2\end{pmatrix}.$$

Then, $(M_1\oplus M_2,(\tau_1,\tau_2))$ is a (minimal) toric pair,

$$P^{(\tau_1,\tau_2)}_{M_1\oplus M_2}=P^{\tau_1}_{M_1}\times P^{\tau_2}_{M_2},\qquad\text{and}\qquad \widetilde{X}^{(\tau_1,\tau_2)}_{M_1\oplus M_2}=\widetilde{X}^{\tau_1}_{M_1}\times\widetilde{X}^{\tau_2}_{M_2}.$$

The projections $\pi_j:\mathbb{C}^{N_1+N_2}\longrightarrow\mathbb{C}^{N_j}$ induce a $\mathbb{T}^{N_1+N_2}$-equivariant biholomorphism

$$X^{(\tau_1,\tau_2)}_{M_1\oplus M_2}\ni[z]\xrightarrow{\ \sim\ }\Big([\pi_1(z)],[\pi_2(z)]\Big)\in X^{\tau_1}_{M_1}\times X^{\tau_2}_{M_2},$$

where the action of $\mathbb{T}^{N_1+N_2}$ on $X^{\tau_1}_{M_1}\times X^{\tau_2}_{M_2}$ is the product of the standard actions of $\mathbb{T}^{N_1}$ on $X^{\tau_1}_{M_1}$ and of $\mathbb{T}^{N_2}$ on $X^{\tau_2}_{M_2}$. By (2.1.4) and Lemma 2.1.4$(a)(g)$,

$$\mathscr{V}^{(\tau_1,\tau_2)}_{M_1\oplus M_2}=\mathscr{V}^{\tau_1}_{M_1}\times\mathscr{V}^{\tau_2}_{M_2}.$$

Thus, by Corollary 2.3.2(a), the $\mathbb{T}^{N_1+N_2}$-fixed points of $X^{(\tau_1,\tau_2)}_{M_1\oplus M_2}$ are the points $([I_1],[I_2])$ for all $I_j\in\mathscr{V}^{\tau_j}_{M_j}$, with $[I_j]$ defined by (2.1.13). By Corollary 2.3.2(b) and the second statement in Lemma 2.1.4(b), the closed $\mathbb{T}^{N_1+N_2}$-fixed curves in $X^{(\tau_1,\tau_2)}_{M_1\oplus M_2}$ are all curves of the form $C_1\times[I_2]$ and $[I_1]\times C_2$, where C_j is any closed $\mathbb{T}^{N_j}$-fixed curve in $X^{\tau_j}_{M_j}$ and $I_j\in\mathscr{V}^{\tau_j}_{M_j}$ is arbitrary.

In particular, $\mathbb{P}^{N_1-1}\times\ldots\times\mathbb{P}^{N_s-1}$ is given by the minimal toric pair

$$s\text{ rows}\left\{\left(\begin{array}{c|c|c}
\overbrace{1\,1\ldots1\,1}^{N_1\text{ columns}} & \cdots & \overbrace{0\,0\ldots0\,0}^{N_s\text{ columns}}\\ \hline
0\,0\ldots0\,0 & \cdots & 0\,0\ldots0\,0\\ \hline
\vdots & \ddots & \vdots\\ \hline
0\,0\ldots0\,0 & & 0\,0\ldots0\,0\\ \hline
0\,0\ldots0\,0 & \cdots & 1\,1\,\cdots\,1\,1
\end{array}\right)\right.,\qquad \tau=\begin{pmatrix}\tau_1\\ \tau_2\\ \vdots\\ \tau_s\end{pmatrix}\in(\mathbb{R}^{>0})^s. \tag{2.4.2}$$

By Proposition 2.3.3(b),

$$H^*_{\mathbb{T}^N}\big(\mathbb{P}^{N_1-1}\times\ldots\times\mathbb{P}^{N_s-1}\big)=\frac{\mathbb{Q}[\alpha^{(i)}_j,1\leqslant i\leqslant s,1\leqslant j\leqslant N_i][x_1,\ldots,x_s]}{\left(\prod_{j=1}^{N_i}\Big(x_i-\alpha^{(i)}_j\Big),1\leqslant i\leqslant s\right)}. \tag{2.4.3}$$

Remark 2.4.4. Let $\sigma:[N]\longrightarrow[N]$ be a permutation and (M,τ) be a (minimal) toric pair. Let

$$M^\sigma\equiv\big(m_{i\sigma(j)}\big)_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant N}}\equiv M\circ\mathrm{Id}^\sigma$$

be the matrix obtained from M by permuting its columns as dictated by σ. Then (M^σ,τ) is a (minimal) toric pair as well and $\mathrm{Id}^{\sigma^{-1}}$ induces a biholomorphism between X^τ_M and $X^\tau_{M^\sigma}$ (since $\mu_{M^\sigma}=\mu_M\circ\mathrm{Id}^\sigma$) equivariant with respect to

$$\mathbb{T}^N\longrightarrow\mathbb{T}^N,\quad(t_1,t_2,\ldots,t_N)\longrightarrow\big(t_{\sigma(1)},t_{\sigma(2)},\ldots,t_{\sigma(N)}\big).$$

In particular, taking $k=0$ in Example 2.4.2 gives - via (2.4.2) - $\mathbb{P}^1\times\mathbb{P}^1$ as expected, since the corresponding matrices differ by a permutation of columns.

Chapter 3

Explicit Gromov-Witten formulas

For the remaining part of this dissertation, X_M^τ is the compact projective manifold defined by (2.1.2), where (M,τ) is a minimal toric pair as in Definition 2.1.1. Theorem 3.2.1 in Section 3.2 below computes the one-point GW generating functions $\dot{Z}_\eta$ and $\ddot{Z}_\eta$ of (1.1.6) if $\eta\in H^*(X_M^\tau)$ is of the form $\eta=\mathrm{H}^{\mathbf{p}}$, where $\{\mathrm{H}_1,\ldots,\mathrm{H}_k\}$ is the basis for $H^2(X_M^\tau;\mathbb{Z})$ referred to in Proposition 2.2.2 and

$$\mathrm{H}^{\mathbf{p}} \equiv \mathrm{H}_1^{p_1}\ldots\mathrm{H}_k^{p_k} \qquad \forall\,\mathbf{p}=(p_1,\ldots,p_k)\in(\mathbb{Z}^{\geqslant 0})^k.$$

We denote

$$\dot{Z}_{\mathbf{p}} \equiv \dot{Z}_{\mathrm{H}^{\mathbf{p}}} \qquad \text{and} \qquad \ddot{Z}_{\mathbf{p}} \equiv \ddot{Z}_{\mathrm{H}^{\mathbf{p}}}. \tag{3.0.1}$$

Section 3.1 constructs the explicit formal power series in terms of which $\dot{Z}_{\mathbf{p}}$ and $\ddot{Z}_{\mathbf{p}}$ are expressed in Theorem 3.2.1. Throughout this construction, which extends the constructions in [Z1, Section 2.3] and [PoZ, Sections 2,3] from $\mathbb{P}^{n-1}$ to an arbitrary toric manifold X_M^τ, we assume that

$$\nu_E(\mathbf{d}) \geqslant 0 \qquad \forall\,\mathbf{d}\in\Lambda, \tag{3.0.2}$$

with ν_E as in (3.1.1), and identify

$$H_2(X_M^\tau;\mathbb{Z}) \cong \mathbb{Z}^k$$

via the basis $\{\mathrm{H}_1,\ldots,\mathrm{H}_k\}$. Via this identification $\Lambda\hookrightarrow\mathbb{Z}^k$, with Λ as in (1.1.3).

3.1 Notation and construction of explicit power series

If R is a ring, we denote by $R[\![\hbar]\!]$ the ring of formal Laurent series in $\hbar^{-1}$ with finite principal part:

$$R[\![\hbar]\!] \equiv R[[\hbar^{-1}]] + R[\hbar].$$

Given $f,g\in R[\![\hbar]\!]$ and $s\in\mathbb{Z}^{\geqslant 0}$, we write

$$f \cong g \mod \hbar^{-s} \qquad \text{if} \qquad f-g \in R[\hbar] + \left\{\sum_{i=1}^{s-1} a_i\hbar^{-i} : a_i\in R \;\; \forall\, i\in[s-1]\right\}.$$

If R is a field, we view $R(\hbar)$ as a subring of $R[\![\hbar]\!]$ by associating to each element of $R(\hbar)$ its Laurent series at $\hbar^{-1}=0$.

With the line bundles $L_i^{\pm}$ as in (1.1.2) and U_j as in (2.2.3) and $\mathbf{d}\in H_2(X_M^\tau;\mathbb{Z})$, we define

$$D_j(\mathbf{d}) \equiv \langle \mathrm{U}_j, \mathbf{d}\rangle, \qquad L_i^{\pm}(\mathbf{d}) \equiv \langle c_1(L_i^{\pm}), \mathbf{d}\rangle,$$
$$\nu_E(\mathbf{d}) \equiv \sum_{j=1}^{N} D_j(\mathbf{d}) - \sum_{i=1}^{a} L_i^+(\mathbf{d}) + \sum_{i=1}^{b} L_i^-(\mathbf{d})\,.^{1} \tag{3.1.1}$$

If in addition $Y\subset X$ is a one-dimensional submanifold, let

$$D_j(Y) \equiv D_j\left([Y]_{X_M^\tau}\right), \qquad L_i^{\pm}(Y) \equiv L_i^{\pm}\left([Y]_{X_M^\tau}\right),$$

where $[Y]_{X_M^\tau}\in H_2(X_M^\tau;\mathbb{Z})$ is the homology class represented by Y. By (2.2.7), our assumption (3.0.2), and Footnote 2,

$$c_1(TX_M^\tau) - \sum_{i=1}^{a} c_1(L_i^+) + \sum_{i=1}^{b} c_1(L_i^-) \in \overline{\mathcal{K}}_M^\tau.$$

Thus, if $E\neq E^+$, then X_M^τ is Fano. In this case, the Cone Theorem [La, Theorem 1.5.33] implies that the closed $\mathbb{R}$-cone of curves is a polytope spanned by classes of rational curves. By [La, Proposition 1.4.28] and [La, Theorem 1.4.23(i)], this closed cone is the $\mathbb{R}$-cone spanned by Λ. Thus, $L_i^-(\mathbf{d})<0$ for all $\mathbf{d}\in\Lambda-\{0\}$ and all $i\in[b]$.[2]

Let R be a ring. Similarly to Section 1.1, we denote by $R[[\Lambda-0]]$ and $R[[\Lambda;\nu_E=0]]$ the subalgebras of $R[[\Lambda]]$ given by

$$R[[\Lambda-0]] \equiv \left\{\sum_{\mathbf{d}\in\Lambda} a_{\mathbf{d}}Q^{\mathbf{d}} \in R[[\Lambda]] : a_{\mathbf{0}} = 0\right\},$$
$$R[[\Lambda;\nu_E=0]] \equiv \left\{\sum_{\mathbf{d}\in\Lambda} a_{\mathbf{d}}Q^{\mathbf{d}} \in R[[\Lambda]] : a_{\mathbf{d}} = 0 \text{ if } \nu_E(\mathbf{d})\neq 0\right\}.$$

In some cases, the formal variables whose powers are indexed by Λ within $R[[\Lambda]]$ will be denoted by $Q\equiv(Q_1,\ldots,Q_k)$ as in Section 1.1, while in other cases the formal variables will be $q\equiv(q_1,\ldots,q_k)$. If $f\in R[[\Lambda]]$ and $\mathbf{d}\in\Lambda$, we write $[\![f]\!]_{q;\mathbf{d}}\in R$ for the coefficient of $q^{\mathbf{d}}$ in f. By Proposition 2.2.4, the set $\{\mathbf{s}\in\Lambda : \mathbf{d}-\mathbf{s}\in\Lambda\}$ is finite for every $\mathbf{d}\in\Lambda$; thus,

$$f\in R[[\Lambda]] \quad \text{is invertible} \qquad \Longleftrightarrow \qquad [\![f]\!]_{q;\mathbf{0}}\in R \quad \text{is invertible.}$$

If $f\equiv\sum_{\mathbf{d}\in\Lambda} f_{\mathbf{d}}q^{\mathbf{d}}\in R[[\Lambda]]$, we define

$$[\![f]\!]_{\nu_E=0} \equiv \sum_{\substack{\mathbf{d}\in\Lambda\\ \nu_E(\mathbf{d})=0}} f_{\mathbf{d}}q^{\mathbf{d}} \in R[[\Lambda;\nu_E=0]].$$

[1]By (1.1.2) and (2.2.7), $\nu_E(\mathbf{d})=\langle c_1(T(E^-|_Y)),\mathbf{d}\rangle$ if Y is a smooth complete intersection defined by a holomorphic section of E^+ and $T(E^-|_Y)$ is the tangent bundle of the total space of $E^-|_Y$.

[2]In the notation of [La], $N^1(X_M^\tau)_{\mathbb{R}} = H^{1,1}(X_M^\tau)\cap H^2(X_M^\tau;\mathbb{R})$ as can be seen from Poincaré Duality, Lefschetz Theorem on $(1,1)$-classes, and Hard Lefschetz Theorem.

Let $\mathrm{A}=(\mathrm{A}_1,\ldots,\mathrm{A}_k)$ be a tuple of formal variables. If $f\equiv\sum\limits_{\mathbf{d}\in\Lambda}f_{\mathbf{d}}(\mathrm{A})q^{\mathbf{d}}\in R[[\mathrm{A}]][[\Lambda]]$ and $p\geqslant 0$, we write

$$[\![f]\!]_{\mathrm{A};p}\equiv\sum_{\mathbf{d}\in\Lambda}[\![f_{\mathbf{d}}(\mathrm{A})]\!]_{\mathrm{A};p}\,q^{\mathbf{d}}\in R[\mathrm{A}]_p[[\Lambda]],$$

where $[\![f_{\mathbf{d}}(\mathrm{A})]\!]_{\mathrm{A};p}\in R[\mathrm{A}]$ denotes the degree p homogeneous part of $f_{\mathbf{d}}(\mathrm{A})$ and $R[\mathrm{A}]_p$ the space of homogeneous polynomials of degree p in $\mathrm{A}_1,\ldots,\mathrm{A}_k$ with coefficients in R. Finally, we write

$$|\mathbf{p}|\equiv p_1+p_2+\ldots+p_k\qquad\forall\,\mathbf{p}=(p_1,p_2,\ldots,p_k)\in(\mathbb{Z}^{\geqslant 0})^k.$$

For each $\mathbf{d}\in\Lambda$, let

$$U(\mathbf{d};\mathrm{A},\hbar)\equiv\frac{\prod\limits_{\substack{j\in[N]\\D_j(\mathbf{d})<0}}\prod\limits_{s=D_j(\mathbf{d})+1}^{0}\left(\sum\limits_{i=1}^{k}m_{ij}\mathrm{A}_i+s\hbar\right)}{\prod\limits_{\substack{j\in[N]\\D_j(\mathbf{d})\geqslant 0}}\prod\limits_{s=1}^{D_j(\mathbf{d})}\left(\sum\limits_{i=1}^{k}m_{ij}\mathrm{A}_i+s\hbar\right)}\in\mathbb{Q}[\mathrm{A}][\![\hbar]\!]\,. \tag{3.1.2}$$

By Proposition 2.2.5, the line bundles γ_i^* of (2.2.3) form a basis for the Picard group of X_M^τ. Thus, there are well-defined integers $\ell_{ri}^{\pm}$ such that

$$L_i^{\pm}=\gamma_1^{*\ell_{1i}^{\pm}}\otimes\ldots\otimes\gamma_k^{*\ell_{ki}^{\pm}}. \tag{3.1.3}$$

With A and $\mathbf{d}$ as above, let

$$\begin{aligned}\dot{E}(\mathbf{d};A,\hbar)&\equiv\prod_{i=1}^{a}\prod_{s=1}^{L_i^+(\mathbf{d})}\left(\sum_{r=1}^{k}\ell_{ri}^{+}\mathrm{A}_r+s\hbar\right)\prod_{i=1}^{b}\prod_{s=0}^{-L_i^-(\mathbf{d})-1}\left(\sum_{r=1}^{k}\ell_{ri}^{-}\mathrm{A}_r-s\hbar\right)\in\mathbb{Z}[A,\hbar],\\ \ddot{E}(\mathbf{d};A,\hbar)&\equiv\prod_{i=1}^{a}\prod_{s=0}^{L_i^+(\mathbf{d})-1}\left(\sum_{r=1}^{k}\ell_{ri}^{+}\mathrm{A}_r+s\hbar\right)\prod_{i=1}^{b}\prod_{s=1}^{-L_i^-(\mathbf{d})}\left(\sum_{r=1}^{k}\ell_{ri}^{-}\mathrm{A}_r-s\hbar\right)\in\mathbb{Z}[A,\hbar].\end{aligned} \tag{3.1.4}$$

The formal power series computing $\dot{Z}_{\mathbf{p}}$ and $\ddot{Z}_{\mathbf{p}}$ in Theorem 3.2.1 are obtained from

$$\begin{aligned}\dot{Y}(\mathrm{A},\hbar,q)&\equiv\sum_{\mathbf{d}\in\Lambda}q^{\mathbf{d}}U(\mathbf{d};\mathrm{A},\hbar)\dot{E}(\mathbf{d};\mathrm{A},\hbar)\in\mathbb{Q}[A]\big[\big[\hbar^{-1},\Lambda\big]\big],\\ \ddot{Y}(\mathrm{A},\hbar,q)&\equiv\sum_{\mathbf{d}\in\Lambda}q^{\mathbf{d}}U(\mathbf{d};\mathrm{A},\hbar)\ddot{E}(\mathbf{d};\mathrm{A},\hbar)\in\mathbb{Q}[A]\big[\big[\hbar^{-1},\Lambda\big]\big].\end{aligned} \tag{3.1.5}$$

We define

$$\dot{I}_{\mathbf{0}}(q)\equiv\dot{Y}(\mathrm{A},\hbar,q)\mod\hbar^{-1},\qquad\ddot{I}_{\mathbf{0}}(q)\equiv\ddot{Y}(\mathrm{A},\hbar,q)\mod\hbar^{-1}, \tag{3.1.6}$$

and so

$$\dot{I}_{\mathbf{0}}(q) \equiv 1 + \delta_{b,0} \sum_{\substack{\mathbf{d}\in\Lambda-0,\ \nu_E(\mathbf{d})=0 \\ D_j(\mathbf{d})\geqslant 0\ \forall\, j\in[N]}} q^{\mathbf{d}} \frac{\prod_{i=1}^{a} \big(L_i^+(\mathbf{d})!\big)}{\prod_{j=1}^{N} \big(D_j(\mathbf{d})!\big)},$$

$$\ddot{I}_{\mathbf{0}}(q) \equiv 1 + \sum_{\substack{\mathbf{d}\in\Lambda-0,\ \nu_E(\mathbf{d})=0 \\ D_j(\mathbf{d})\geqslant 0\ \forall\, j\in[N] \\ L_i^+(\mathbf{d})=0\,\forall\, i\in[a]}} q^{\mathbf{d}} (-1)^{\sum_{i=1}^{b} L_i^-(\mathbf{d})} \frac{\prod_{i=1}^{b} \big((-L_i^-(\mathbf{d}))!\big)}{\prod_{j=1}^{N} \big(D_j(\mathbf{d})!\big)}.$$

We next describe an operator $\mathbf{D}^{\mathbf{p}}$ acting on a subset of $\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]]$ and certain associated "structure coefficients" in $\mathbb{Q}[[\Lambda]]$ which occur in the formulas for $\dot{Z}_{\mathbf{p}}$ and $\ddot{Z}_{\mathbf{p}}$. Fix an element $Y(\mathrm{A},\hbar,q)$ of $\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]]$ such that for all $\mathbf{d}\in\Lambda$

$$[\![Y(\mathrm{A},\hbar,q)]\!]_{q;\mathbf{d}} \equiv \frac{f_{\mathbf{d}}(\mathrm{A},\hbar)}{g_{\mathbf{d}}(\mathrm{A},\hbar)}$$

for some homogeneous polynomials $f_{\mathbf{d}}(\mathrm{A},\hbar), g_{\mathbf{d}}(\mathrm{A},\hbar)\in\mathbb{Q}[\mathrm{A},\hbar]$ satisfying

$$f_{\mathbf{0}}(\mathrm{A},\hbar) = g_{\mathbf{0}}(\mathrm{A},\hbar)\,, \qquad \deg f_{\mathbf{d}} - \deg g_{\mathbf{d}} = -\nu_E(\mathbf{d}), \qquad g_{\mathbf{d}}\big|_{\mathrm{A}=0} \neq 0 \quad \forall\, \mathbf{d}\in\Lambda. \tag{3.1.7}$$

This condition is satisfied by the power series $\dot{Y}$ and $\ddot{Y}$ of (3.1.5) and so the construction below applies to $Y=\dot{Y}$ and $Y=\ddot{Y}$. We inductively define

$$J_p(Y) \in \mathrm{End}_{\mathbb{Q}[[\Lambda;\nu_E=0]]}\big(\mathbb{Q}[[\Lambda;\nu_E=0]][\mathrm{A}]_p\big) \ \ \forall\, p\in\mathbb{Z}^{\geqslant 0}, \quad \mathbf{D}^{\mathbf{p}}Y(\mathrm{A},\hbar,q) \in \mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]] \ \ \forall\, \mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$$

satisfying

(P1) for every $\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$ with $|\mathbf{p}|=p$, $[\![\{J_p(Y)\}(\mathrm{A}^{\mathbf{p}})]\!]_{q;\mathbf{0}} = \mathrm{A}^{\mathbf{p}}$;

(P2) there exist $\mathrm{C}_{\mathbf{p},s}^{(\mathbf{r})}\equiv\mathrm{C}_{\mathbf{p},s}^{(\mathbf{r})}(Y) \in \mathbb{Q}[[\Lambda]]$ with $\mathbf{p},\mathbf{r}\in(\mathbb{Z}^{\geqslant 0})^k$ and $s\in\mathbb{Z}^{\geqslant 0}$ such that

$$\mathbf{D}^{\mathbf{p}}Y(\mathrm{A},\hbar,q) = \hbar^{|\mathbf{p}|} \sum_{s=0}^{\infty} \sum_{\mathbf{r}\in(\mathbb{Z}^{\geqslant 0})^k} \mathrm{C}_{\mathbf{p},s}^{(\mathbf{r})}(q)\mathrm{A}^{\mathbf{r}}\hbar^{-s}\,, \tag{3.1.8}$$

$$\Big[\!\!\Big[\mathrm{C}_{\mathbf{p},s}^{(\mathbf{r})}\Big]\!\!\Big]_{q;\mathbf{d}}\!=0 \text{ if } s\neq\nu_E(\mathbf{d})+|\mathbf{r}|, \quad \Big[\!\!\Big[\mathrm{C}_{\mathbf{p},s}^{(\mathbf{r})}\Big]\!\!\Big]_{\nu_E=0}\!=\delta_{\mathbf{p},\mathbf{r}}\delta_{|\mathbf{r}|,s} \text{ if } s\leqslant|\mathbf{p}|, \quad \Big[\!\!\Big[\mathrm{C}_{\mathbf{p},|\mathbf{r}|}^{(\mathbf{r})}\Big]\!\!\Big]_{q;\mathbf{0}}\!=\delta_{\mathbf{p},\mathbf{r}}. \tag{3.1.9}$$

By (3.1.7), we can define $J_0(Y) \in \mathbb{Q}[[\Lambda;\nu_E=0]]$ and $\mathbf{D}^{\mathbf{0}}Y\in\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]]$ by

$$\{J_0(Y)\}(1) \equiv Y(\mathrm{A},\hbar,q) \mod \hbar^{-1}, \qquad \mathbf{D}^{\mathbf{0}}Y(\mathrm{A},\hbar,q) \equiv [\{J_0(Y)\}(1)]^{-1}\,Y(\mathrm{A},\hbar,q). \tag{3.1.10}$$

Suppose next that $p\geqslant 0$ and we have constructed an operator $J_p(Y)$ and power series $\mathbf{D}^{\mathbf{p}'}Y$ for all $\mathbf{p}'\in(\mathbb{Z}^{\geqslant 0})^k$ with $|\mathbf{p}'|=p$ satisfying the above properties. For each $\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$

with $|\mathbf{p}|=p+1$, let

$$\widetilde{\mathbf{D}}^{\mathbf{p}}Y(\mathrm{A},\hbar,q) \equiv \frac{1}{|\mathrm{supp}(\mathbf{p})|}\sum_{i\in\mathrm{supp}(\mathbf{p})}\bigg\{\mathrm{A}_i+\hbar\, q_i\frac{\mathrm{d}}{\mathrm{d}q_i}\bigg\}\mathbf{D}^{\mathbf{p}-e_i}Y(\mathrm{A},\hbar,q)\in\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]],$$
$$\{J_{p+1}(Y)\}(\mathrm{A}^{\mathbf{p}}) \equiv \Big[\!\!\Big[\widetilde{\mathbf{D}}^{\mathbf{p}}Y(\mathrm{A},\hbar,q)\mod \hbar^{-1}\Big]\!\!\Big]_{\mathrm{A};p+1}, \tag{3.1.11}$$

where $\{e_1,\ldots,e_k\}$ is the canonical basis for $\mathbb{Z}^k$. By (P2),

$$\begin{aligned}\{J_{p+1}(Y)\}(\mathrm{A}^{\mathbf{p}}) &= \frac{1}{|\mathrm{supp}(\mathbf{p})|}\sum_{i\in\mathrm{supp}(\mathbf{p})}\Bigg[\sum_{|\mathbf{r}|=p}\mathrm{C}^{(\mathbf{r})}_{\mathbf{p}-e_i,p}\mathrm{A}_i\cdot\mathrm{A}^{\mathbf{r}}+\sum_{|\mathbf{r}|=p+1}q_i\frac{\mathrm{d}\mathrm{C}^{(\mathbf{r})}_{\mathbf{p}-e_i,p+1}}{\mathrm{d}q_i}\mathrm{A}^{\mathbf{r}}\Bigg]\\ &= \frac{1}{|\mathrm{supp}(\mathbf{p})|}\sum_{|\mathbf{r}|=p+1}\Bigg[\sum_{i\in\mathrm{supp}(\mathbf{p})}\bigg(\mathrm{C}^{(\mathbf{r}-e_i)}_{\mathbf{p}-e_i,p}+q_i\frac{\mathrm{d}\mathrm{C}^{(\mathbf{r})}_{\mathbf{p}-e_i,p+1}}{\mathrm{d}q_i}\bigg)\Bigg]\mathrm{A}^{\mathbf{r}},\end{aligned} \tag{3.1.12}$$

where we set $\mathrm{C}^{(\mathbf{r}-e_i)}_{\mathbf{p}-e_i,p}\equiv 0$ if $i\notin\mathrm{supp}(\mathbf{r})$. By (3.1.12) and (3.1.9),

$$\{J_{p+1}(Y)\}(\mathrm{A}^{\mathbf{p}})\in\mathbb{Q}[[\Lambda;\nu_E=0]][\mathrm{A}]_{p+1} \quad\text{and}\quad [\![\{J_{p+1}(Y)\}(\mathrm{A}^{\mathbf{p}})]\!]_{q;\mathbf{0}}=\mathrm{A}^{\mathbf{p}}\,; \tag{3.1.13}$$

in particular, $J_{p+1}(Y)$ is invertible. With $\mathrm{c}_{\mathbf{p};\mathbf{p}'}(q)\in\mathbb{Q}[[\Lambda;\nu_E=0]]$ for $\mathbf{p},\mathbf{p}'\in(\mathbb{Z}^{\geqslant 0})^k$ with $|\mathbf{p}|,|\mathbf{p}'|=p+1$ given by

$$\{J_{p+1}(Y)\}^{-1}(\mathrm{A}^{\mathbf{p}})\equiv\sum_{\mathbf{p}'\in(\mathbb{Z}^{\geqslant 0})^k,|\mathbf{p}'|=p+1}\mathrm{c}_{\mathbf{p};\mathbf{p}'}(q)\mathrm{A}^{\mathbf{p}'}, \tag{3.1.14}$$

we define

$$\mathbf{D}^{\mathbf{p}}Y(\mathrm{A},\hbar,q)\equiv\sum_{\mathbf{p}'\in(\mathbb{Z}^{\geqslant 0})^k,|\mathbf{p}'|=p+1}\mathrm{c}_{\mathbf{p};\mathbf{p}'}(q)\widetilde{\mathbf{D}}^{\mathbf{p}'}Y(\mathrm{A},\hbar,q). \tag{3.1.15}$$

By (3.1.15) and the inductive assumption (3.1.8),

$$\mathbf{D}^{\mathbf{p}}Y(\mathrm{A},\hbar,q)=\hbar^{p+1}\sum_{s=0}^{\infty}\sum_{\mathbf{r}\in(\mathbb{Z}^{\geqslant 0})^k}\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}(q)\mathrm{A}^{\mathbf{r}}\hbar^{-s}, \qquad\text{where}$$
$$\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}=\sum_{\mathbf{p}'\in(\mathbb{Z}^{\geqslant 0})^k,|\mathbf{p}'|=p+1}\frac{\mathrm{c}_{\mathbf{p};\mathbf{p}'}}{|\mathrm{supp}(\mathbf{p}')|}\sum_{i\in\mathrm{supp}(\mathbf{p}')}\bigg(\mathrm{C}^{(\mathbf{r}-e_i)}_{\mathbf{p}'-e_i,s-1}+q_i\frac{\mathrm{d}\mathrm{C}^{(\mathbf{r})}_{\mathbf{p}'-e_i,s}}{\mathrm{d}q_i}\bigg), \tag{3.1.16}$$

where we set $\mathrm{C}^{(\mathbf{r}-e_i)}_{\mathbf{p}'-e_i,s-1}=0$ if $i\notin\mathrm{supp}(\mathbf{r})$ or $s=0$. By the first property in (3.1.9) with $\mathbf{p}$ replaced by $\mathbf{p}'-e_i$ with $|\mathbf{p}'|=p+1$ and $i\in\mathrm{supp}(\mathbf{p}')$, $\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}$ satisfies this property as well. By the second property in (3.1.9) with $\mathbf{p}$ replaced by $\mathbf{p}'-e_i$ with $|\mathbf{p}'|=p+1$ and $i\in\mathrm{supp}(\mathbf{p}')$, $\Big[\!\!\Big[\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}\Big]\!\!\Big]_{\nu_E=0}=0$ if $s\leqslant p$. Since $\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},p+1}=\delta_{\mathbf{p},\mathbf{r}}$ whenever $|\mathbf{r}|=p+1$ by (3.1.16) and (3.1.14), $\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}$ also satisfies the second property in (3.1.9). By the second statement in (3.1.13) and (3.1.14), $[\![\mathrm{c}_{\mathbf{p};\mathbf{p}'}]\!]_{q;\mathbf{0}}=\delta_{\mathbf{p},\mathbf{p}'}$. Thus, by the third property in (3.1.9) with $\mathbf{p}$ replaced by $\mathbf{p}'-e_i$ with $|\mathbf{p}'|=p+1$ and $i\in\mathrm{supp}(\mathbf{p}')$, $\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}$ satisfies the last property in (3.1.9) as well.

Define $\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}} \equiv \widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(Y) \in \mathbb{Q}[[\Lambda]]$ for $\mathbf{p},\mathbf{s} \in (\mathbb{Z}^{\geqslant 0})^k$ and $r \in \mathbb{Z}^{\geqslant 0}$ with $|\mathbf{s}| \leqslant |\mathbf{p}| - r$ and $r \leqslant |\mathbf{p}|$ by

$$\sum_{t=0}^{r} \sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant 0})^k \\ |\mathbf{s}|\leqslant|\mathbf{p}|-t}} \widetilde{\mathrm{C}}^{(t)}_{\mathbf{p},\mathbf{s}} \mathrm{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|+r-t} = \delta_{\mathbf{p},\mathbf{r}}\delta_{r,0} \qquad \forall\, \mathbf{r}\in(\mathbb{Z}^{\geqslant 0})^k,\ |\mathbf{r}| \leqslant |\mathbf{p}|-r. \tag{3.1.17}$$

Equations (3.1.17) indeed uniquely determine $\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}}$, since

$$\sum_{t=0}^{r} \sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant 0})^k \\ |\mathbf{s}|\leqslant|\mathbf{p}|-t}} \widetilde{\mathrm{C}}^{(t)}_{\mathbf{p},\mathbf{s}} \mathrm{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|+r-t} = \sum_{t=0}^{r-1} \sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant 0})^k \\ |\mathbf{s}|\leqslant|\mathbf{p}|-t}} \widetilde{\mathrm{C}}^{(t)}_{\mathbf{p},\mathbf{s}} \mathrm{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|+r-t} + \sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant 0})^k \\ |\mathbf{s}|<|\mathbf{r}|}} \widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}} \mathrm{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|} + \widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{r}}, \tag{3.1.18}$$

as follows from (3.1.9). By (3.1.17) together with the first and third statements in (3.1.9),

$$\left[\!\!\left[\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}} \right]\!\!\right]_{q;\mathbf{0}} = \delta_{\mathbf{p},\mathbf{s}}\delta_{r,0}\,. \tag{3.1.19}$$

By (3.1.17), (3.1.18), and induction on $|\mathbf{s}|$,

$$\widetilde{\mathrm{C}}^{(0)}_{\mathbf{p},\mathbf{s}}(q) = \delta_{\mathbf{p},\mathbf{s}} \qquad \forall\, \mathbf{p},\mathbf{s}\in(\mathbb{Z}^{\geqslant 0})^k \quad \text{with} \quad |\mathbf{s}| \leqslant |\mathbf{p}|. \tag{3.1.20}$$

By (3.1.17), (3.1.18), the first statement in (3.1.9), and induction on $|\mathbf{s}|$ and r,

$$\left[\!\!\left[\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}} \right]\!\!\right]_{q;\mathbf{d}} = 0 \qquad \text{if} \quad \nu_E(\mathbf{d}) \neq r. \tag{3.1.21}$$

Remark 3.1.1. With $\dot{Y}$, $\ddot{Y}$ as in (3.1.5) and $\dot{I}_{\mathbf{0}}$, $\ddot{I}_{\mathbf{0}}$ as in (3.1.6),

$$\left\{J_0(\dot{Y})\right\}(1) = \dot{I}_{\mathbf{0}}(q), \qquad \mathbf{D}^{\mathbf{0}}\dot{Y}(\mathrm{A},\hbar,q) = \frac{1}{\dot{I}_{\mathbf{0}}(q)}\dot{Y}(\mathrm{A},\hbar,q),$$
$$\left\{J_0(\ddot{Y})\right\}(1) = \ddot{I}_{\mathbf{0}}(q), \qquad \mathbf{D}^{\mathbf{0}}\ddot{Y}(\mathrm{A},\hbar,q) = \frac{1}{\ddot{I}_{\mathbf{0}}(q)}\ddot{Y}(\mathrm{A},\hbar,q),$$

by (3.1.10).

Define

$$\left\{\mathrm{A} + \hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}} \equiv \left\{\mathrm{A}_1 + \hbar\, q_1\frac{\mathrm{d}}{\mathrm{d}q_1}\right\}^{p_1} \ldots \left\{\mathrm{A}_k + \hbar\, q_k\frac{\mathrm{d}}{\mathrm{d}q_k}\right\}^{p_k} \qquad \forall\, \mathbf{p} = (p_1,\ldots,p_k)\in(\mathbb{Z}^{\geqslant 0})^k.$$

Remark 3.1.2. If $\nu_E(\mathbf{d}) > 0$ for all $\mathbf{d}\in\Lambda - \{0\}$ and $Y(\mathrm{A},\hbar,q)\in\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]]$ satisfies (3.1.7), then $J_p(Y) = \mathrm{Id}$ for all $p\in\mathbb{Z}^{\geqslant 0}$ by (P1) above. Along with the first equation in (3.1.11), (3.1.14), (3.1.15), and induction on $|\mathbf{p}|$, this implies that

$$\mathbf{D}^{\mathbf{p}}Y(\mathrm{A},\hbar,q) = \left\{\mathrm{A} + \hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}} Y(\mathrm{A},\hbar,q)$$

for all $\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$.

Remark 3.1.3. Suppose $p^*\in\mathbb{Z}^{\geqslant 0}$, $Y(\mathrm{A},\hbar,q)\in\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]]$ satisfies (3.1.7), and

$$\deg_\hbar f_{\mathbf{d}}(\mathrm{A},\hbar) - \deg_\hbar g_{\mathbf{d}}(\mathrm{A},\hbar) < -p^* \qquad \forall\, \mathbf{d}\in\Lambda-\{0\}.$$

By the same reasoning as in Remark 3.1.2, we again find that

$$J_p(Y) = \mathrm{Id}, \qquad \mathbf{D}^{\mathbf{p}}Y(\mathrm{A},\hbar,q) \equiv \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}} Y(\mathrm{A},\hbar,q),$$

for all $p\in\mathbb{Z}^{\geqslant 0}$ and $\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$ such that $p,|\mathbf{p}|\leqslant p^*$.

Remark 3.1.4. Let (M,τ) be the toric pair of Example 2.4.1 with $N=n$ so that $X_M^\tau=\mathbb{P}^{n-1}$. Let

$$E \equiv \bigoplus_{i=1}^{a}\mathcal{O}_{\mathbb{P}^{n-1}}(\ell_i^+)\oplus\bigoplus_{i=1}^{b}\mathcal{O}_{\mathbb{P}^{n-1}}(\ell_i^-)$$

with $a,b\geqslant 0$, $\ell_i^+>0$ for all $i\in[a]$, $\ell_i^-<0$ for all $i\in[b]$, and $\sum\limits_{i=1}^{a}\ell_i^+-\sum\limits_{i=1}^{b}\ell_i^-\leqslant n$. Thus,

$$\nu_E(d) = \left(n-\sum_{i=1}^{a}\ell_i^+ +\sum_{i=1}^{b}\ell_i^-\right)d$$

for all $d\in\mathbb{Z}^{\geqslant 0}$. By (3.1.5),

$$\dot{Y}(\mathrm{A},\hbar,q) = \sum_{d=0}^{\infty} q^d\frac{\prod\limits_{i=1}^{a}\prod\limits_{s=1}^{\ell_i^+ d}\big(\ell_i^+\mathrm{A}+s\hbar\big)\prod\limits_{i=1}^{b}\prod\limits_{s=0}^{-\ell_i^- d-1}\big(\ell_i^-\mathrm{A}-s\hbar\big)}{\prod\limits_{s=1}^{d}(\mathrm{A}+s\hbar)^n},$$

$$\ddot{Y}(\mathrm{A},\hbar,q) = \sum_{d=0}^{\infty} q^d\frac{\prod\limits_{i=1}^{a}\prod\limits_{s=0}^{\ell_i^+ d-1}\big(\ell_i^+\mathrm{A}+s\hbar\big)\prod\limits_{i=1}^{b}\prod\limits_{s=1}^{-\ell_i^- d}\big(\ell_i^-\mathrm{A}-s\hbar\big)}{\prod\limits_{s=1}^{d}(\mathrm{A}+s\hbar)^n}.$$

By Remark 3.1.3,

$$J_p(\dot{Y}) = \mathrm{Id},\quad \mathbf{D}^p\dot{Y}(\mathrm{A},\hbar,q) = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p \dot{Y}(\mathrm{A},\hbar,q) \qquad \forall\, p<b,$$
$$J_p(\ddot{Y}) = \mathrm{Id},\quad \mathbf{D}^p\ddot{Y}(\mathrm{A},\hbar,q) = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p \ddot{Y}(\mathrm{A},\hbar,q) \qquad \forall\, p<a.$$

If $\sum\limits_{i=1}^{a}\ell_i^+-\sum\limits_{i=1}^{b}\ell_i^-<n$, then

$$J_p(\dot{Y}),J_p(\ddot{Y})=\mathrm{Id},\quad \mathbf{D}^p\dot{Y} = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p\dot{Y},\quad \mathbf{D}^p\ddot{Y} = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p\ddot{Y}$$

for all p by Remark 3.1.2. If $\sum_{i=1}^{a}\ell_i^+ - \sum_{i=1}^{b}\ell_i^- = n$, then we follow [PoZ, (1.1)] and set

$$F(\mathrm{w},q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{i=1}^{a}\prod_{r=1}^{\ell_i^+ d}\left(\ell_i^+\mathrm{w}+r\right)\prod_{i=1}^{b}\prod_{r=1}^{-\ell_i^- d}\left(\ell_i^-\mathrm{w}-r\right)}{\prod_{r=1}^{d}(\mathrm{w}+r)^n},$$
$$\mathbf{M}F(\mathrm{w},q) \equiv \left\{1+\frac{q}{\mathrm{w}}\frac{\mathrm{d}}{\mathrm{d}q}\right\}\left(\frac{F(\mathrm{w},q)}{F(0,q)}\right), \quad I_p(q) \equiv \mathbf{M}^p F(0,q). \tag{3.1.22}$$

By (3.1.11) and (3.1.15) above,

$$J_p(\dot{Y}) = I_{p-b}(q)\mathrm{Id}, \quad \mathbf{D}^p\dot{Y}(\mathrm{A},\hbar,q) = \mathrm{A}^p\frac{1}{I_{p-b}(q)}\mathbf{M}^{p-b}F\left(\frac{\mathrm{A}}{\hbar},q\right) \qquad \forall\, p\geqslant b,$$
$$J_p(\ddot{Y}) = I_{p-a}(q)\mathrm{Id}, \quad \mathbf{D}^p\ddot{Y}(\mathrm{A},\hbar,q) = \mathrm{A}^p\frac{1}{I_{p-a}(q)}\mathbf{M}^{p-a}F\left(\frac{\mathrm{A}}{\hbar},q\right) \qquad \forall\, p\geqslant a.$$

3.2 Statements

The statements and proofs of the theorems below rely on the one-point mirror formula (5.1.2) below, which is proved in [LLY3]. We begin by defining the mirror map occurring in this formula.

For each $i\in[k]$, let

$$f_i(q)\equiv\frac{1}{\dot{I}_{\mathbf{0}}(q)}\sum_{\substack{\mathbf{d}\in\Lambda\\ \nu_E(\mathbf{d})=0}} q^{\mathbf{d}}\frac{\partial\left\{U(\mathbf{d};\mathrm{A},1)\dot{E}(\mathbf{d};\mathrm{A},1)\right\}}{\partial \mathrm{A}_i}\bigg|_{\mathrm{A}=0}\in\mathbb{Q}[[\Lambda-0]], \tag{3.2.1}$$

with $\dot{I}_{\mathbf{0}}(q)$ defined by (3.1.6). The mirror map is the change of variables $q\longrightarrow Q$, where

$$(Q_1,\ldots,Q_k) = \left(q_1\mathrm{e}^{f_1(q)},\ldots,q_k\mathrm{e}^{f_k(q)}\right). \tag{3.2.2}$$

Finally, let

$$G(q)\equiv\frac{\delta_{b,0}}{\dot{I}_{\mathbf{0}}(q)}\sum_{\substack{\mathbf{d}\in\Lambda,\ \nu_E(\mathbf{d})=1\\ D_j(\mathbf{d})\geqslant 0\,\forall\, j\in[N]}} q^{\mathbf{d}}\frac{\prod_{i=1}^{a}\left(L_i^+(\mathbf{d})!\right)}{\prod_{j=1}^{N}\left(D_j(\mathbf{d})!\right)}\in\mathbb{Q}[[\Lambda-0]]. \tag{3.2.3}$$

Theorem 3.2.1. *If $\nu_E(\mathbf{d})\geqslant 0$ for all $\mathbf{d}\in\Lambda$, then $\dot{Z}_{\mathbf{p}}$ and $\ddot{Z}_{\mathbf{p}}$ of (3.0.1) and (1.1.6) are given by*

$$\dot{Z}_{\mathbf{p}}(\hbar,Q) = \mathrm{e}^{-\frac{1}{\hbar}\left[G(q)+\sum_{i=1}^{k}\mathrm{H}_i f_i(q)\right]}\dot{Y}_{\mathbf{p}}(\mathrm{H},\hbar,q)\in H^*(X_M^\tau)[\hbar^{-1}][[\Lambda]],$$
$$\ddot{Z}_{\mathbf{p}}(\hbar,Q) = \mathrm{e}^{-\frac{1}{\hbar}\left[G(q)+\sum_{i=1}^{k}\mathrm{H}_i f_i(q)\right]}\ddot{Y}_{\mathbf{p}}(\mathrm{H},\hbar,q)\in H^*(X_M^\tau)[\hbar^{-1}][[\Lambda]],$$

where

$$\begin{aligned} \dot{Y}_{\mathbf{p}}(\mathrm{A},\hbar,q) &\equiv \mathbf{D}^{\mathbf{p}}\dot{Y}(\mathrm{A},\hbar,q) + \sum_{r=1}^{|\mathbf{p}|}\sum_{|\mathbf{s}|=0}^{|\mathbf{p}|-r}\dot{\widetilde{\mathrm{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}(q)\hbar^{|\mathbf{p}|-r-|\mathbf{s}|}\mathbf{D}^{\mathbf{s}}\dot{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]], \\ \ddot{Y}_{\mathbf{p}}(\mathrm{A},\hbar,q) &\equiv \mathbf{D}^{\mathbf{p}}\ddot{Y}(\mathrm{A},\hbar,q) + \sum_{r=1}^{|\mathbf{p}|}\sum_{|\mathbf{s}|=0}^{|\mathbf{p}|-r}\ddot{\widetilde{\mathrm{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}(q)\hbar^{|\mathbf{p}|-r-|\mathbf{s}|}\mathbf{D}^{\mathbf{s}}\ddot{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]], \end{aligned} \tag{3.2.4}$$

with $\dot{\widetilde{\mathrm{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}\equiv\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(\dot{Y})$ *and* $\ddot{\widetilde{\mathrm{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}\equiv\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(\ddot{Y})$ *defined by (3.1.17), Q and q related by the mirror map (3.2.2), G and f_i given by (3.2.3) and (3.2.1), and the operator $\mathbf{D}^{\mathbf{p}}$ defined by (3.1.11) and (3.1.15).*

If $|\mathbf{p}|<b$, $\mathbf{D}^{\mathbf{p}}\dot{Y}=\{\mathrm{A}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\}^{\mathbf{p}}\dot{Y}$ and $\dot{\widetilde{\mathrm{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}=0$ for all $r\in[|\mathbf{p}|]$. If $|\mathbf{p}|<a$ and $L_i^+(\mathbf{d})\geqslant 1$ for all $i\in[a]$ and $\mathbf{d}\in\Lambda-\{0\}$, then $\mathbf{D}^{\mathbf{p}}\ddot{Y}=\{\mathrm{A}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\}^{\mathbf{p}}\ddot{Y}$ and $\ddot{\widetilde{\mathrm{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}=0$ for all $r\in[|\mathbf{p}|]$.

This follows from Theorem 4.2.3 together with (4.1.15) and (EP1) below; Theorem 4.2.3 is an equivariant version of Theorem 3.2.1.

Remark 3.2.2. In the inductive construction of $\mathbf{D}^{\mathbf{p}}Y$ with $Y=\dot{Y}$ or $Y=\ddot{Y}$, the first equation in (3.1.11) may be replaced by

$$\widetilde{\mathbf{D}}^{\mathbf{p}}Y(\mathrm{A},\hbar,q)\equiv\sum_{i\in\mathrm{supp}(\mathbf{p})}c_{\mathbf{p};i}\left\{\mathrm{A}_i+\hbar\,q_i\frac{\mathrm{d}}{\mathrm{d}q_i}\right\}\mathbf{D}^{\mathbf{p}-e_i}Y(\mathrm{A},\hbar,q)\in\mathbb{Q}(\mathrm{A},\hbar)[[\Lambda]]$$

for any tuple $(c_{\mathbf{p};i})_{i\in\mathrm{supp}(\mathbf{p})}$ of rational numbers with $\sum\limits_{i\in\mathrm{supp}(\mathbf{p})}c_{\mathbf{p};i}=1$. The endomorphism $J_{p+1}(Y)$ and the power series $\mathbf{D}^{\mathbf{p}}Y$ defined by the second equation in (3.1.11) and (3.1.15) in terms of the new "weighted" $\widetilde{\mathbf{D}}^{\mathbf{p}}Y$ satisfy (P1) and (P2) by the same arguments as in the case when $c_{\mathbf{p};i}=\frac{1}{|\mathrm{supp}(\mathbf{p})|}$ for all $i\in\mathrm{supp}(\mathbf{p})$. Therefore, (3.1.17) continues to define power series $\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(Y)$ in terms of the "new weighted" $\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},\mathbf{s}}(Y)$. The resulting "weighted" power series $Y_{\mathbf{p}}$ of (3.2.4) do not depend on the "weights" $c_{\mathbf{p};i}$ as elements of $H^*(X_M^\tau)[\![\hbar]\!][[\Lambda]]$; this follows from Remark 4.2.6.

Corollary 3.2.3. *If $\nu_E(\mathbf{d})=0$ or $\nu_E(\mathbf{d})>|\mathbf{p}|$ for all $\mathbf{d}\in\Lambda-\{0\}$, then*

$$\dot{Z}_{\mathbf{p}}(\hbar,Q)=\mathrm{e}^{-\frac{1}{\hbar}\left[G(q)+\sum\limits_{i=1}^{k}\mathrm{H}_if_i(q)\right]}\mathbf{D}^{\mathbf{p}}\dot{Y}(\mathrm{H},\hbar,q),\qquad \ddot{Z}_{\mathbf{p}}(\hbar,Q)=\mathrm{e}^{-\frac{1}{\hbar}\left[G(q)+\sum\limits_{i=1}^{k}\mathrm{H}_if_i(q)\right]}\mathbf{D}^{\mathbf{p}}\ddot{Y}(\mathrm{H},\hbar,q),$$

with Q and q related by the mirror map (3.2.2) and G and f_i given by (3.2.3) and (3.2.1).

This follows from Theorem 3.2.1 together with (3.1.21).

Let $\mathrm{pr}_i\colon X_M^\tau\times X_M^\tau\longrightarrow X_M^\tau$ denote the projection onto the i-th component.

Corollary 3.2.4. *Let $g_{\mathbf{ps}}\in\mathbb{Q}$ be such that $\sum\limits_{|\mathbf{p}|+|\mathbf{s}|=N-k}g_{\mathbf{ps}}\mathrm{pr}_1^*\mathrm{H}^{\mathbf{p}}\mathrm{pr}_2^*\mathrm{H}^{\mathbf{s}}$ is the Poincaré dual to the diagonal class in X_M^τ, where $N-k$ is the complex dimension of X_M^τ. If $N>k$ and $\nu_E(\mathbf{d})>N-k$ for all $\mathbf{d}\in\Lambda-\{0\}$, then the two-point function $\dot{Z}$ of (1.1.5) is given by*

$$\dot{Z}(\hbar_1,\hbar_2,q)=\frac{1}{\hbar_1+\hbar_2}\sum_{|\mathbf{p}|+|\mathbf{s}|=N-k}g_{\mathbf{ps}}\mathrm{pr}_1^*\left\{\mathrm{H}+\hbar_1\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\dot{Y}(\mathrm{H},\hbar_1,q)\mathrm{pr}_2^*\left\{\mathrm{H}+\hbar_2\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{s}}\ddot{Y}(\mathrm{H},\hbar_2,q).$$

This follows from Theorem 1.1.3, Corollary 3.2.3, and Remark 3.1.2.

Remark 3.2.5. If

$$P(\mathrm{A}) \equiv \frac{\prod_{i=1}^{a}\left(\sum_{r=1}^{k} \ell_{ri}^{+}\mathrm{A}_r\right)}{\prod_{i=1}^{b}\left(\sum_{r=1}^{k} \ell_{ri}^{-}\mathrm{A}_r\right)} \in \mathbb{Q}[\mathrm{A}],$$

then

$$Z^*(\hbar_1, \hbar_2, Q) = \dot{Z}^*(\hbar_1, \hbar_2, Q)\mathrm{pr}_1^* P(\mathrm{H}),$$

where $\mathrm{pr}_1 : X_M^\tau \times X_M^\tau \longrightarrow X_M^\tau$ is the projection onto the first component, while $\dot{Z}^*$ and Z^* are as in Remark 1.1.4. Via Theorem 1.1.3, this expresses the two-point function Z^* in terms of the one-point functions $\dot{Z}_\eta$, $\ddot{Z}_\eta$. In this case and if $\nu_E(\mathbf{d}) \geqslant 0$ for all $\mathbf{d} \in \Lambda$, Z^* can be computed explicitly in terms of $\dot{Y}$ and $\ddot{Y}$ via Theorem 3.2.1.

We next use an idea from [CoZ] to express Z^* in terms of one-point GW generating functions and then show how to compute the latter in the $b>0$ case. If $\mathrm{pr}_i : X_M^\tau \times X_M^\tau \longrightarrow X_M^\tau$ and $g_{\mathbf{ps}} \in \mathbb{Q}$ are as in Corollary 3.2.4, then

$$\begin{aligned} Z^*(\hbar_1, \hbar_2, Q) = \frac{1}{\hbar_1 + \hbar_2} \sum_{|\mathbf{p}|+|\mathbf{s}|=N-k} g_{\mathbf{ps}} \big[&\mathrm{pr}_1^* \mathrm{H}^{\mathbf{p}} \mathrm{pr}_2^* Z_{\mathbf{s}}^*(\hbar_2, Q) \\ &+ \mathrm{pr}_1^* Z_{\mathbf{p}}^*(\hbar_1, Q) \mathrm{pr}_2^* \ddot{Z}_{\mathbf{s}}(\hbar_2, Q)\big], \end{aligned} \tag{3.2.5}$$

where

$$Z_{\mathbf{p}}^*(\hbar, Q) \equiv \sum_{\mathbf{d} \in \Lambda - 0} Q^{\mathbf{d}} \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}_E)\mathrm{ev}_2^* \mathrm{H}^{\mathbf{p}}}{\hbar - \psi_1}\right] \in H^*(X_M^\tau)\,[\hbar^{-1}][[\Lambda]]$$

and $\mathrm{ev}_1 : \overline{\mathfrak{M}}_{0,2}(X_M^\tau, \mathbf{d}) \longrightarrow X_M^\tau$. This follows from (4.2.5).

We next assume that $b>0$ and $\nu_E(\mathbf{d}) \geqslant 0$ for all $\mathbf{d} \in \Lambda$ and express $Z_{\mathbf{p}}^*(\hbar, Q)$ in terms of explicit power series. Along with (3.2.5) and Theorem 3.2.1, this will conclude the computation of Z^*.

With $U(\mathbf{d}; \mathrm{A}, \hbar)$ given by (3.1.2), we define

$$\widehat{Y}(\mathrm{A}, \hbar, q) \equiv \sum_{\mathbf{d} \in \Lambda} q^{\mathbf{d}} U(\mathbf{d}; \mathrm{A}, \hbar) \prod_{i=1}^{a} \prod_{s=1}^{L_i^+(\mathbf{d})} \left(\sum_{r=1}^{k} \ell_{ri}^+ \mathrm{A}_r + s\hbar\right) \prod_{i=1}^{b} \prod_{s=1}^{-L_i^-(\mathbf{d})} \left(\sum_{r=1}^{k} \ell_{ri}^- \mathrm{A}_r - s\hbar\right). \tag{3.2.6}$$

As $\widehat{Y}$ satisfies (3.1.7), we may define $\mathbf{D}^{\mathbf{p}}\widehat{Y}$ and $\widetilde{\mathrm{C}}_{\mathbf{p},\mathbf{s}}^{(r)} \equiv \widetilde{\mathrm{C}}_{\mathbf{p},\mathbf{s}}^{(r)}(\widehat{Y})$ by (3.1.15) and (3.1.17). We define $\widehat{Y}_{\mathbf{p}}(\mathrm{A}, \hbar, q)$ by the right-hand side of (3.2.4) above, with $\dot{Y}$ replaced by $\widehat{Y}$ and $\dot{\widetilde{\mathrm{C}}}_{\mathbf{p},\mathbf{s}}^{(r)}$ by $\widetilde{\mathrm{C}}_{\mathbf{p},\mathbf{s}}^{(r)}$.

Let

$$\widetilde{Y}^*(\mathrm{A}, \hbar, q) \equiv \sum_{\mathbf{d} \in \Lambda - 0} q^{\mathbf{d}} U(\mathbf{d}; \mathrm{A}, \hbar) \prod_{i=1}^{a} \prod_{s=1}^{L_i^+(\mathbf{d})} \left(\sum_{r=1}^{k} \ell_{ri}^+ \mathrm{A}_r + s\hbar\right) \prod_{i=1}^{b} \prod_{s=1}^{-L_i^-(\mathbf{d})-1} \left(\sum_{r=1}^{k} \ell_{ri}^- \mathrm{A}_r - s\hbar\right). \tag{3.2.7}$$

We define $\mathrm{E}^{(\mathbf{r})}_{\mathbf{p},\mathbf{s}}\in\mathbb{Q}[[\Lambda]]$ by

$$\left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\widetilde{Y}^*(\mathrm{A},\hbar,q)\cong\sum_{s=0}^{|\mathbf{p}|-b}\sum_{|\mathbf{r}|=0}^{|\mathbf{p}|-b-s}\mathrm{E}^{(\mathbf{r})}_{\mathbf{p},s}\mathrm{A}^{\mathbf{r}}\hbar^s\quad\mathrm{mod}\ \hbar^{-1}. \tag{3.2.8}$$

It follows that $\left[\!\left[\mathrm{E}^{(\mathbf{r})}_{\mathbf{p},\mathbf{s}}\right]\!\right]_{q;\mathbf{d}}=0$ unless $|\mathbf{p}|=b+s+\nu_E(\mathbf{d})+|\mathbf{r}|$. Whenever $b\geqslant 2$,

$$Z^*_{\mathbf{p}}(\hbar,q)=e(E^+)\left[\left\{\mathrm{H}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\widetilde{Y}^*(\mathrm{H},\hbar,q)-\sum_{s=0}^{|\mathbf{p}|-b}\sum_{|\mathbf{r}|=0}^{|\mathbf{p}|-b-s}\mathrm{E}^{(\mathbf{r})}_{\mathbf{p},s}\hbar^s\widehat{Y}_{\mathbf{r}}(\mathrm{H},\hbar,q)\right]. \tag{3.2.9}$$

If $b=1$ and Q and q are related by the mirror map (3.2.2),

$$\begin{split}Z^*_{\mathbf{p}}(\hbar,Q)=e(E^+)\mathrm{e}^{-\frac{e(E^-)f_0(q)}{\hbar}}\Bigg[\left\{\mathrm{H}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\widetilde{Y}^*(\mathrm{H},\hbar,q)\\ -\sum_{s=0}^{|\mathbf{p}|-b}\sum_{|\mathbf{r}|=0}^{|\mathbf{p}|-b-s}\mathrm{E}^{(\mathbf{r})}_{\mathbf{p},s}\hbar^s\widehat{Y}_{\mathbf{r}}(\mathrm{H},\hbar,q)\Bigg]-\frac{e(E^+)\mathrm{H}^{\mathbf{p}}f_0(q)}{\hbar}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left[-\frac{e(E^-)f_0(q)}{\hbar}\right]^n,\end{split} \tag{3.2.10}$$

where

$$f_0(q)\equiv\sum_{\substack{\mathbf{d}\in\Lambda-0,\,\nu_E(\mathbf{d})=0\\ D_j(\mathbf{d})\geqslant 0\,\forall\, j\in[N]}}q^{\mathbf{d}}(-1)^{L_1^-(\mathbf{d})+1}\big(-L_1^-(\mathbf{d})-1\big)!\frac{\prod_{i=1}^{a}\big(L_i^+(\mathbf{d})!\big)}{\prod_{j=1}^{N}(D_j(\mathbf{d}))!}.\text{[3]} \tag{3.2.11}$$

Identities (3.2.9) and (3.2.10) follow by setting $\alpha=0$ in (4.2.13) and (4.2.14).

As in [CoZ], if $X^\tau_M=\mathbb{P}^{n-1}$ and $b\geqslant 2$, (3.2.9) can be be replaced by a simpler formula in terms of the power series $F(\mathrm{w},q)$ in (3.1.22) above. Assume that $E\longrightarrow\mathbb{P}^{n-1}$ is as in Remark 3.1.4 and $\sum_{i=1}^{a}\ell_i^+-\sum_{i=1}^{b}\ell_i^-=n$. Similarly to [CoZ],

$$Z^*_p(\hbar,q)=\frac{e(E^+)}{e(E^-)}\times\begin{cases}\left\{\mathrm{H}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p\dot{Y}(\mathrm{H},\hbar,q)-\mathrm{H}^p, & \text{if}\quad p<b,\\ \mathrm{H}^p\frac{\mathbf{M}^{p-b}F(\frac{\mathrm{H}}{\hbar},q)}{I_{p-b}(q)}-\mathrm{H}^p, & \text{if}\quad p\geqslant b,\end{cases}$$

where the right-hand side should be first simplified in $\mathbb{Q}(\mathrm{H},\hbar)[[q]]$ to eliminate division by H and only afterwards viewed as an element in $H^*(\mathbb{P}^{n-1})[\hbar^{-1}][[q]]$. This follows from Remarks 4.1.4 and 3.1.4 together with Theorem 4.2.3. By Theorem 3.2.1 and Remark 3.1.4,

$$\ddot{Z}_p(\hbar,q)=\begin{cases}\left\{\mathrm{H}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p\ddot{Y}(\mathrm{H},\hbar,q), & \text{if}\quad p<a,\\ \mathrm{H}^p\frac{M^{p-a}F(\frac{\mathrm{H}}{\hbar},q)}{I_{p-a}(q)}, & \text{if}\quad p\geqslant a.\end{cases}$$

[3] In this case, $f_i(q)=\ell_{i1}^-f_0(q)$ with $\ell_{i1}^-\in\mathbb{Z}$ given by (3.1.3).

The last two displayed equations together with (3.2.5) imply that

$$\sum_{d=1}^{\infty} dq^d \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_E)\mathrm{ev}_1^*\mathrm{H}^{c_1}\mathrm{ev}_2^*\mathrm{H}^{c_2} = \frac{\prod_{i=1}^{a}\ell_i^+}{\prod_{i=1}^{b}\ell_i^-}\left(I_{c_1+1-b}(q)-1\right) \quad\text{and}\quad I_{c_1+1-b}=I_{c_2+1-b}, \tag{3.2.12}$$

whenever $c_1+c_2=n-2-a+b$ and with $I_p(q)$ defined by (3.1.22) if $p\geqslant 0$ and $I_p(q)\equiv 1$ if $p<0$.

Chapter 4

Equivariant theorems

In this chapter we introduce equivariant versions of the GW generating functions $\dot{Z}$, $\dot{Z}_\eta$, and $\ddot{Z}_\eta$ of (1.1.5) and (1.1.6). We then present theorems about them which imply the non-equivariant statements of Section 3.2.

With $\alpha \equiv (\alpha_1, \ldots, \alpha_N)$ denoting the $\mathbb{T}^N$-weights of Section 2.3, $H^*_{\mathbb{T}^N}(X^\tau_M)$ is generated over $\mathbb{Q}[\alpha]$ by $\{x_1, \ldots, x_k\}$; see Proposition 2.3.3. The classes x_i of (2.3.7) satisfy

$$H^2_{\mathbb{T}^N}(X^\tau_M) \ni x_i \xrightarrow{\text{restriction}} \mathrm{H}_i \in H^2(X^\tau_M) \qquad \forall\, i \in [k], \quad \mathbf{e}(\gamma_i^*) = x_i \qquad \forall\, i \in [k],$$

where $\mathbf{e}(\gamma_i^*)$ is defined by the lift (2.3.3) of the action of $\mathbb{T}^N$ on X^τ_M to the line bundle γ_i^*. Let

$$x \equiv (x_1, \ldots, x_k), \quad x^{\mathbf{p}} \equiv x_1^{p_1} \cdot \ldots \cdot x_k^{p_k} \qquad \forall\, \mathbf{p} = (p_1, \ldots, p_k) \in \left(\mathbb{Z}^{\geqslant 0}\right)^k.$$

The action of $\mathbb{T}^N$ on X^τ_M induces an action on $\overline{\mathfrak{M}}_{0,m}(X^\tau_M, \mathbf{d})$ which lifts to an action on the vector orbi-bundles $\mathcal{V}_E$, $\dot{\mathcal{V}}_E$, and $\ddot{\mathcal{V}}_E$ of (1.0.1) and (1.1.4). It also lifts to an action on the universal cotangent line bundle to the i-th marked point whose equivariant Euler class will also be denoted by ψ_i. The evaluation maps $\mathrm{ev}_i : \overline{\mathfrak{M}}_{0,m}(X^\tau_M, \mathbf{d}) \longrightarrow X^\tau_M$ are $\mathbb{T}^N$-equivariant.

With $\mathrm{ev}_1, \mathrm{ev}_2 : \overline{\mathfrak{M}}_{0,3}(X^\tau_M, \mathbf{d}) \longrightarrow X^\tau_M$ denoting the evaluation maps at the first two marked points, let

$$\dot{\mathcal{Z}}(\hbar_1, \hbar_2, Q) \equiv \frac{\hbar_1 \hbar_2}{\hbar_1 + \hbar_2} \sum_{\mathbf{d} \in \Lambda} Q^{\mathbf{d}} (\mathrm{ev}_1 \times \mathrm{ev}_2)_* \left[\frac{\mathbf{e}(\dot{\mathcal{V}}_E)}{(\hbar_1 - \psi_1)\,(\hbar_2 - \psi_2)} \right]. \tag{4.0.1}$$

With $\mathrm{ev}_1, \mathrm{ev}_2 : \overline{\mathfrak{M}}_{0,2}(X^\tau_M, \mathbf{d}) \longrightarrow X^\tau_M$ denoting the evaluation maps at the two marked points and for all $\eta \in H^*_{\mathbb{T}^N}(X^\tau_M)$, let

$$\begin{aligned} \dot{\mathcal{Z}}_\eta(\hbar, Q) &\equiv \eta + \sum_{\mathbf{d} \in \Lambda - 0} Q^{\mathbf{d}} \mathrm{ev}_{1*} \left[\frac{\mathbf{e}(\dot{\mathcal{V}}_E) \mathrm{ev}_2^* \eta}{\hbar - \psi_1} \right] \in H^*_{\mathbb{T}^N}(X^\tau_M)\,[[\hbar^{-1}, \Lambda]], \\ \ddot{\mathcal{Z}}_\eta(\hbar, Q) &\equiv \eta + \sum_{\mathbf{d} \in \Lambda - 0} Q^{\mathbf{d}} \mathrm{ev}_{1*} \left[\frac{\mathbf{e}(\ddot{\mathcal{V}}_E) \mathrm{ev}_2^* \eta}{\hbar - \psi_1} \right] \in H^*_{\mathbb{T}^N}(X^\tau_M)\,[[\hbar^{-1}, \Lambda]]. \end{aligned} \tag{4.0.2}$$

In the $\eta = x^{\mathbf{p}}$ cases, these are equivariant versions of $\dot{Z}_{\mathbf{p}}$ and $\ddot{Z}_{\mathbf{p}}$ in (3.0.1):

$$\dot{\mathcal{Z}}_{\mathbf{p}}(\hbar, Q) \equiv \dot{\mathcal{Z}}_{x^{\mathbf{p}}}(\hbar, Q), \ \ddot{\mathcal{Z}}_{\mathbf{p}}(\hbar, Q) \equiv \ddot{\mathcal{Z}}_{x^{\mathbf{p}}}(\hbar, Q) \in H^*_{\mathbb{T}^N}(X^\tau_M)\,[[\hbar^{-1}, \Lambda]]. \tag{4.0.3}$$

In particular, $\dot{\mathcal{Z}}_{\mathbf{0}} \equiv \dot{\mathcal{Z}}_1$, with $\mathbf{0}\in\mathbb{Z}^k$ and $1\in H^*_{\mathbb{T}^N}(X^\tau_M)$.

Section 4.1 below constructs the explicit formal power series in terms of which $\dot{\mathcal{Z}}_{\mathbf{p}}$ and $\ddot{\mathcal{Z}}_{\mathbf{p}}$ are expressed in Theorem 4.2.3. Throughout this construction, which extends the constructions in [Z1, Section 2.3] and [PoZ, Section 3] from $\mathbb{P}^{n-1}$ to an arbitrary toric manifold X^τ_M, we assume that $\nu_E(\mathbf{d}) \geqslant 0$ for all $\mathbf{d}\in\Lambda$ and identify $H_2(X^\tau_M;\mathbb{Z}) \cong \mathbb{Z}^k$ via the basis $\{\mathrm{H}_1,\ldots,\mathrm{H}_k\}$ of $H^2(X^\tau_M;\mathbb{Z})$. Via this identification $\Lambda\hookrightarrow\mathbb{Z}^k$.

4.1 Construction of equivariant power series

We begin by defining equivariant versions $\dot{\mathcal{Y}}$ and $\ddot{\mathcal{Y}}$ of the power series $\dot{Y}$ and $\ddot{Y}$ in (3.1.5) as these will compute $\dot{\mathcal{Z}}_{\mathbf{p}}$ and $\ddot{\mathcal{Z}}_{\mathbf{p}}$ in Theorem 4.2.3. We consider the lift (2.3.3) of the $\mathbb{T}^N$-action on X^τ_M to the line bundles $L^\pm_i$ of (1.1.2) and (3.1.3) so that

$$\lambda^\pm_i \equiv \mathbf{e}(L^\pm_i) = \sum_{r=1}^k \ell^\pm_{ri} x_r. \tag{4.1.1}$$

An equivariant version of the power series $U(\mathbf{d};\mathrm{A},\hbar)$ in (3.1.2) is given by

$$u(\mathbf{d};\mathrm{A},\hbar) \equiv \frac{\prod\limits_{\substack{j\in[N]\\ D_j(\mathbf{d})<0}} \prod\limits_{s=D_j(\mathbf{d})+1}^{0} \left(\sum\limits_{i=1}^k m_{ij}\mathrm{A}_i - \alpha_j + s\hbar\right)}{\prod\limits_{\substack{j\in[N]\\ D_j(\mathbf{d})\geqslant 0}} \prod\limits_{s=1}^{D_j(\mathbf{d})} \left(\sum\limits_{i=1}^k m_{ij}\mathrm{A}_i - \alpha_j + s\hbar\right)} \in \mathbb{Q}[\alpha,\mathrm{A}][\![\hbar]\!]. \tag{4.1.2}$$

By (2.3.9),

$$u(\mathbf{d};x,\hbar) = \frac{\prod\limits_{\substack{j\in[N]\\ D_j(\mathbf{d})<0}} \prod\limits_{s=D_j(\mathbf{d})+1}^{0} (u_j + s\hbar)}{\prod\limits_{\substack{j\in[N]\\ D_j(\mathbf{d})\geqslant 0}} \prod\limits_{s=1}^{D_j(\mathbf{d})} (u_j + s\hbar)}. \tag{4.1.3}$$

With $\dot{E}(\mathbf{d};\mathrm{A},\hbar)$ and $\ddot{E}(\mathbf{d};\mathrm{A},\hbar)$ as in (3.1.4), let

$$\begin{aligned} \dot{\mathcal{Y}}(\mathrm{A},\hbar,q) &\equiv \sum_{\mathbf{d}\in\Lambda} q^{\mathbf{d}} u(\mathbf{d};\mathrm{A},\hbar)\dot{E}(\mathbf{d};\mathrm{A},\hbar) \in \mathbb{Q}[\alpha,\mathrm{A}][[\hbar^{-1},\Lambda]],\\ \ddot{\mathcal{Y}}(\mathrm{A},\hbar,q) &\equiv \sum_{\mathbf{d}\in\Lambda} q^{\mathbf{d}} u(\mathbf{d};\mathrm{A},\hbar)\ddot{E}(\mathbf{d};\mathrm{A},\hbar) \in \mathbb{Q}[\alpha,\mathrm{A}][[\hbar^{-1},\Lambda]]. \end{aligned} \tag{4.1.4}$$

In the above definitions of $\dot{\mathcal{Y}}$ and $\ddot{\mathcal{Y}}$ and throughout the construction below, the torus weights α should be thought of as formal variables, in the same way in which A of Section 3.1 are formal variables. With A replaced by x, $\dot{\mathcal{Y}}$ and $\ddot{\mathcal{Y}}$ become well-defined elements in $H^*_{\mathbb{T}^N}(X^\tau_M)[[\hbar^{-1},\Lambda]]$. However, this is irrelevant for the purposes of this section and becomes relevant only when we use $\dot{\mathcal{Y}}$ and $\ddot{\mathcal{Y}}$ in the formulas for $\dot{\mathcal{Z}}_{\mathbf{p}}$ and $\ddot{\mathcal{Z}}_{\mathbf{p}}$.

As before, $\mathbb{Q}_\alpha \equiv \mathbb{Q}(\alpha)$. We next describe an operator $\mathfrak{D}^{\mathbf{p}}$ acting on a subset of $\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]]$ and certain associated "equivariant structure coefficients" in $\mathbb{Q}[\alpha][[\Lambda]]$ which occur in the formulas for $\dot{\mathcal{Z}}_{\mathbf{p}}$ and $\ddot{\mathcal{Z}}_{\mathbf{p}}$. Fix an element $\mathcal{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]]$ such that for all $\mathbf{d}\in\Lambda$

$$[\![\mathcal{Y}(\mathrm{A},\hbar,q)]\!]_{q;\mathbf{d}} \equiv \frac{f_{\mathbf{d}}(\mathrm{A},\hbar)}{g_{\mathbf{d}}(\mathrm{A},\hbar)}$$

for some homogeneous polynomials $f_{\mathbf{d}}(\mathrm{A},\hbar), g_{\mathbf{d}}(\mathrm{A},\hbar)\in\mathbb{Q}[\alpha,\mathrm{A},\hbar]$, symmetric in α, and satisfying

$$f_{\mathbf{0}}(\mathrm{A},\hbar)=g_{\mathbf{0}}(\mathrm{A},\hbar), \qquad \deg f_{\mathbf{d}}-\deg g_{\mathbf{d}} = -\nu_E(\mathbf{d}), \qquad g_{\mathbf{d}}\Big|_{\substack{\mathrm{A}=0\\ \alpha=0}}\neq 0 \quad \forall\, \mathbf{d}\in\Lambda. \tag{4.1.5}$$

This condition is satisfied by the power series $\dot{\mathcal{Y}}$ and $\ddot{\mathcal{Y}}$ of (4.1.4) and so the construction below applies to $\mathcal{Y}=\dot{\mathcal{Y}}$ and $\mathcal{Y}=\ddot{\mathcal{Y}}$. We inductively define $\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)$ in $\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]]$ satisfying

(EP1) with $\mathbf{D}^{\mathbf{p}}$ defined in Section 3.1,

$$\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)\Big|_{\alpha=0} = \mathbf{D}^{\mathbf{p}}\left(\mathcal{Y}(\mathrm{A},\hbar,q)\Big|_{\alpha=0}\right);$$

(EP2) there exist $\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},s}\equiv\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},s}(\mathcal{Y})\in\mathbb{Q}[\alpha][[\Lambda]]$ with $\mathbf{p},\mathbf{r}\in(\mathbb{Z}^{\geqslant 0})^k$, $s\in\mathbb{Z}^{\geqslant 0}$, such that $\left[\!\left[\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},s}\right]\!\right]_{q;\mathbf{d}}$ is a homogeneous symmetric polynomial in α of degree $-\nu_E(\mathbf{d})-|\mathbf{r}|+s$,

$$\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q) = \hbar^{|\mathbf{p}|}\sum_{s=0}^{\infty}\sum_{\mathbf{r}\in(\mathbb{Z}^{\geqslant 0})^k}\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},s}(q)\mathrm{A}^{\mathbf{r}}\hbar^{-s}, \tag{4.1.6}$$

$$\left[\!\left[\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},s}\right]\!\right]_{q;\mathbf{0}} = \delta_{\mathbf{p},\mathbf{r}}\delta_{|\mathbf{r}|,s} \quad \forall\, \mathbf{p},\mathbf{r}\in(\mathbb{Z}^{\geqslant 0})^k,\ s\in\mathbb{Z}^{\geqslant 0}. \tag{4.1.7}$$

By (4.1.5), (3.1.10), and since $\left[\!\left[\{J_0(\mathcal{Y}|_{\alpha=0})\}(1)\right]\!\right]_{q;\mathbf{0}}=1$ by (P1), we can define

$$\mathfrak{D}^{\mathbf{0}}\mathcal{Y}(\mathrm{A},\hbar,q) \equiv \left[\{J_0(\mathcal{Y}|_{\alpha=0})\}(1)\right]^{-1}\mathcal{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]]. \tag{4.1.8}$$

Suppose next that $p\geqslant 0$ and we have constructed power series $\mathfrak{D}^{\mathbf{p}'}\mathcal{Y}(\mathrm{A},\hbar,q)$ for all $\mathbf{p}'\in(\mathbb{Z}^{\geqslant 0})^k$ with $|\mathbf{p}'|=p$ satisfying the above properties. For each $\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$ with $|\mathbf{p}|=p+1$, let

$$\begin{aligned}
\widetilde{\mathfrak{D}}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q) &\equiv \frac{1}{|\mathrm{supp}(\mathbf{p})|}\sum_{i\in\mathrm{supp}(\mathbf{p})}\left\{\mathrm{A}_i+\hbar\, q_i\frac{\mathrm{d}}{\mathrm{d}q_i}\right\}\mathfrak{D}^{\mathbf{p}-e_i}\mathcal{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]],\\
\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q) &\equiv \sum_{\mathbf{p}'\in(\mathbb{Z}^{\geqslant 0})^k,|\mathbf{p}'|=p+1}\mathrm{c}_{\mathbf{p};\mathbf{p}'}(q)\widetilde{\mathfrak{D}}^{\mathbf{p}'}\mathcal{Y}(\mathrm{A},\hbar,q),
\end{aligned} \tag{4.1.9}$$

where $\mathrm{c}_{\mathbf{p};\mathbf{p}'}(q)\in\mathbb{Q}[[\Lambda;\nu_E=0]]$ are defined by (3.1.14) with $Y\equiv\mathcal{Y}|_{\alpha=0}$ and where $\{e_1,\ldots,e_k\}$ is the standard basis of $\mathbb{Z}^k$. Since (EP1) holds with $\mathbf{p}$ replaced by any $\mathbf{p}'$ with $|\mathbf{p}'|=p$,

$$\widetilde{\mathfrak{D}}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)\Big|_{\alpha=0} = \widetilde{\mathbf{D}}^{\mathbf{p}}\left(\mathcal{Y}(\mathrm{A},\hbar,q)\Big|_{\alpha=0}\right)$$

by (3.1.11); thus, by the second equation in (4.1.9) and (3.1.15), $\mathfrak{D}^{\mathbf{p}}\mathcal{Y}$ satisfies (EP1). It is immediate to verify that $\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)$ admits an expansion as in (4.1.6). Since $[\![\mathrm{c}_{\mathbf{p};\mathbf{p}'}]\!]_{q;\mathbf{0}}=\delta_{\mathbf{p},\mathbf{p}'}$ by the second statement in (3.1.13) and (3.1.14) and (4.1.7) holds for $\mathbf{p}-e_i$ with $i\in\mathrm{supp}(\mathbf{p})$ instead of $\mathbf{p}$, (4.1.7) also holds for $\mathbf{p}$ with $|\mathbf{p}|=p+1$.

By (EP1), (3.1.8), and (4.1.6),

$$\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},s}(\mathcal{Y})\big|_{\alpha=0}=\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}\left(\mathcal{Y}\Big|_{\alpha=0}\right), \tag{4.1.10}$$

with $\mathrm{C}^{(\mathbf{r})}_{\mathbf{p},s}$ as in (P2).

Define $\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}\equiv\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(\mathcal{Y})\in\mathbb{Q}[\alpha][[\Lambda]]$ for $\mathbf{p},\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k$ and $r\in\mathbb{Z}^{\geqslant0}$ with $|\mathbf{s}|\leqslant|\mathbf{p}|-r$ and $r\leqslant|\mathbf{p}|$ by

$$\sum_{t=0}^{r}\sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k\\|\mathbf{s}|\leqslant|\mathbf{p}|-t}}\widetilde{\mathcal{C}}^{(t)}_{\mathbf{p},\mathbf{s}}\mathcal{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|+r-t}=\delta_{\mathbf{p},\mathbf{r}}\delta_{r,0}\qquad\forall\,\mathbf{r}\in(\mathbb{Z}^{\geqslant0})^k,\ |\mathbf{r}|\leqslant|\mathbf{p}|-r. \tag{4.1.11}$$

Equations (4.1.11) indeed uniquely determine $\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}$, since

$$\sum_{t=0}^{r}\sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k\\|\mathbf{s}|\leqslant|\mathbf{p}|-t}}\widetilde{\mathcal{C}}^{(t)}_{\mathbf{p},\mathbf{s}}\mathcal{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|+r-t}=\sum_{t=0}^{r-1}\sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k\\|\mathbf{s}|\leqslant|\mathbf{p}|-t}}\widetilde{\mathcal{C}}^{(t)}_{\mathbf{p},\mathbf{s}}\mathcal{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|+r-t}+\sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k\\|\mathbf{s}|<|\mathbf{r}|}}\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}\mathcal{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|}+\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{r}}. \tag{4.1.12}$$

This follows from

$$\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},|\mathbf{r}|}=\delta_{\mathbf{p},\mathbf{r}}\quad\text{if}\quad|\mathbf{r}|\leqslant|\mathbf{p}|, \tag{4.1.13}$$

which in turn follows from (4.1.10), the second equation in (3.1.9), and the first property in (EP2). By (4.1.11) and (4.1.7),

$$\left[\!\!\left[\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}\right]\!\!\right]_{q;\mathbf{0}}=\delta_{\mathbf{p},\mathbf{s}}\delta_{r,0}. \tag{4.1.14}$$

By (4.1.10), (3.1.17), (3.1.18), (4.1.11), (4.1.12), and induction,

$$\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(\mathcal{Y})\big|_{\alpha=0}=\widetilde{\mathrm{C}}^{(r)}_{\mathbf{p},\mathbf{s}}\left(\mathcal{Y}\big|_{\alpha=0}\right). \tag{4.1.15}$$

By (4.1.14) in the $\mathbf{d}=\mathbf{0}$ case and (4.1.11), (4.1.12), (EP2), and induction in all other cases, $\left[\!\!\left[\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(q)\right]\!\!\right]_{q;\mathbf{d}}$ is a degree $r-\nu_E(\mathbf{d})$ homogeneous symmetric polynomial in α. In particular, $\widetilde{\mathcal{C}}^{(0)}_{\mathbf{p},\mathbf{s}}(q)\in\mathbb{Q}[[\Lambda]]$. This together with (4.1.15) and (3.1.20) implies that,

$$\widetilde{\mathcal{C}}^{(0)}_{\mathbf{p},\mathbf{s}}(q)=\delta_{\mathbf{p},\mathbf{s}}\qquad\forall\mathbf{p},\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k\quad\text{with}\quad|\mathbf{s}|\leqslant|\mathbf{p}|. \tag{4.1.16}$$

Remark 4.1.1. By (4.1.8), $\dot{\mathcal{Y}}\big|_{\alpha=0}=\dot{Y}$, $\ddot{\mathcal{Y}}\big|_{\alpha=0}=\ddot{Y}$, and Remark 3.1.1,

$$\mathfrak{D}^{\mathbf{0}}\dot{\mathcal{Y}}(\mathrm{A},\hbar,q)=\frac{1}{\dot{I}_{\mathbf{0}}(q)}\dot{\mathcal{Y}}(\mathrm{A},\hbar,q),\qquad\mathfrak{D}^{\mathbf{0}}\ddot{\mathcal{Y}}(\mathrm{A},\hbar,q)=\frac{1}{\ddot{I}_{\mathbf{0}}(q)}\ddot{\mathcal{Y}}(\mathrm{A},\hbar,q).$$

Remark 4.1.2. If $\nu_E(\mathbf{d})>0$ for all $\mathbf{d}\in\Lambda-\{0\}$ and $\mathcal{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]]$ satisfies (4.1.5), then

$$\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)=\left\{\mathrm{A}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)\qquad\forall\,\mathbf{p}=(p_1,\ldots,p_k)\in(\mathbb{Z}^{\geqslant0})^k.$$

This follows by induction on $|\mathbf{p}|$ from (4.1.9) since $\mathrm{c}_{\mathbf{p};\mathbf{p}'}(\mathcal{Y}\big|_{\alpha=0})=\delta_{\mathbf{p},\mathbf{p}'}$ with $\mathrm{c}_{\mathbf{p};\mathbf{p}'}$ defined by (3.1.14). The latter follows since $J_p(\mathcal{Y}\big|_{\alpha=0})=\mathrm{Id}$ by Remark 3.1.2.

Remark 4.1.3. Suppose $p^*\in\mathbb{Z}^{\geqslant0}$, $\mathcal{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]]$ satisfies (4.1.5), and

$$\deg_\hbar f_{\mathbf{d}}(\mathrm{A},\hbar)-\deg_\hbar g_{\mathbf{d}}(\mathrm{A},\hbar)<-p^*\qquad\forall\,\mathbf{d}\in\Lambda-0.$$

By the same reasoning as in Remark 4.1.2, but using Remark 3.1.3 instead of 3.1.2,

$$\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)=\left\{\mathrm{A}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)\qquad\text{if}\quad|\mathbf{p}|\leqslant p^*.\tag{4.1.17}$$

By (4.1.11), (4.1.12), and (4.1.16),

$$\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},|\mathbf{r}|+r}+\sum_{t=1}^{r-1}\sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k\\|\mathbf{s}|\leqslant|\mathbf{p}|-t}}\widetilde{\mathcal{C}}^{(t)}_{\mathbf{p},\mathbf{s}}\mathcal{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|+r-t}+\sum_{\substack{\mathbf{s}\in(\mathbb{Z}^{\geqslant0})^k\\|\mathbf{s}|<|\mathbf{r}|}}\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}\mathcal{C}^{(\mathbf{r})}_{\mathbf{s},|\mathbf{r}|}+\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{r}}=0\tag{4.1.18}$$

if $r\geqslant1$ and $|\mathbf{r}|\leqslant|\mathbf{p}|-r$. By (4.1.17) and (4.1.7),

$$\mathfrak{D}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)\cong\mathrm{A}^{\mathbf{p}}\mod\hbar^{-1}\qquad\text{if}\quad|\mathbf{p}|\leqslant p^*.$$

This together with (4.1.6) implies that whenever $|\mathbf{p}|\leqslant p^*$,

$$\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},|\mathbf{r}|+r}=0\qquad\text{if}\quad r\geqslant1\quad\text{and}\quad|\mathbf{r}|\leqslant|\mathbf{p}|-r.\tag{4.1.19}$$

Finally, using (4.1.18), (4.1.19), and induction, we find that

$$\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}=0\qquad\text{if}\quad r\geqslant1,\quad|\mathbf{p}|\leqslant p^*,\quad|\mathbf{s}|\leqslant|\mathbf{p}|-r.$$

Remark 4.1.4. Let (M,τ) be the toric pair of Example 2.4.1 with $N=n$ so that $X_M^\tau=\mathbb{P}^{n-1}$ and $E\longrightarrow\mathbb{P}^{n-1}$ be as in Remark 3.1.4. By (4.1.4),

$$\dot{\mathcal{Y}}(\mathrm{A},\hbar,q)=\sum_{d=0}^{\infty}q^d\frac{\prod\limits_{i=1}^{a}\prod\limits_{s=1}^{\ell_i^+d}\big(\ell_i^+\mathrm{A}+s\hbar\big)\prod\limits_{i=1}^{b}\prod\limits_{s=0}^{-\ell_i^-d-1}\big(\ell_i^-\mathrm{A}-s\hbar\big)}{\prod\limits_{j=1}^{n}\prod\limits_{s=1}^{d}(\mathrm{A}-\alpha_j+s\hbar)},$$

$$\ddot{\mathcal{Y}}(\mathrm{A},\hbar,q)=\sum_{d=0}^{\infty}q^d\frac{\prod\limits_{i=1}^{a}\prod\limits_{s=0}^{\ell_i^+d-1}\big(\ell_i^+\mathrm{A}+s\hbar\big)\prod\limits_{i=1}^{b}\prod\limits_{s=1}^{-\ell_i^-d}\big(\ell_i^-\mathrm{A}-s\hbar\big)}{\prod\limits_{j=1}^{n}\prod\limits_{s=1}^{d}(\mathrm{A}-\alpha_j+s\hbar)}.$$

By Remark 4.1.3,

$$\mathfrak{D}^p\dot{\mathcal{Y}}(\mathrm{A},\hbar,q) = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p \dot{\mathcal{Y}}(\mathrm{A},\hbar,q) \qquad \text{and} \qquad \widetilde{\mathcal{C}}^{(r)}_{p,s}(\dot{\mathcal{Y}}) = 0 \qquad \forall\, p<b,\ 1\leqslant r\leqslant p,$$
$$\mathfrak{D}^p\ddot{\mathcal{Y}}(\mathrm{A},\hbar,q) = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p \ddot{\mathcal{Y}}(\mathrm{A},\hbar,q) \qquad \text{and} \qquad \widetilde{\mathcal{C}}^{(r)}_{p,s}(\ddot{\mathcal{Y}}) = 0 \qquad \forall\, p<a,\ 1\leqslant r\leqslant p.$$

If $\sum\limits_{i=1}^{a}\ell_i^+ - \sum\limits_{i=1}^{b}\ell_i^- < n$, then

$$\mathfrak{D}^p\dot{\mathcal{Y}} = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p \dot{\mathcal{Y}}, \quad \mathfrak{D}^p\ddot{\mathcal{Y}} = \left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p \ddot{\mathcal{Y}},$$

for all p by Remark 4.1.2. If $\sum\limits_{i=1}^{a}\ell_i^+ - \sum\limits_{i=1}^{b}\ell_i^- = n$, then

$$\mathfrak{D}^b\dot{\mathcal{Y}} = \frac{1}{I_0(q)}\left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^b \dot{\mathcal{Y}}, \qquad \mathfrak{D}^p\dot{\mathcal{Y}} = \frac{1}{I_{p-b}(q)}\left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}\mathfrak{D}^{p-1}\dot{\mathcal{Y}} \qquad \forall\, p>b,$$
$$\mathfrak{D}^a\ddot{\mathcal{Y}} = \frac{1}{I_0(q)}\left\{\mathrm{A}+h\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^a \ddot{\mathcal{Y}}, \qquad \mathfrak{D}^p\ddot{\mathcal{Y}} = \frac{1}{I_{p-a}(q)}\left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}\mathfrak{D}^{p-1}\ddot{\mathcal{Y}} \qquad \forall\, p>a,$$

by (4.1.9) and Remark 3.1.4.

4.2 Equivariant statements

Theorem 4.2.1. *Suppose (M,τ) is a minimal toric pair and $\mathrm{pr}_i : X_M^\tau \times X_M^\tau \longrightarrow X_M^\tau$ is the projection onto the i-th component. If $\eta_j, \breve{\eta}_j \in H^*_{\mathbb{T}^N}(X_M^\tau)$ are such that*

$$\sum_{j=1}^{s} \mathrm{pr}_1^*\eta_j\, \mathrm{pr}_2^*\breve{\eta}_j \in H^{2(N-k)}_{\mathbb{T}^N}(X_M^\tau \times X_M^\tau)$$

is the equivariant Poincaré dual of the diagonal, then

$$\dot{\mathcal{Z}}(\hbar_1,\hbar_2,Q) = \frac{1}{\hbar_1+\hbar_2}\sum_{j=1}^{s} \mathrm{pr}_1^*\dot{\mathcal{Z}}_{\eta_j}(\hbar_1,Q)\,\mathrm{pr}_2^*\ddot{\mathcal{Z}}_{\breve{\eta}_j}(\hbar_2,Q). \tag{4.2.1}$$

Corollary 4.2.2. *Let (M,τ) be the minimal toric pair (2.4.2) so that $X_M^\tau = \prod\limits_{i=1}^{s}\mathbb{P}^{N_i-1}$, $N = \sum\limits_{i=1}^{s} N_i$, and $H^*_{\mathbb{T}^N}\Big(\prod\limits_{i=1}^{s}\mathbb{P}^{N_i-1}\Big)$ is given by (2.4.3). Let $\mathrm{pr}_j : X_M^\tau \times X_M^\tau \longrightarrow X_M^\tau$ denote the projection onto the j-th component. For all $i\in[s]$ and $r\in\mathbb{Z}^{\geqslant 0}$, denote by $\sigma_r^{(i)}$ the r-th elementary symmetric polynomial in $\alpha_1^{(i)},\ldots,\alpha_{N_i}^{(i)}$. Then,*

$$\dot{\mathcal{Z}}(\hbar_1,\hbar_2,Q) = \frac{1}{\hbar_1+\hbar_2}\sum_{\substack{r_i+a_i+b_i=N_i-1\,\forall\, i\in[s]\\ r_i,a_i,b_i\geqslant 0\,\forall\, i\in[s]}} (-1)^{\sum\limits_{i=1}^{s} r_i}\sigma_{r_1}^{(1)}\ldots\sigma_{r_s}^{(s)}\,\mathrm{pr}_1^*\dot{\mathcal{Z}}_{(a_1,\ldots,a_s)}(\hbar_1,Q)\,\mathrm{pr}_2^*\ddot{\mathcal{Z}}_{(b_1,\ldots,b_s)}(\hbar_2,Q).$$

This follows from Theorem 4.2.1 as the equivariant Poincaré dual to the diagonal in $\prod_{i=1}^{s}\mathbb{P}^{N_i-1}$ is

$$\sum_{\substack{r_i+a_i+b_i=N_i-1\,\forall\, i\in[s]\\ r_i,a_i,b_i\geqslant 0\,\forall\, i\in[s]}} (-1)^{\sum_{i=1}^{s} r_i}\sigma_{r_1}^{(1)}\ldots\sigma_{r_s}^{(s)}\mathrm{pr}_1^*(x_1^{a_1}\ldots x_s^{a_s})\,\mathrm{pr}_2^*(x_1^{b_1}\ldots x_s^{b_s}).$$

Theorem 4.2.3. *Let (M,τ) be a minimal toric pair. If $\nu_E(\mathbf{d})\geqslant 0$ for all $\mathbf{d}\in\Lambda$, then $\dot{\mathcal{Z}}_{\mathbf{p}}$ and $\ddot{\mathcal{Z}}_{\mathbf{p}}$ of (4.0.3) and (4.0.2) are given by*

$$\begin{aligned}\dot{\mathcal{Z}}_{\mathbf{p}}(\hbar,Q) &= \mathrm{e}^{-\frac{1}{\hbar}\left[G(q)+\sum_{i=1}^{k}x_if_i(q)+\sum_{j=1}^{N}\alpha_jg_j(q)\right]}\dot{\mathcal{Y}}_{\mathbf{p}}(x,\hbar,q)\in H^*_{\mathbb{T}^N}(X_M^\tau)[\![\hbar]\!][[\Lambda]],\\ \ddot{\mathcal{Z}}_{\mathbf{p}}(\hbar,Q) &= \mathrm{e}^{-\frac{1}{\hbar}\left[G(q)+\sum_{i=1}^{k}x_if_i(q)+\sum_{j=1}^{N}\alpha_jg_j(q)\right]}\ddot{\mathcal{Y}}_{\mathbf{p}}(x,\hbar,q)\in H^*_{\mathbb{T}^N}(X_M^\tau)[\![\hbar]\!][[\Lambda]],\end{aligned}\tag{4.2.2}$$

where

$$\begin{aligned}\dot{\mathcal{Y}}_{\mathbf{p}}(x,\hbar,q) &\equiv \mathfrak{D}^{\mathbf{p}}\dot{\mathcal{Y}}(x,\hbar,q)+\sum_{r=1}^{|\mathbf{p}|}\sum_{|\mathbf{s}|=0}^{|\mathbf{p}|-r}\dot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}(q)\hbar^{|\mathbf{p}|-r-|\mathbf{s}|}\mathfrak{D}^{\mathbf{s}}\dot{\mathcal{Y}}(x,\hbar,q),\\ \ddot{\mathcal{Y}}_{\mathbf{p}}(x,\hbar,q) &\equiv \mathfrak{D}^{\mathbf{p}}\ddot{\mathcal{Y}}(x,\hbar,q)+\sum_{r=1}^{|\mathbf{p}|}\sum_{|\mathbf{s}|=0}^{|\mathbf{p}|-r}\ddot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}(q)\hbar^{|\mathbf{p}|-r-|\mathbf{s}|}\mathfrak{D}^{\mathbf{s}}\ddot{\mathcal{Y}}(x,\hbar,q),\end{aligned}\tag{4.2.3}$$

with $\dot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}\equiv\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(\dot{\mathcal{Y}})$ and $\ddot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}\equiv\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(\ddot{\mathcal{Y}})$ defined by (4.1.11), Q and q related by the mirror map (3.2.2), G, f_i and $g_j\in\mathbb{Q}[[\Lambda-0;\nu_E=0]]$ [1]given by (3.2.3), (3.2.1), and (5.1.1), and the operator $\mathfrak{D}^{\mathbf{p}}$ defined by (4.1.9). The coefficient of $q^{\mathbf{d}}$ within each of $\dot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}$ and $\ddot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}$ is a degree $r-\nu_E(\mathbf{d})$ homogeneous symmetric polynomial in $\alpha_1,\ldots,\alpha_N$.

If $|\mathbf{p}|<b$, $\mathfrak{D}^{\mathbf{p}}\dot{\mathcal{Y}}=\{\mathrm{A}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\}^{\mathbf{p}}\dot{\mathcal{Y}}$ and $\dot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}=0$ for all $r\in[|\mathbf{p}|]$. If $|\mathbf{p}|<a$ and $L_i^+(\mathbf{d})\geqslant 1$ for all $i\in[a]$ and $\mathbf{d}\in\Lambda-\{0\}$, then $\mathfrak{D}^{\mathbf{p}}\ddot{\mathcal{Y}}=\{\mathrm{A}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\}^{\mathbf{p}}\ddot{\mathcal{Y}}$ and $\ddot{\widetilde{\mathcal{C}}}^{(r)}_{\mathbf{p},\mathbf{s}}=0$ for all $r\in[|\mathbf{p}|]$.

Corollary 4.2.4. *If $\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$ and $\max(|\mathbf{p}|,1)<\nu_E(\mathbf{d})$ for all $\mathbf{d}\in\Lambda-\{0\}$, then*

$$\dot{\mathcal{Z}}_{\mathbf{p}}(\hbar,q)=\left\{x+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\dot{\mathcal{Y}}(x,\hbar,q),\quad \ddot{\mathcal{Z}}_{\mathbf{p}}(\hbar,q)=\left\{x+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\ddot{\mathcal{Y}}(x,\hbar,q).$$

This follows from Theorem 4.2.3 and Remark 4.1.2.

Corollary 4.2.5. *Let $g_{\mathbf{ps}}\in\mathbb{Q}[\alpha]$ be homogeneous polynomials such that $\sum_{|\mathbf{p}|+|\mathbf{s}|\leqslant N-k} g_{\mathbf{ps}}\mathrm{pr}_1^*x^{\mathbf{p}}\mathrm{pr}_2^*x^{\mathbf{s}}$ is the equivariant Poincaré dual to the diagonal in X_M^τ, where $N-k$ is the complex dimension*

[1]Furthermore, $g_j=0$ if $b>0$ or $D_j(\mathbf{d})\in\{-1,0\}$ for all $\mathbf{d}\in\Lambda$ with $\nu_E(\mathbf{d})=0$.

of X_M^τ. If $N>k$ and $\nu_E(\mathbf{d})>N-k$ for all $\mathbf{d}\in\Lambda-\{0\}$, then the two-point function $\dot{\mathcal{Z}}$ of (4.0.1) is given by

$$\dot{\mathcal{Z}}(\hbar_1,\hbar_2,q)=\frac{1}{\hbar_1+\hbar_2}\sum_{|\mathbf{p}|+|\mathbf{s}|\leqslant N-k} g_{\mathbf{ps}}\mathrm{pr}_1^*\left\{x+\hbar_1\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\dot{\mathcal{Y}}(x,\hbar_1,q)\mathrm{pr}_2^*\left\{x+\hbar_2\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{s}}\ddot{\mathcal{Y}}(x,\hbar_2,q).$$

This follows from Theorem 4.2.1 and Corollary 4.2.4.

Remark 4.2.6. In the inductive construction of $\mathfrak{D}^{\mathbf{p}}\mathcal{Y}$ with $\mathcal{Y}=\dot{\mathcal{Y}}$ or $\mathcal{Y}=\ddot{\mathcal{Y}}$, the first equation in (4.1.9) may be replaced by

$$\widetilde{\mathfrak{D}}^{\mathbf{p}}\mathcal{Y}(\mathrm{A},\hbar,q)\equiv\sum_{i\in\mathrm{supp}(\mathbf{p})}c_{\mathbf{p};i}\left\{\mathrm{A}_i+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\right\}\mathfrak{D}^{\mathbf{p}-e_i}\mathcal{Y}(\mathrm{A},\hbar,q)\in\mathbb{Q}_\alpha(\mathrm{A},\hbar)[[\Lambda]],$$

for any tuple $(c_{\mathbf{p};i})_{i\in\mathrm{supp}(\mathbf{p})}$ of rational numbers with $\sum\limits_{i\in\mathrm{supp}(\mathbf{p})}c_{\mathbf{p};i}=1$. The power series $\mathfrak{D}^{\mathbf{p}}\mathcal{Y}$ defined by the second equation in (4.1.9) in terms of the "new weighted" $\widetilde{\mathfrak{D}}^{\mathbf{p}}\mathcal{Y}$ satisfy (EP1) and (EP2) with $\mathbf{D}^{\mathbf{p}}$ correspondingly "weighted" as in Remark 3.2.2. This follows by the same arguments as in the case when $c_{\mathbf{p};i}=\frac{1}{|\mathrm{supp}(\mathbf{p})|}$ for all $i\in\mathrm{supp}(\mathbf{p})$. Therefore, (4.1.11) continues to define power series $\widetilde{\mathcal{C}}^{(r)}_{\mathbf{p},\mathbf{s}}(\mathcal{Y})$ in terms of the "new weighted" $\mathcal{C}^{(\mathbf{r})}_{\mathbf{p},\mathbf{s}}(\mathcal{Y})$. The resulting "weighted" power series $\mathcal{Y}_{\mathbf{p}}$ of (4.2.3) do not depend on the "weights" $c_{\mathbf{p};i}$ as elements of $H^*_{\mathbb{T}^N}(X_M^\tau)[\![\hbar]\!][[\Lambda]]$ by the proof of Theorem 4.2.3 outlined in Section 5.1.

Remark 4.2.7. We define an equivariant version of Z^* in (1.1.7). Let

$$\mathcal{Z}^*(\hbar_1,\hbar_2,Q)\equiv\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}(\mathrm{ev}_1\times\mathrm{ev}_2)_*\left[\frac{\mathbf{e}(\mathcal{V}_E)}{(\hbar_1-\psi_1)\,(\hbar_2-\psi_2)}\right], \tag{4.2.4}$$

where $\mathrm{ev}_1,\mathrm{ev}_2:\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d})\longrightarrow X_M^\tau$. Since $\mathbf{e}(\dot{\mathcal{V}}_E)\mathrm{ev}_1^*\mathbf{e}(E^+)=\mathbf{e}(\mathcal{V}_E)\mathrm{ev}_1^*\mathbf{e}(E^-)$,

$$\dot{\mathcal{Z}}^*(\hbar_1,\hbar_2,Q)\mathrm{pr}_1^*\mathbf{e}(E^+)=\mathcal{Z}^*(\hbar_1,\hbar_2,Q)\mathrm{pr}_1^*\mathbf{e}(E^-),$$

where $\dot{\mathcal{Z}}^*$ is obtained from $\dot{\mathcal{Z}}$ by disregarding the $Q^{\mathbf{0}}$ term and $\mathrm{pr}_1:X_M^\tau\times X_M^\tau\longrightarrow X_M^\tau$ is the projection onto the first component. This together with Theorem 4.2.1 expresses $\mathcal{Z}^*$ in terms of $\dot{\mathcal{Z}}_\eta$ and $\ddot{\mathcal{Z}}_\eta$ in the $E=E^+$ case.

Using an idea from [CoZ], we derive a formula for $\mathcal{Z}^*$ in terms of one-point GW generating functions that holds in all cases. Following [CoZ], we then show how to express the latter in terms of explicit power series if $b>0$. If $\mathrm{pr}_i:X_M^\tau\times X_M^\tau\longrightarrow X_M^\tau$ and $g_{\mathbf{ps}}\in\mathbb{Q}[\alpha]$ are as in Corollary 4.2.5, then

$$\begin{aligned}\mathcal{Z}^*(\hbar_1,\hbar_2,Q)=\frac{1}{\hbar_1+\hbar_2}\sum_{|\mathbf{p}|+|\mathbf{s}|\leqslant N-k}g_{\mathbf{ps}}\big[&\mathrm{pr}_1^*x^{\mathbf{p}}\mathrm{pr}_2^*\mathcal{Z}^*_{\mathbf{s}}(\hbar_2,Q)\\&+\mathrm{pr}_1^*\mathcal{Z}^*_{\mathbf{p}}(\hbar_1,Q)\mathrm{pr}_2^*\ddot{\mathcal{Z}}_{\mathbf{s}}(\hbar_2,Q)\big],\end{aligned} \tag{4.2.5}$$

where

$$\mathcal{Z}^*_{\mathbf{p}}(\hbar,Q)\equiv\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}_E)\mathrm{ev}_2^*x^{\mathbf{p}}}{\hbar-\psi_1}\right]\in H^*_{\mathbb{T}^N}(X_M^\tau)\,[[\hbar^{-1},\Lambda]].$$

This follows from Theorem 4.2.1, using that $\mathrm{pr}_1^*(\mathbf{e}(E^+)/\mathbf{e}(E^-))(\dot{\mathcal{Z}}-[\![\dot{\mathcal{Z}}]\!]_{Q;\mathbf{0}})=\mathcal{Z}^*$, and

$$\mathrm{pr}_1^*\left(\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)}\right)\sum_{|\mathbf{p}|+|\mathbf{s}|\leqslant N-k} g_{\mathbf{ps}}\mathrm{pr}_1^*x^{\mathbf{p}}\mathrm{pr}_2^*(\ddot{\mathcal{Z}}_{\mathbf{s}}(\hbar,Q)-x^{\mathbf{s}})=\sum_{|\mathbf{p}|+|\mathbf{s}|\leqslant N-k} g_{\mathbf{ps}}\mathrm{pr}_1^*x^{\mathbf{p}}\mathrm{pr}_2^*\mathcal{Z}_{\mathbf{s}}^*(\hbar,Q). \quad (4.2.6)$$

In the $X_M^\tau=\mathbb{P}^{n-1}$ case, (4.2.5) is [CoZ, (2.19)] and the proof of the $X_M^\tau=\mathbb{P}^{n-1}$ case of (4.2.6) in [CoZ] extends to the case of an arbitrary toric manifold.

We give another proof of (4.2.6), using the Virtual Localization Theorem (5.4.1) on $\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d})$ as in Section 5.4. We prove that (4.2.6) holds when restricted to $[I]\times[J]$ for arbitrary $I,J\in\mathscr{V}_M^\tau$. The left-hand side of (4.2.6) restricted to $[I]\times[J]$ is

$$\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)}\Big|_{[I]}\sum_{|\mathbf{p}|+|\mathbf{s}|\leqslant N-k} g_{\mathbf{ps}}x^{\mathbf{p}}\Big|_{[I]}\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}\int_{[\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d})]^{vir}}\frac{\mathbf{e}(\ddot{\mathcal{V}}_E)\mathrm{ev}_2^*x^{\mathbf{s}}\mathrm{ev}_1^*\phi_J}{\hbar-\psi_1}. \quad (4.2.7)$$

The right-hand side of (4.2.6) restricted to $[I]\times[J]$ is

$$\sum_{|\mathbf{p}|+|\mathbf{s}|\leqslant N-k} g_{\mathbf{ps}}x^{\mathbf{p}}\Big|_{[I]}\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}\int_{[\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d})]^{vir}}\frac{\mathbf{e}(\mathcal{V}_E)\mathrm{ev}_2^*x^{\mathbf{s}}\mathrm{ev}_1^*\phi_J}{\hbar-\psi_1}. \quad (4.2.8)$$

Since ϕ_I is the equivariant Poincaré dual of $[I]$,

$$\sum_{\mathbf{p},\mathbf{s}} g_{\mathbf{ps}}\mathrm{pr}_1^*x^{\mathbf{p}}\mathrm{pr}_2^*x^{\mathbf{s}}\big|_{[I]\times[J]}=\int_{\Delta(X_M^\tau)}\mathrm{pr}_1^*\phi_I\mathrm{pr}_2^*\phi_J=\int_{X_M^\tau}\phi_I\phi_J=\phi_I(J)=0 \qquad \forall\, I\neq J\in\mathscr{V}_M^\tau,$$

where $\Delta(X_M^\tau)\subset X_M^\tau\times X_M^\tau$ denotes the diagonal. Thus, by the Virtual Localization Theorem (5.4.1), a graph Γ as in Section 5.4 may contribute to (4.2.7) or (4.2.8) only if its second marked point is mapped into $[I]$. Finally, (4.2.6) follows from the above since

$$\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)}\Big|_{[I]}\mathbf{e}(\ddot{\mathcal{V}}_E)\Big|_{\mathcal{Z}_\Gamma}=\mathbf{e}(\mathcal{V}_E)\Big|_{\mathcal{Z}_\Gamma}$$

whenever $\mathcal{Z}_\Gamma\subset\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d})$ is the $\mathbb{T}^N$-pointwise fixed locus corresponding to a graph Γ whose second marked point is mapped into $[I]$.

We next assume that $b>0$ and $\nu_E(\mathbf{d})\geqslant 0$ for all $\mathbf{d}\in\Lambda$ and express $\mathcal{Z}_{\mathbf{p}}^*(\hbar,Q)$ in terms of explicit power series. Along with (4.2.5) and Theorem 4.2.3, this will conclude the computation of $\mathcal{Z}^*$.

We define

$$\widehat{\mathcal{Y}}(\mathrm{A},\hbar,q)\equiv\sum_{\mathbf{d}\in\Lambda}q^{\mathbf{d}}u(\mathbf{d};\mathrm{A},\hbar)\prod_{i=1}^{a}\prod_{s=1}^{L_i^+(\mathbf{d})}\left(\sum_{r=1}^{k}\ell_{ri}^+\mathrm{A}_r+s\hbar\right)\prod_{i=1}^{b}\prod_{s=1}^{-L_i^-(\mathbf{d})}\left(\sum_{r=1}^{k}\ell_{ri}^-\mathrm{A}_r-s\hbar\right). \quad (4.2.9)$$

As $\widehat{\mathcal{Y}}$ satisfies (4.1.5), we may define $\mathfrak{D}^{\mathbf{p}}\widehat{\mathcal{Y}}$ and $\widetilde{\mathcal{C}}_{\mathbf{p},\mathbf{s}}^{(r)}\equiv\widetilde{\mathcal{C}}_{\mathbf{p},\mathbf{s}}^{(r)}(\widehat{\mathcal{Y}})$ by (4.1.9) and (4.1.11). We define $\widehat{\mathcal{Y}}_{\mathbf{p}}(\mathrm{A},\hbar,q)$ by the right-hand side of (4.2.3) above, with $\dot{\mathcal{Y}}$ replaced by $\widehat{\mathcal{Y}}$ and $\dot{\widetilde{\mathcal{C}}}_{\mathbf{p},\mathbf{s}}^{(r)}$ by $\widetilde{\mathcal{C}}_{\mathbf{p},\mathbf{s}}^{(r)}$. Let

$$\widetilde{\mathcal{Y}}^*(\mathrm{A},\hbar,q)\equiv\sum_{\mathbf{d}\in\Lambda-0}q^{\mathbf{d}}u(\mathbf{d};\mathrm{A},\hbar)\prod_{i=1}^{a}\prod_{s=1}^{L_i^+(\mathbf{d})}\left(\sum_{r=1}^{k}\ell_{ri}^+\mathrm{A}_r+s\hbar\right)\prod_{i=1}^{b}\prod_{s=1}^{-L_i^-(\mathbf{d})-1}\left(\sum_{r=1}^{k}\ell_{ri}^-\mathrm{A}_r-s\hbar\right). \quad (4.2.10)$$

Define $\mathcal{E}_{\mathbf{p},\mathbf{s}}^{(\mathbf{r})}\in\mathbb{Q}[\alpha][[\Lambda]]$ by

$$\left\{\mathrm{A}+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\widetilde{\mathcal{Y}}^*(\mathrm{A},\hbar,q)\cong\sum_{s=0}^{|\mathbf{p}|-b}\sum_{|\mathbf{r}|=0}^{|\mathbf{p}|-b-s}\mathcal{E}_{\mathbf{p},s}^{(\mathbf{r})}\mathrm{A}^{\mathbf{r}}\hbar^s \mod \hbar^{-1}. \tag{4.2.11}$$

It follows that $\left[\!\left[\mathcal{E}_{\mathbf{p},\mathbf{s}}^{(\mathbf{r})}\right]\!\right]_{q;\mathbf{d}}$ is a degree $|\mathbf{p}|-b-s-\nu_E(\mathbf{d})-|\mathbf{r}|$ symmetric homogeneous polynomial in α. Then,

$$\dot{\mathcal{Y}}_{\mathbf{p}}(x,\hbar,Q)=\left\{x+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\dot{\mathcal{Y}}(x,\hbar,q)-\mathbf{e}(E^-)\sum_{s=0}^{|\mathbf{p}|-b}\sum_{|\mathbf{r}|=0}^{|\mathbf{p}|-b-s}\mathcal{E}_{\mathbf{p},s}^{(\mathbf{r})}\hbar^s\widehat{\mathcal{Y}}_{\mathbf{r}}(x,\hbar,q), \tag{4.2.12}$$

where $\dot{\mathcal{Y}}_{\mathbf{p}}$ is defined by (4.2.3); see Section 5.1 for a proof of (4.2.12).

Whenever $b\geqslant 2$,

$$\mathcal{Z}_{\mathbf{p}}^*(\hbar,q)=\mathbf{e}(E^+)\left[\left\{x+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\widetilde{\mathcal{Y}}^*(x,\hbar,q)-\sum_{s=0}^{|\mathbf{p}|-b}\sum_{|\mathbf{r}|=0}^{|\mathbf{p}|-b-s}\mathcal{E}_{\mathbf{p},s}^{(\mathbf{r})}\hbar^s\widehat{\mathcal{Y}}_{\mathbf{r}}(x,\hbar,q)\right]. \tag{4.2.13}$$

If $b=1$,

$$\begin{aligned}\mathcal{Z}_{\mathbf{p}}^*(\hbar,Q)&=\mathbf{e}(E^+)\mathrm{e}^{-\frac{\mathbf{e}(E^-)f_0(q)}{\hbar}}\Bigg[\left\{x+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\widetilde{\mathcal{Y}}^*(x,\hbar,q)\\&-\sum_{s=0}^{|\mathbf{p}|-b}\sum_{|\mathbf{r}|=0}^{|\mathbf{p}|-b-s}\mathcal{E}_{\mathbf{p},s}^{(\mathbf{r})}\hbar^s\widehat{\mathcal{Y}}_{\mathbf{r}}(x,\hbar,q)\Bigg]-\frac{\mathbf{e}(E^+)x^{\mathbf{p}}f_0(q)}{\hbar}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left[-\frac{\mathbf{e}(E^-)f_0(q)}{\hbar}\right]^n,\end{aligned} \tag{4.2.14}$$

with Q and q related by the mirror map (3.2.2) and $f_0(q)\in\mathbb{Q}[[\Lambda]]$ given by (3.2.11). Equations (4.2.13) and (4.2.14) follow from $\mathcal{Z}_{\mathbf{p}}^*=\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)}\left(\dot{\mathcal{Z}}_{\mathbf{p}}-x^{\mathbf{p}}\right)$, (4.2.2), and (4.2.12).

Chapter 5

Proofs

5.1 Outline

In this chapter, we prove Theorems 4.2.1 and 4.2.3 and the identity (4.2.12). The proofs of the two theorems are in the spirit of the proof of mirror symmetry in [Gi1] and [Gi2] but with a twist. Similarly to [Gi1] and [Gi2], our argument revolves around the restrictions on power series imposed by certain recursivity and polynomiality conditions. The concept of C-recursivity was first introduced in [Gi1] in the $X_M^\tau = \mathbb{P}^{n-1}$ case, extended to an arbitrary X_M^τ in [Gi2], and re-defined in [Z1]; all these definitions involve an explicit collection C of structure coefficients. Our concept of C-recursivity introduced in Definition 5.3.1 extends the notion of C-recursivity with an arbitrary collection of structure coefficients from the $X_M^\tau = \mathbb{P}^{n-1}$ case considered in [PoZ] to an arbitrary X_M^τ. The concept of (self-) polynomiality introduced in [Gi1] in the $X_M^\tau = \mathbb{P}^{n-1}$ case and extended to an arbitrary X_M^τ in [Gi2] was modified into the concept of mutual polynomiality for a pair of power series in the $X_M^\tau = \mathbb{P}^{n-1}$ case in [Z1]; we extend the latter to an arbitrary X_M^τ in Definition 5.3.4. By Proposition 5.3.5, which extends [Z1, Proposition 2.1] from the $X_M^\tau = \mathbb{P}^{n-1}$ case to an arbitrary X_M^τ, C-recursivity and mutual polynomiality impose severe restrictions on power series, more severe than the restrictions imposed by recursivity and self-polynomiality as discovered in [Gi1].

Analogous to [Z1] and [PoZ], the proof of Theorem 4.2.3 relies on the one-point mirror theorem of [LLY3]. We begin by stating it. The coefficient of $\alpha_j/\hbar$ for $j \in [N]$ in the Laurent expansion of $\frac{1}{\dot{I}_{\mathbf{0}}(q)} \dot{\mathcal{Y}}\big|_{\mathrm{A}=0}$ at $\hbar^{-1}=0$ is given by

$$g_j(q) \equiv \frac{\delta_{b,0}}{\dot{I}_{\mathbf{0}}(q)} \left[\sum_{\substack{\mathbf{d}\in\Lambda,\, \nu_E(\mathbf{d})=0 \\ D_s(\mathbf{d})\geqslant 0\ \forall\, s\in[N]}} q^{\mathbf{d}} \frac{\prod\limits_{i=1}^{a} \left[L_i^+(\mathbf{d})!\right]}{\prod\limits_{r=1}^{N} \left[D_r(\mathbf{d})!\right]} \left(\sum_{s=1}^{D_j(\mathbf{d})} \frac{1}{s} \right) \right. \\ \left. + \sum_{\substack{\mathbf{d}\in\Lambda,\, \nu_E(\mathbf{d})=0 \\ D_j(\mathbf{d})<-1 \\ D_s(\mathbf{d})\geqslant 0\ \forall\, s\in[N]-\{j\}}} q^{\mathbf{d}} (-1)^{D_j(\mathbf{d})} \left[-D_j(\mathbf{d})-1\right]! \frac{\prod\limits_{i=1}^{a} \left[L_i^+(\mathbf{d})!\right]}{\prod\limits_{s\in[N]-\{j\}} \left[D_s(\mathbf{d})!\right]} \right]. \tag{5.1.1}$$

By [LLY3, Theorem 4.7] together with [LLY3, Section 5.2], if $\nu_E(\mathbf{d})\geqslant 0$ for all $\mathbf{d}\in\Lambda$, then

$$\dot{\mathcal{Z}}_{\mathbf{0}}(\hbar,Q)=\frac{1}{\dot{I}_{\mathbf{0}}(q)}\mathrm{e}^{-\frac{1}{\hbar}\left[G(q)+\sum\limits_{i=1}^{k}x_if_i(q)+\sum\limits_{j=1}^{N}\alpha_jg_j(q)\right]}\dot{\mathcal{Y}}(x,\hbar,q), \tag{5.1.2}$$

with Q and q related by the mirror map (3.2.2), G, f_i, and g_j defined by (3.2.3), (3.2.1), and (5.1.1).[1]

Remark 5.1.1. By (4.1.4), (3.2.3), and (3.2.1),

$$\dot{I}_{\mathbf{0}}(q)G(q)\equiv\left[\!\!\left[\dot{\mathcal{Y}}(\mathrm{A},\hbar,q)\Big|_{\substack{\alpha=0\\ \mathrm{A}=0}}\right]\!\!\right]_{\hbar^{-1};1}\quad\text{and}\quad\dot{I}_{\mathbf{0}}(q)f_i(q)\equiv\left[\!\!\left[\dot{\mathcal{Y}}(\mathrm{A},\hbar,q)\right]\!\!\right]_{\frac{\mathrm{A}_i}{\hbar};1}\qquad\forall\,i\in[k],$$

where $[\![\quad]\!]_{\hbar^{-1};1}$ and $[\![\quad]\!]_{\frac{\mathrm{A}_i}{\hbar};1}$ denote the coefficients of $\hbar^{-1}$ and $\frac{\mathrm{A}_i}{\hbar}$ respectively within the Laurent expansion around $\hbar^{-1}=0$ of the power series inside of the brackets. Thus,

$$\left[\!\!\left[\dot{\mathcal{Y}}(\mathrm{A},\hbar,q)\right]\!\!\right]_{\hbar^{-1};1}\equiv\dot{I}_{\mathbf{0}}(q)\left[G(q)+\sum_{i=1}^{k}\mathrm{A}_if_i(q)+\sum_{j=1}^{N}\alpha_jg_j(q)\right].$$

Some of the proofs in this chapter also hold if we replace $\mathbb{Q}$ by any field $R\supseteq\mathbb{Q}$. Given such a field R, let

$$R_\alpha\equiv\mathbb{Q}_\alpha\otimes_{\mathbb{Q}}R=R[\alpha_1,\ldots,\alpha_N]_{\langle P:P\in\mathbb{Q}[\alpha]-0\rangle}\quad\text{and}\quad H^*_{\mathbb{T}^N}(X^\tau_M;R)\equiv H^*_{\mathbb{T}^N}(X^\tau_M)\otimes_{\mathbb{Q}}R.$$

An element in $H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ admits a lift to an element in $R[\alpha,x][\![\hbar]\!][[\Lambda]]$ and an element in $R[\alpha,x][\![\hbar]\!][[\Lambda]]$ induces an element in $H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ via Proposition 2.3.3. Given $Y(\hbar,Q)\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ and $J\in\mathscr{V}^\tau_M$, we write

$$Y(\hbar,Q)\big|_{[J]}\quad\text{or}\quad Y(\hbar,Q)\big|_J\quad\text{or}\quad Y(x(J),\hbar,Q)\in R[\alpha][\![\hbar]\!][[\Lambda]]$$

for the power series obtained from Y by replacing each coefficient of $\hbar^sQ^{\mathbf{d}}$ in Y by its image via the restriction map $\cdot(J)$ of (2.3.8).

In proving Theorem 4.2.3, we follow the steps outlined in [Z1, Section 1.3] and used for proving [Z1, Theorem 1.1]:

(M1) if $R\supseteq\mathbb{Q}$ is any field, $Y,Z\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$, $Z(\hbar,Q)$ is C-recursive in the sense of Definition 5.3.1 and satisfies the mutual polynomiality condition (MPC) of Definition 5.3.4 with respect to $Y(\hbar,Q)$, the transforms of $Z(\hbar,Q)$ of Lemma 5.3.7 are also C-recursive and satisfy the MPC with respect to appropriate transforms of $Y(\hbar,Q)$;

(M2) if $R\supseteq\mathbb{Q}$ is any field, $Z\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ is recursive in the sense of Definition 5.3.1 and (Y,Z) satisfies the MPC for some $Y\in H^*_{\mathbb{T}}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ with $\left[\!\!\left[Y(\hbar,Q)\big|_I\right]\!\!\right]_{Q;\mathbf{0}}\in R^*_\alpha$ for all $I\in\mathscr{V}^\tau_M$, then Z is determined by its 'mod $\hbar^{-1}$ part' (see Proposition 5.3.5);

[1]See Appendix A for the correspondence between the relevant notation in [LLY3] and ours and detailed references within [LLY3] indicating how [LLY3, Theorem 4.7] together with [LLY3, Section 5.2] implies (5.1.2).

(M3) $\dot{\mathcal{Y}}_{\mathbf{p}}$ of (4.2.3) and $\dot{\mathcal{Z}}_\zeta$ of (4.0.2) are $\dot{\mathfrak{C}}$-recursive in the sense of Definition 5.3.1 with $\dot{\mathfrak{C}}$ given by (5.5.2), while $\ddot{\mathcal{Y}}_{\mathbf{p}}$ of (4.2.3) and $\ddot{\mathcal{Z}}_\zeta$ of (4.0.2) are $\ddot{\mathfrak{C}}$-recursive with $\ddot{\mathfrak{C}}$ given by (5.5.2);

(M4) $(\dot{\mathcal{Y}},\ddot{\mathcal{Y}}_{\mathbf{p}})$, $(\ddot{\mathcal{Y}},\dot{\mathcal{Y}}_{\mathbf{p}})$, $(\dot{\mathcal{Z}}_1,\ddot{\mathcal{Z}}_\zeta)$, and $(\ddot{\mathcal{Z}}_1,\dot{\mathcal{Z}}_\zeta)$ satisfy the MPC;

(M5) the two sides of (4.2.2) viewed as powers series in $\hbar^{-1}$, agree mod $\hbar^{-1}$.

The proof of Theorem 4.2.1 described below follows the same ideas and extends the proof of [Z1, (1.17)].

Claims (M3) and (M4) concerning $\dot{\mathcal{Z}}_\zeta$ and $\ddot{\mathcal{Z}}_\zeta$ follow from Lemmas 5.5.1 and 5.6.1, since by the string equation of [MirSym, Section 26.3] and (5.5.3),

$$\dot{\mathcal{Z}}_\zeta(\hbar,Q)=\hbar\dot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q) \quad \text{and} \quad \ddot{\mathcal{Z}}_\zeta(\hbar,Q)=\hbar\ddot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q),$$

if $m=3$, $\beta_2=\beta_3=0$, $\eta_2=\zeta$, and $\eta_3=1$.

By Lemmas 5.7.1, 5.7.2, and 5.3.6, $\dot{\mathcal{Y}}$ is $\dot{\mathfrak{C}}$-recursive and $\ddot{\mathcal{Y}}$ is $\ddot{\mathfrak{C}}$-recursive, while $(\dot{\mathcal{Y}},\ddot{\mathcal{Y}})$ and $(\ddot{\mathcal{Y}},\dot{\mathcal{Y}})$ satisfy the MPC. This together with the admissibility of transforms (a) and (b) of Lemma 5.3.7 proves claims (M3) and (M4) for $\dot{\mathcal{Y}}_{\mathbf{p}}$ and $\ddot{\mathcal{Y}}_{\mathbf{p}}$.

Claims (M3) and (M4) together with (5.1.2), the admissibility of transforms (c) and (d) of Lemma 5.3.7, and Proposition 5.3.5, prove that verifying (4.2.2) amounts to showing that the two sides of each of these equations agree mod $\hbar^{-1}$; this is in turn equivalent to (4.1.11).

Lemma 5.3.6, Lemma 5.3.7, and Proposition 5.3.5 are proved in Section 5.3; the preparations for this section and the ones following it are made in Section 5.2. Lemmas 5.5.1 and 5.6.1 are proved in Sections 5.5 and 5.6, respectively. Both proofs rely on the Virtual Localization Theorem [GraPa, (1)]. The localization data provided by [Sp] is presented in Section 5.4. Lemmas 5.7.1 and 5.7.2 are proved in Section 5.7.

Proof of (4.2.12). Define $E^-_{\mathbf{p}}\in\mathbb{Z}$ with $\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k$ by

$$\prod_{i=1}^{b}\left(\sum_{r=1}^{k}\ell^-_{ri}\mathrm{A}_r\right) \equiv \sum_{\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k} E^-_{\mathbf{p}}\mathrm{A}^{\mathbf{p}}.$$

By (4.1.4) and (4.2.9),

$$\mathbf{e}(E^-)\widehat{\mathcal{Y}}(x,\hbar,q) = \sum_{\mathbf{p}\in(\mathbb{Z}^{\geqslant 0})^k} E^-_{\mathbf{p}}\left\{x+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^{\mathbf{p}}\dot{\mathcal{Y}}(x,h,q). \tag{5.1.3}$$

Since $\dot{\mathcal{Y}}$ is $\dot{\mathfrak{C}}$-recursive by Lemma 5.7.1 and $(\ddot{\mathcal{Y}},\dot{\mathcal{Y}})$ satisfies the MPC by Lemmas 5.3.6 and 5.7.2, $\mathbf{e}(E^-)\widehat{\mathcal{Y}}$ is $\dot{\mathfrak{C}}$-recursive and $(\ddot{\mathcal{Y}},\mathbf{e}(E^-)\widehat{\mathcal{Y}})$ satisfies the MPC by (5.1.3) and Lemma 5.3.7(a). This together with Lemma 5.3.7$(a)(b)$ implies that the right-hand side of (4.2.12) is $\dot{\mathfrak{C}}$-recursive and satisfies the MPC with respect to $\ddot{\mathcal{Y}}$. Since $\dot{\mathcal{Y}}_{\mathbf{p}}$ also satisfies these two properties by (M3) and (M4), the claim follows from (M2) and the fact that both sides of (4.2.12) are congruent to $x^{\mathbf{p}}$ modulo $\hbar^{-1}$. The latter follows from the fact that $\dot{\mathcal{Y}}_{\mathbf{p}}(x,h,Q)$ and $\widehat{\mathcal{Y}}_{\mathbf{p}}(x,h,Q)$ are congruent to $x^{\mathbf{p}}$ modulo $\hbar^{-1}$ by (4.1.11) together with (4.1.4), (4.2.10), and (4.2.11). $\square$

Proof of Theorem 4.2.1. By (4.0.1), (2.3.15), and (2.3.1),

$$(\hbar_1+\hbar_2)\dot{\mathcal{Z}}(\hbar_1,\hbar_2,Q)\Big|_{[I]\times[J]}=\hbar_1\hbar_2\sum_{\mathbf{d}\in\Lambda}Q^{\mathbf{d}}\int_{[\overline{\mathfrak{M}}_{0,3}(X_M^\tau,\mathbf{d})]^{vir}}\frac{\mathbf{e}(\dot{\mathcal{V}}_E)\mathrm{ev}_1^*\phi_I\mathrm{ev}_2^*\phi_J}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)} \tag{5.1.4}$$

for all $I,J\in\mathcal{V}_M^\tau$. Applying Lemmas 5.5.1 and 5.6.1 for $\dot{\mathcal{Z}}_{\eta,\beta}(\hbar_1,Q)$ with

$$m=3,\quad \beta_2=n,\quad \beta_3=0,\quad \eta_2=\phi_J,\quad \eta_3=1,$$

along with Lemma 5.3.7(b), we obtain that the coefficient of $\hbar_2^{-n}$ in $(\hbar_1+\hbar_2)\dot{\mathcal{Z}}(\hbar_1,\hbar_2,Q)$ is $\dot{\mathfrak{C}}$-recursive with $\dot{\mathfrak{C}}$ given by (5.5.2) and satisfies the MPC with respect to $\ddot{\mathcal{Z}}_1(\hbar_1,Q)$ for all $n\geqslant 0$. Using this, Proposition 2.3.3(b), (M3), (M4), and (M2), it follows that in order to prove (4.2.1) it suffices to show that

$$(\hbar_1+\hbar_2)\dot{\mathcal{Z}}(\hbar_1,\hbar_2,Q)\Big|_{[I]\times[J]}\cong\sum_{j=1}^{s}\dot{\mathcal{Z}}_{\eta_j}(\hbar_1,Q)\Big|_{[I]}\ddot{\mathcal{Z}}_{\check{\eta}_j}(\hbar_2,Q)\Big|_{[J]}\mod \hbar_1^{-1} \tag{5.1.5}$$

for all $I,J\in\mathcal{V}_M^\tau$. By (5.1.4) and the string equation, the left-hand side of (5.1.5) mod $\hbar_1^{-1}$ is

$$\begin{aligned}\phi_I(J)+\hbar_2\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}&\int_{[\overline{\mathfrak{M}}_{0,3}(X_M^\tau,\mathbf{d})]^{vir}}\frac{\mathbf{e}(\dot{\mathcal{V}}_E)\mathrm{ev}_1^*\phi_I\mathrm{ev}_2^*\phi_J}{\hbar_2-\psi_2}\\ &=\Delta_*1\Big|_{[I]\times[J]}+\sum_{\mathbf{d}\in\Lambda-0}Q^{\mathbf{d}}\int_{[\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d})]^{vir}}\frac{\mathbf{e}(\dot{\mathcal{V}}_E)\mathrm{ev}_1^*\phi_I\mathrm{ev}_2^*\phi_J}{\hbar_2-\psi_2},\end{aligned} \tag{5.1.6}$$

where $\Delta\colon X_M^\tau\longrightarrow X_M^\tau\times X_M^\tau$, $\Delta[z]\equiv([z],[z])$. The right-hand side of (5.1.5) mod $\hbar_1^{-1}$ is

$$\sum_{j=1}^{s}\eta_j\Big|_{[I]}\ddot{\mathcal{Z}}_{\check{\eta}_j}(\hbar_2,Q)\Big|_{[J]}. \tag{5.1.7}$$

Applying Lemmas 5.5.1 and 5.6.1 for $\ddot{\mathcal{Z}}_{\eta,\beta}(\hbar_2,Q)$ with

$$m=3,\quad \beta_2=\beta_3=0,\quad \eta_2=\phi_I,\quad \eta_3=1,$$

along with Lemma 5.3.7(b), we obtain that (5.1.6) is the restriction to $[J]$ of a $\ddot{\mathfrak{C}}$-recursive formal power series which satisfies the MPC with respect to $\dot{\mathcal{Z}}_1(\hbar_2,Q)$. Since (5.1.7) also satisfies these two properties, by Proposition 5.3.5 the power series (5.1.6) and (5.1.7) agree if and only if they agree mod $\hbar_2^{-1}$. The latter is the case since (5.1.7) mod $\hbar^{-1}$ is the equivariant Poincaré dual to the diagonal in $X_M^\tau\times X_M^\tau$ restricted to the point $[I]\times[J]$. □

5.2 Notation for fixed points and curves

With $\mathcal{V}_M^\tau$ as in (2.1.4) and for all $I,J\in\mathcal{V}_M^\tau$ with $|I\cap J|=k-1$, we denote by

$$\overline{IJ}\equiv X_M^\tau(I\cup J)\subseteq X_M^\tau\quad\text{and}\quad \deg\overline{IJ}\equiv\big[\overline{IJ}\big]_{[X_M^\tau]}\in\Lambda \tag{5.2.1}$$

the $\mathbb{P}^1$ passing through the points $[I]$ and $[J]$ and its homology class, respectively; see Corollary 2.3.2. Given $I\in\mathscr{V}_M^\tau$ and $j\in[N]-I$, we denote by

$$\overline{Ij}\equiv X_M^\tau(I\cup\{j\})\quad\text{and}\quad \deg\overline{Ij}\equiv\big[\overline{Ij}\big]_{[X_M^\tau]}\in\Lambda$$

the compact one-dimensional complex submanifold of X_M^τ defined by Remark 2.1.10 and its homology class, respectively. Since X_M^τ admits a Kähler form,

$$\deg\overline{IJ},\ \deg\overline{Ij}\in\Lambda-\{0\}$$

by [GriH, Chapter 0, Section 7]. By the last part of Remark 2.1.10, there exists a unique element $v(I,j)$ of $\mathscr{V}_M^\tau$ such that

$$v(I,j)\neq I \text{ and } v(I,j)\subset I\cup\{j\}.$$

Since $v(I,j)\cup I=\{j\}\cup I$, $j\in v(I,j)$ and $\overline{Ij}=\overline{Iv(I,j)}$. Let $\{\hat{j}\}\equiv I-v(I,j)$.

Applying the Localization Theorem (2.3.2) to the integral of 1 over $\overline{Ij}\cong\mathbb{P}^1$ and using (2.3.14) and Corollary 2.3.2, we find that

$$u_j(I)+u_{\hat{j}}(v(I,j))=0\qquad\forall\, I\in\mathscr{V}_M^\tau,\ j\in[N]-I. \tag{5.2.2}$$

Applying the Localization Theorem (2.3.2) to the integrals of x_i, $\lambda_i^\pm$, and u_s over $\overline{Ij}$ and using Corollary 2.3.2, (2.3.14), and (5.2.2), we find that

$$x_i(I)-x_i(v(I,j)) = \langle \mathrm{H}_i,\deg\overline{Ij}\rangle u_j(I)\qquad\forall\, I\in\mathscr{V}_M^\tau,\ j\in[N]-I,\ i\in[k], \tag{5.2.3}$$

$$\lambda_i^\pm(I)-\lambda_i^\pm(v(I,j)) = L_i^\pm\big(\overline{Ij}\big)\,u_j(I)\qquad\forall\, I\in\mathscr{V}_M^\tau,\ j\in[N]-I,\ i\in[a]\ (i\in[b]), \tag{5.2.4}$$

$$u_s(I)-u_s(v(I,j)) = D_s\big(\overline{Ij}\big)\,u_j(I)\qquad\forall\, I\in\mathscr{V}_M^\tau,\ j\in[N]-I,\ s\in[N]. \tag{5.2.5}$$

By (5.2.5), (5.2.2), (2.3.10), and (2.3.13),

$$D_j\big(\overline{Ij}\big)=D_{\hat{j}}\big(\overline{Ij}\big)=1,\qquad D_s\big(\overline{Ij}\big)=0\quad\forall\, s\in I\cap v(I,j). \tag{5.2.6}$$

The last five identities are stated in [Gi2].

5.3 Recursivity, polynomiality, and admissible transforms

As in [Gi2], we introduce a partial order on Λ: if $\mathbf{s},\mathbf{d}\in\Lambda$, we define $\mathbf{s}\preceq\mathbf{d}$ if $\mathbf{d}-\mathbf{s}\in\Lambda$. By Proposition 2.2.4,

$$\mathbf{d}\in\Lambda\qquad\Longrightarrow\qquad\{\mathbf{s}\in\Lambda:\mathbf{s}\preceq\mathbf{d}\}\ \text{ is finite.} \tag{5.3.1}$$

This implies that for every non-empty subset S of Λ, there exists $\mathbf{d}\in S$ such that

$$\mathbf{s}\in\Lambda,\mathbf{s}\prec\mathbf{d}\qquad\Longrightarrow\qquad\mathbf{s}\notin S.$$

Definition 5.3.1. *Let $R\supseteq\mathbb{Q}$ be any field and $C\equiv(C_{I,j}(d))^{d\geqslant1}_{I\in\mathscr{V}^\tau_M,\,j\in[N]-I}$ any collection of elements of R_α. A power series $Z\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ is C-***recursive** *if the following holds: if $\mathbf{d}^*\in\Lambda$ is such that*

$$[\![Z\left(x(v(I,j)),\hbar,Q\right)]\!]_{Q;\mathbf{d}^*-d\cdot\deg\overline{Ij}}\in R_\alpha(\hbar)\quad\forall\, I\in\mathscr{V}^\tau_M, j\in[N]-I, d\geqslant1,$$

and $[\![Z\left(x(v(I,j)),\hbar,Q\right)]\!]_{Q;\mathbf{d}^*-d\cdot\deg\overline{Ij}}$ is regular at $\hbar=-u_j(I)/d$ for all $I\in\mathscr{V}^\tau_M$, $j\in[N]-I$, and $d\geqslant1$, then

$$[\![Z\left(x(I),\hbar,Q\right)]\!]_{Q;\mathbf{d}^*}-\sum_{d\geqslant1}\sum_{\substack{j\in[N]-I\\ d\cdot\deg\overline{Ij}\leq\mathbf{d}^*}}\frac{C_{I,j}(d)}{\hbar+\frac{u_j(I)}{d}}\,[\![Z\left(x\left(v(I,j)\right),\hbar,Q\right)]\!]_{Q;\mathbf{d}^*-d\cdot\deg\overline{Ij}}\Big|_{\hbar=-\frac{u_j(I)}{d}}\in R_\alpha[\hbar,\hbar^{-1}],$$

for all $I\in\mathscr{V}^\tau_M$. A power series $Z\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ is called **recursive** *if it is C-recursive for some collection $C\equiv(C_{I,j}(d))^{d\geqslant1}_{I\in\mathscr{V}^\tau_M,j\in[N]-I}$ of elements of R_α.*

By Remark 5.3.2 below, if $Z\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ is $(C_{I,j}(d))^{d\geqslant1}_{I\in\mathscr{V}^\tau_M,j\in[N]-I}$-recursive, then for each $I\in\mathscr{V}^\tau_M$

$$\begin{aligned}Z(x(I),\hbar,Q)=&\sum_{\mathbf{d}\in\Lambda}\sum_{r=-N_{\mathbf{d}}}^{N_{\mathbf{d}}}Z^{(r)}_{I;\mathbf{d}}\hbar^{-r}Q^{\mathbf{d}}\\&+\sum_{d=1}^{\infty}\sum_{j\in[N]-I}\frac{C_{I,j}(d)Q^{d\cdot\deg\overline{Ij}}}{\hbar+\frac{u_j(I)}{d}}Z\left(x\left(v(I,j)\right),-\frac{u_j(I)}{d},Q\right)\end{aligned}$$

for some integers $N_{\mathbf{d}}$ and some $Z^{(r)}_{I;\mathbf{d}}\in R_\alpha$.

Remark 5.3.2. Let $R\supseteq\mathbb{Q}$ be any field. If $Z\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$ is recursive, then $Z\big|_I\in R_\alpha(\hbar)[[\Lambda]]$ and $[\![Z(x(v(I,j)),\hbar,Q)]\!]_{Q;\mathbf{d}}$ is regular at $\hbar=\frac{-u_j(I)}{d}$ for all $I\in\mathscr{V}^\tau_M$, $\mathbf{d}\in\Lambda$, $j\in[N]-I$, and $d\geqslant1$; this follows by induction on $\mathbf{d}\in\Lambda$. The regularity claim also uses Remark 5.3.3 below.

The C-recursivity is an R_α-linear property (that is, if Z_1 and Z_2 are C-recursive, then so is $f_1Z_1+f_2Z_2$ for any $f_1,f_2\in R_\alpha$). By Lemma 5.3.7(b), C-recursivity is actually an $R_\alpha[\hbar][[\Lambda]]$-linear property.

Remark 5.3.3. For all $I\in\mathscr{V}^\tau_M$, $j\in[N]-I$, all $d\in\mathbb{Q}-\{1\}$, and all $s\in[N]$,

$$u_j(I)+d\cdot u_s(v(I,j))\neq0.$$

Proof. Assume that

$$u_j(I)+d\cdot u_s(v(I,j))=0\tag{5.3.2}$$

for some $I\in\mathscr{V}^\tau_M$, $j\in[N]-I$, $d\in\mathbb{Q}-\{1\}$, and $s\in[N]$. If $d=0$ or $s\in v(I,j)$, then $u_j(I)=0$ by (2.3.10) which contradicts (2.3.13). If $d\neq0$ and $s\in(I-v(I,j))$, then $u_j(I)(1-d)=0$ by (5.3.2) and (5.2.2), which again contradicts (2.3.13). If $d\neq0$ and $s\notin(I\cup v(I,j))$, then setting $\alpha_i=0$ for all $i\in(I\cup v(I,j))$ in (5.3.2) and using (2.3.13), we find that $-d\alpha_s=0$, which is false. □

For the purposes of Definition 5.3.4 and the transforms (a) and (d) in Lemma 5.3.7 below as well as all statements involving them, we identify $H_2(X_M^\tau;\mathbb{Z})$ with $\mathbb{Z}^k$ via the dual basis to $\{\mathrm{H}_1,\ldots,\mathrm{H}_k\}$ so that $\Lambda\subset\mathbb{Z}^k$.

Definition 5.3.4. *For any $Y\equiv Y(\hbar,Q), Z\equiv Z(\hbar,Q)\in H^*_{\mathbb{T}^N}(X_M^\tau;R)[\![\hbar]\!][[\Lambda]]$, define $\Phi_{Y,Z}\in R_\alpha[\![\hbar]\!][[z,\Lambda]]$ by*

$$\Phi_{Y,Z}(\hbar,z,Q)\equiv\sum_{I\in\mathscr{V}_M^\tau}\frac{\mathrm{e}^{x(I)\cdot z}}{\prod\limits_{j\in[N]-I}u_j(I)}Y\left(x(I),\hbar,Q\mathrm{e}^{\hbar z}\right)Z\left(x(I),-\hbar,Q\right),$$

where $z\equiv(z_1,\ldots,z_k)$, $x(I)\cdot z\equiv\sum\limits_{i=1}^{k}x_i(I)z_i$, and $Q\mathrm{e}^{\hbar z}\equiv\left(Q_1\mathrm{e}^{\hbar z_1},\ldots,Q_k\mathrm{e}^{\hbar z_k}\right)$.

If $Y,Z\in H^*_{\mathbb{T}^N}(X_M^\tau;R)[\![\hbar]\!][[\Lambda]]$, the pair (Y,Z) satisfies the mutual polynomiality condition (MPC) if $\Phi_{Y,Z}\in R_\alpha[\hbar][[z,\Lambda]]$.

Proposition 5.3.5. *Let $R\supseteq\mathbb{Q}$ be a field. Assume that $Z\in H^*_{\mathbb{T}^N}(X_M^\tau;R)[\![\hbar]\!][[\Lambda]]$ is recursive and that (Y,Z) satisfies the MPC for some $Y\in H^*_{\mathbb{T}^N}(X_M^\tau;R)[\![\hbar]\!][[\Lambda]]$ with*

$$[\![Y(\hbar,Q)\big|_I]\!]_{Q;\mathbf{0}}\in R_\alpha^*\qquad\forall\,I\in\mathscr{V}_M^\tau.$$

Then, $Z(\hbar,Q)\cong 0\mod\hbar^{-1}$ if and only if $Z(\hbar,Q)=0$.

Proof. By the second statement in Proposition 2.3.3(b),

$$Z(\hbar,Q)=0\quad\Longleftrightarrow\quad Z(\hbar,Q)\big|_I=0\qquad\forall\,I\in\mathscr{V}_M^\tau.$$

Set $f_I\equiv[\![Y(-\hbar,Q)\big|_I]\!]_{Q;\mathbf{0}}$ and assume that $[\![Z(\hbar,Q)\big|_I]\!]_{Q;\mathbf{d}'}=0$ for all $0\preceq\mathbf{d}'\prec\mathbf{d}$ and all $I\in\mathscr{V}_M^\tau$. Since Z is recursive and $Z(\hbar,Q)\cong 0$ modulo $\hbar^{-1}$,

$$[\![Z(\hbar,Q)\big|_I]\!]_{Q;\mathbf{d}}=\sum_{r=1}^{N_\mathbf{d}}Z_{I;\mathbf{d}}^{(r)}\hbar^{-r}$$

for some $N_\mathbf{d}\geqslant 0$ and some $Z_{I;\mathbf{d}}^{(r)}\in R_\alpha$. Thus,

$$[\![\Phi_{Y,Z}(-\hbar,z,Q)]\!]_{Q;\mathbf{d}}=\sum_{I\in\mathscr{V}_M^\tau}\frac{\mathrm{e}^{x(I)\cdot z}}{\prod\limits_{j\in[N]-I}u_j(I)}f_I\left(\sum_{r=1}^{N_\mathbf{d}}Z_{I;\mathbf{d}}^{(r)}\hbar^{-r}\right)\in R_\alpha[\hbar][[z]].$$

This implies that

$$\sum_{I\in\mathscr{V}_M^\tau}\frac{(x(I)\cdot z)^m}{\prod\limits_{j\in[N]-I}u_j(I)}f_I\left(\sum_{r=1}^{N_\mathbf{d}}Z_{I;\mathbf{d}}^{(r)}\hbar^{-r}\right)\in R_\alpha[\hbar,z]\qquad\forall\,m\geqslant 0.$$

In particular,

$$\sum_{I\in\mathscr{V}_M^\tau}\frac{(x(I)\cdot z)^m}{\prod\limits_{j\in[N]-I}u_j(I)}f_I Z_{I;\mathbf{d}}^{(r)}=0\qquad\forall\,0\leqslant m\leqslant|\mathscr{V}_M^\tau|-1,\;\forall\,r\in[N_\mathbf{d}].$$

For each $r\in[N_{\mathbf{d}}]$, this is a linear system in the 'unknowns' $f_I Z^{(r)}_{I;\mathbf{d}}\Big/\prod_{j\in[N]-I} u_j(I)$ with $I\in\mathscr{V}^\tau_M$. Its coefficient matrix has a non-zero Vandermonde determinant, since

$$x(I)\neq x(J)\qquad \forall\, I\neq J\in\mathscr{V}^\tau_M$$

by Proposition 2.3.3(a). It follows that $Z^{(r)}_{I;\mathbf{d}}=0$ for all $I\in\mathscr{V}^\tau_M$ and all $r\in[N_{\mathbf{d}}]$. □

Lemmas 5.3.6 and 5.3.7 below extend [Z1, Lemmas 2.2, 2.3] from the $X^\tau_M=\mathbb{P}^{n-1}$ case to an arbitrary X^τ_M. Our proof of the former is completely different from and much simpler than the one in [Z1]. For the latter, the arguments in [Z1] go through with only two significant changes required.

Lemma 5.3.6. *Let $R\supseteq\mathbb{Q}$ be a field and $Y,Z\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$. Then,*

$$\Phi_{Y,Z}\in R_\alpha[\hbar][[z,\Lambda]]\quad\Longleftrightarrow\quad \Phi_{Z,Y}\in R_\alpha[\hbar][[z,\Lambda]].$$

Proof. Let $Y_{\mathbf{d}}(\hbar)\equiv[\![Y(\hbar,Q)]\!]_{Q;\mathbf{d}}$ and $Z_{\mathbf{d}}(\hbar)\equiv[\![Z(\hbar,Q)]\!]_{Q;\mathbf{d}}$. It follows that $[\![\Phi_{Y,Z}(\hbar,z,Q)]\!]_{Q;\mathbf{d}}$ is

$$\sum_{\substack{0\le\mathbf{d}'\le\mathbf{d}\\ I\in\mathscr{V}^\tau_M}}\frac{\mathrm{e}^{x(I)\cdot z}}{\prod\limits_{j\in[N]-I}u_j(I)}Y_{\mathbf{d}'}(\hbar)\Big|_I\mathrm{e}^{\hbar z\mathbf{d}'}Z_{\mathbf{d}-\mathbf{d}'}(-\hbar)\Big|_I=\sum_{\substack{0\le\mathbf{d}'\le\mathbf{d}\\ I\in\mathscr{V}^\tau_M}}\frac{\mathrm{e}^{x(I)\cdot z}}{\prod\limits_{j\in[N]-I}u_j(I)}Y_{\mathbf{d}-\mathbf{d}'}(\hbar)\Big|_I\mathrm{e}^{\hbar z(\mathbf{d}-\mathbf{d}')}Z_{\mathbf{d}'}(-\hbar)\Big|_I,$$

where $\mathrm{e}^{\hbar z}\equiv(\mathrm{e}^{\hbar z_1},\ldots,\mathrm{e}^{\hbar z_k})$. The right-hand side is $\mathrm{e}^{\hbar z\mathbf{d}}$ times $[\![\Phi_{Z,Y}(-\hbar,z,Q)]\!]_{Q;\mathbf{d}}$. □

Lemma 5.3.7. *Let $R\supseteq\mathbb{Q}$ be any field and $C\equiv(C_{I,j}(d))^{d\geqslant1}_{I\in\mathscr{V}^\tau_M,\,j\in[N]-I}$ any collection of elements of R_α. Let $Y_1,Y_2,Y_3\in H^*_{\mathbb{T}^N}(X^\tau_M;R)[\![\hbar]\!][[\Lambda]]$. If Y_1 is C-recursive and (Y_2,Y_3) satisfies the MPC, then*

- (a) *if $\overline{Y_i}\equiv\left\{x_s+\hbar\,Q_s\frac{\mathrm{d}}{\mathrm{d}Q_s}\right\}Y_i$ for all i and $s\in[k]$, then $\overline{Y_1}$ is C-recursive and $\Phi_{Y_2,\overline{Y_3}}\in R_\alpha[\hbar][[z,\Lambda]]$;*
- (b) *if $f\in R_\alpha[\hbar][[\Lambda]]$, then fY_1 is C-recursive and $\Phi_{Y_2,fY_3}\in R_\alpha[\hbar][[z,\Lambda]]$;*
- (c) *if $f\in R_\alpha[[\Lambda{-}0]]$ and $\overline{Y_i}\equiv\mathrm{e}^{f/\hbar}Y_i$ for all i, then $\overline{Y_1}$ is C-recursive and $\Phi_{\overline{Y_2},\overline{Y_3}}\in R_\alpha[\hbar][[z,\Lambda]]$;*
- (d) *if $f_r\in R_\alpha[[\Lambda-0]]$ for all $r\in[k]$ and $\overline{Y_i}(\hbar,Q)\equiv\mathrm{e}^{f\cdot x/\hbar}Y_i(\hbar,Q\mathrm{e}^f)$ for all i, where $f\cdot x\equiv\sum\limits_{r=1}^k f_rx_r$ and $Q\mathrm{e}^f\equiv(Q_1\mathrm{e}^{f_1},\ldots,Q_k\mathrm{e}^{f_k})$, then $\overline{Y_1}$ is C-recursive and $\Phi_{\overline{Y_2},\overline{Y_3}}\in R_\alpha[\hbar][[z,\Lambda]]$.*

Proof. For all $I\in\mathscr{V}^\tau_M$,

$$\begin{aligned}&\left\{x_s(I)+\hbar\,Q_s\frac{\mathrm{d}}{\mathrm{d}Q_s}\right\}\left(\frac{C_{I,j}(d)}{\hbar+\frac{u_j(I)}{d}}Q^{d\cdot\deg\overline{Ij}}Y_1\left(x\left(v(I,j)\right),-\frac{u_j(I)}{d},Q\right)\right)=\\&\qquad\frac{C_{I,j}(d)}{\hbar+\frac{u_j(I)}{d}}Q^{d\cdot\deg\overline{Ij}}\overline{Y_1}\left(x\left(v(I,j)\right),-\frac{u_j(I)}{d},Q\right)+\frac{C_{I,j}(d)}{\hbar+\frac{u_j(I)}{d}}Q^{d\cdot\deg\overline{Ij}}\\&\qquad\times\left(\left(\hbar+\frac{u_j(I)}{d}\right)Q_s\frac{\mathrm{d}}{\mathrm{d}Q_s}+\hbar\,d\cdot\deg_s\overline{Ij}+x_s(I)-x_s\left(v(I,j)\right)\right)Y_1\left(x\left(v(I,j)\right),-\frac{u_j(I)}{d},Q\right).\end{aligned}$$

The first claim in (a) now follows from Remark 5.3.2 and (5.2.3). The second claim in (a) and the claims in (b)-(d) follow similarly to the proof of [Z1, Lemma 2.3] for the $X_M^\tau = \mathbb{P}^{n-1}$ case, using Lemma 5.3.6, Remark 5.3.2, (5.3.1), and (5.2.3). Equation (5.2.3) and property (5.3.1) are used in the proof of the recursivity claim in (d) when showing that

$$\frac{1}{\hbar + \frac{u_j(I)}{d}} \left(\mathrm{e}^{df(Q)\cdot \deg \overline{Ij} + \frac{f(Q)x(I)}{\hbar}} - \mathrm{e}^{\frac{-f(Q)x(v(I,j))d}{u_j(I)}} \right) \in R_\alpha[\hbar, \hbar^{-1}][[\Lambda]].$$

Property (5.3.1) is also used to show that transforms (c) and (d) preserve $H^*_{\mathbb{T}^N}(X_M^\tau; R_\alpha)[\hbar, \hbar^{-1}][[\Lambda]]$, that

$$\frac{\mathrm{e}^{f/\hbar} - \mathrm{e}^{-df/u_j(I)}}{\hbar + \frac{u_j(I)}{d}} \in R_\alpha[\hbar, \hbar^{-1}][[\Lambda]], \quad \mathrm{e}^{\frac{f(Q\mathrm{e}^{\hbar z}) - f(Q)}{\hbar}} \in R_\alpha[\hbar][[z, Q]],$$

in the case of (c), and that

$$z_r + \frac{f_r(Q\mathrm{e}^{\hbar z}) - f_r(Q)}{\hbar} \in R_\alpha[\hbar, z][[\Lambda]] \qquad \forall\, r \in [k]$$

in the case of (d). □

5.4 Torus action on the moduli space of stable maps

An action of $\mathbb{T}^N$ on a smooth projective variety X induces an action on $\overline{\mathfrak{M}}_{0,m}(X, \mathbf{d})$ as in Chapter 4 and an integration along the fiber homomorphism as in Section 2.3. The Virtual Localization Theorem [GraPa, (1)] implies that

$$\int_{[\overline{\mathfrak{M}}_{0,m}(X,\mathbf{d})]^{vir}} \eta = \sum_{F \subseteq \overline{\mathfrak{M}}_{0,m}(X,\mathbf{d})^{\mathbb{T}^N}} \int_{[F]^{vir}} \frac{\eta}{\mathbf{e}(N_{F/X}^{vir})} \in \mathbb{Q}[\alpha] \qquad \forall\, \eta \in H^*_{\mathbb{T}^N}\big(\overline{\mathfrak{M}}_{0,m}(X, \mathbf{d})\big), \quad (5.4.1)$$

where the sum runs over the components of the $\mathbb{T}^N$ pointwise fixed locus

$$\overline{\mathfrak{M}}_{0,m}(X, \mathbf{d})^{\mathbb{T}^N} \subseteq \overline{\mathfrak{M}}_{0,m}(X, \mathbf{d}).$$

This section describes $\overline{\mathfrak{M}}_{0,m}(X_M^\tau, d)^{\mathbb{T}^N}$, the equivariant Euler class $\mathbf{e}(N_{F/X}^{vir})$ of the virtual normal bundle to each component F of $\overline{\mathfrak{M}}_{0,m}(X_M^\tau, d)^{\mathbb{T}^N}$, and the restriction of $\mathbf{e}(\mathcal{V}_E)$ to F. We follow [Sp] where the corresponding statements are formulated in the language of fans rather than toric pairs.

If $f : (\Sigma, z_1, \ldots, z_m) \longrightarrow X_M^\tau$ is a $\mathbb{T}^N$-fixed stable map, then the images of its marked points, nodes, contracted components, and ramification points are $\mathbb{T}^N$-fixed points and so points of the form $[I]$ for some $I \in \mathscr{V}_M^\tau$ by Corollary 2.3.2(a). Each non-contracted component Σ_e of Σ maps to a closed $\mathbb{T}^N$-fixed curve which is of the form $\overline{IJ}$ for some $I, J \in \mathscr{V}_M^\tau$ with $|I \cap J| = k-1$ by Corollary 2.3.2(b). Since all such curves $\overline{IJ}$ are biholomorphic to $\mathbb{P}^1$ by Corollary 2.3.2(b), the map

$$f\Big|_{\Sigma_e} : \Sigma_e \longrightarrow \overline{IJ}$$

is a degree $\mathfrak{d}(e)$ covering map ramified only over $[I]$ and $[J]$. To each such map we associate a decorated graph as in Definition 5.4.1 below; the vertices of this graph correspond to the nodes and contracted components of Σ or the ramification points of f; the edges e correspond to non-contracted components Σ_e of Σ, and $\mathfrak{d}(e)$ describes the degree of $f\big|_{\Sigma_e}$.

Definition 5.4.1. *A genus* 0 *m-point decorated graph* Γ *is a collection of vertices* $\mathrm{Ver}(\Gamma)$*, edges* $\mathrm{Edg}(\Gamma)$*, and maps*

$$\mathfrak{d}:\mathrm{Edg}(\Gamma)\longrightarrow\mathbb{Z}^{>0},\qquad \mathfrak{p}:\mathrm{Ver}(\Gamma)\longrightarrow\mathscr{V}_M^{\tau},\qquad \mathrm{dec}:[m]\longrightarrow\mathrm{Ver}(\Gamma)$$

satisfying the following properties:

1. *the underlying graph* $(\mathrm{Ver}(\Gamma),\mathrm{Edg}(\Gamma))$ *has no loops;*
2. *if two vertices* v *and* v' *are connected by an edge, then* $|\mathfrak{p}(\mathrm{v})\cap\mathfrak{p}(\mathrm{v}')|=k-1$.

Such a decorated graph is said to be of degree $\mathbf{d}\in\Lambda$ *if*

$$\sum_{\substack{e\in\mathrm{Edg}(\Gamma)\\ \partial e=\{\mathrm{v},\mathrm{v}'\}}}\mathfrak{d}(e)\deg\left(\overline{\mathfrak{p}(\mathrm{v})\mathfrak{p}(\mathrm{v}')}\right)=\mathbf{d},$$

where $\partial e\equiv\{\mathrm{v},\mathrm{v}'\}$ *for an edge* e *joining vertices* v *and* v'.

For a decorated graph Γ as in Definition 5.4.1, we denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of $(\mathrm{Ver}(\Gamma),\mathrm{Edg}(\Gamma))$. It acts naturally on $\prod\limits_{e\in\mathrm{Edg}(\Gamma)}\mathbb{Z}_{\mathfrak{d}(e)}$; let

$$A_\Gamma\equiv\prod_{e\in\mathrm{Edg}(\Gamma)}\mathbb{Z}_{\mathfrak{d}(e)}\rtimes\mathrm{Aut}(\Gamma)$$

denote the corresponding semidirect product.

For any $\mathrm{v}\in\mathrm{Ver}(\Gamma)$, let

$$\mathrm{Edg}(\mathrm{v})\equiv|\{e\in\mathrm{Edg}(\Gamma):\mathrm{v}\in\partial e\}|\quad\text{and}\quad\mathrm{val}(\mathrm{v})\equiv|\mathrm{dec}^{-1}(\mathrm{v})|+\mathrm{Edg}(\mathrm{v})$$

denote the number of edges to which the vertex v belongs and its valence, repectively. A flag F in Γ is a pair (v,e), where e is an edge and v is a vertex of e. For a flag $F=(\mathrm{v},e)$, let $\mathrm{val}(F)\equiv\mathrm{val}(\mathrm{v})$. For a flag $F=(\mathrm{v},e)$, let $\omega_F\equiv\mathbf{e}(T_{f^{-1}(\mathfrak{p}(\mathrm{v}))}\mathbb{P}^1)$, where $f:\mathbb{P}^1\longrightarrow\overline{\mathfrak{p}(\mathrm{v})\mathfrak{p}(\mathrm{v}')}$ is the degree $\mathfrak{d}(e)$ cover of $\overline{\mathfrak{p}(\mathrm{v})\mathfrak{p}(\mathrm{v}')}$ corresponding to e, $\partial e=\{\mathrm{v},\mathrm{v}'\}$, and the $\mathbb{T}^N$-action on $\mathbb{P}^1$ is induced from the action on X_M^τ via f. If $\{j\}\equiv\mathfrak{p}(\mathrm{v}')-\mathfrak{p}(\mathrm{v})$,

$$\omega_F=\frac{u_j(\mathfrak{p}(\mathrm{v}))}{\mathfrak{d}(e)}\tag{5.4.2}$$

by (2.3.14). If v is a vertex that belongs to exactly 2 edges e_1 and e_2, then we write $F_i(\mathrm{v})\equiv(\mathrm{v},e_i)$.

Given a decorated graph Γ as above, let

$$\mathfrak{M}_\Gamma\equiv\prod_{\mathrm{v}\in\mathrm{Ver}(\Gamma)}\overline{\mathfrak{M}}_{0,\mathrm{val}(\mathrm{v})},$$

where $\overline{\mathfrak{M}}_{0,m}\equiv$ point, whenever $m\leqslant 2$. For a flag $F=(\mathrm{v},e)$, let $\psi_F\in H^2_{\mathbb{T}^N}(\mathfrak{M}_\Gamma)$ denote the equivariant Euler class of the universal cotangent line bundle on $\mathfrak{M}_\Gamma$ corresponding to F (that is, the pull-back of the ψ class on $\overline{\mathfrak{M}}_{0,\mathrm{val}(\mathrm{v})}$ corresponding to e).

Proposition 5.4.2 ([Sp, Lemma 6.9]). *There is a morphism $\gamma:\mathfrak{M}_\Gamma\longrightarrow\overline{\mathfrak{M}}_{0,m}(X_M^\tau;\mathbf{d})$ whose image is a component of $\overline{\mathfrak{M}}_{0,m}(X_M^\tau;\mathbf{d})^{\mathbb{T}^N}$ and every such component occurs as the image of such a morphism corresponding to some degree $\mathbf{d}$ decorated graph. With $\prod\limits_{e\in\mathrm{Edg}(\Gamma)}\mathbb{Z}_{\mathfrak{d}(e)}$ acting trivially on $\mathfrak{M}_\Gamma$, the induced map*

$$\gamma/A_\Gamma:\mathfrak{M}_\Gamma/A_\Gamma\longrightarrow\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})$$

identifies $\mathfrak{M}_\Gamma/A_\Gamma$ with the corresponding component of $\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})^{\mathbb{T}^N}$.

Proposition 5.4.3 ([Sp, Theorem 7.8]). *Let Γ be a degree $\mathbf{d}$ genus 0 m-point decorated graph and N_Γ^{vir} the virtual normal bundle to $\gamma:\mathfrak{M}_\Gamma\longrightarrow\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})$. Then,*

$$\mathbf{e}\left(N_\Gamma^{vir}\right)=\prod_{\substack{\text{flags }F\text{ of }\Gamma\\ \mathrm{val}(F)\geqslant3}}(\omega_F-\psi_F)\frac{1}{\prod\limits_{\mathrm{v}\in\mathrm{Ver}(\Gamma)}\left[\phi_{\mathfrak{p}(\mathrm{v})}(\mathfrak{p}(\mathrm{v}))\right]^{\mathrm{Edg}(\mathrm{v})-1}}\prod_{\substack{\mathrm{v}\in\mathrm{Ver}(\Gamma)\\ \mathrm{val}(\mathrm{v})=2\\ \mathrm{dec}^{-1}(\mathrm{v})=\varnothing}}\left(\omega_{F_1(\mathrm{v})}+\omega_{F_2(\mathrm{v})}\right)\frac{1}{\prod\limits_{\substack{\text{flags }F\text{ of }\Gamma\\ \mathrm{val}(F)=1}}\omega_F}$$

$$\times\prod_{\substack{e\in\mathrm{Edg}(\Gamma)\\ \partial e=\{\mathrm{v},\mathrm{v}'\}}}\left(\frac{(-1)^{\mathfrak{d}(e)}\left(\mathfrak{d}(e)!\right)^2\left(u_j(I)\right)^{2\mathfrak{d}(e)}}{\left(\mathfrak{d}(e)\right)^{2\mathfrak{d}(e)}}\prod_{r\in[N]-(I\cup\{j\})}\frac{\prod\limits_{s=0}^{\mathfrak{d}(e)D_r(\overline{Ij})}\left(u_r(I)-\frac{s}{\mathfrak{d}(e)}u_j(I)\right)}{\prod\limits_{s=\mathfrak{d}(e)D_r(\overline{Ij})+1}^{-1}\left(u_r(I)-\frac{s}{\mathfrak{d}(e)}u_j(I)\right)}\right)\Bigg|_{\substack{I=\mathfrak{p}(\mathrm{v})\\ \{j\}=\mathfrak{p}(\mathrm{v}')-I}}.$$

By (5.2.2) and (5.2.5),

$$(-1)^{\mathfrak{d}(e)}\left(u_j(I)\right)^{2\mathfrak{d}(e)}\Bigg|_{\substack{I=\mathfrak{p}(\mathrm{v})\\ \{j\}=\mathfrak{p}(\mathrm{v}')-I}}=u_{\mathfrak{p}(\mathrm{v}')-\mathfrak{p}(\mathrm{v})}^{\mathfrak{d}(e)}(\mathfrak{p}(\mathrm{v}))u_{\mathfrak{p}(\mathrm{v})-\mathfrak{p}(\mathrm{v}')}^{\mathfrak{d}(e)}(\mathfrak{p}(\mathrm{v}')),$$

$$u_r(I)-\frac{s}{\mathfrak{d}(e)}u_j(I)\Bigg|_{\substack{I=\mathfrak{p}(\mathrm{v})\\ \{j\}=\mathfrak{p}(\mathrm{v}')-I}}=\begin{cases}\frac{\left[\mathfrak{d}(e)D_r\left(\overline{\mathfrak{p}(\mathrm{v})\mathfrak{p}(\mathrm{v}')}\right)-s\right]u_r(\mathfrak{p}(\mathrm{v}))+su_r(\mathfrak{p}(\mathrm{v}'))}{\mathfrak{d}(e)D_r\left(\overline{\mathfrak{p}(\mathrm{v})\mathfrak{p}(\mathrm{v}')}\right)} & \text{if}\quad D_r\left(\overline{\mathfrak{p}(\mathrm{v})\mathfrak{p}(\mathrm{v}')}\right)\neq0,\\ u_r\left(\mathfrak{p}(\mathrm{v})\right)=u_r\left(\mathfrak{p}(\mathrm{v}')\right) & \text{if}\quad D_r\left(\overline{\mathfrak{p}(\mathrm{v})\mathfrak{p}(\mathrm{v}')}\right),s=0;\end{cases}$$

so the edge contributions to $\mathbf{e}(N_\Gamma^{vir})$ in Proposition 5.4.3 are indeed symmetric in the vertices of each edge.

Let $f:(\mathbb{P}^1,z_1,\ldots,z_m)\longrightarrow\overline{IJ}$ be a $\mathbb{T}^N$-fixed stable map. Thus, f is a degree d cover of $\overline{IJ}$ for some $d\in\mathbb{Z}^{>0}$. By (1.0.1),

$$\mathcal{V}_E\big|_{[\mathbb{P}^1,z_1,\ldots,z_m,f]}=H^0\left(\mathbb{P}^1,f^*E^+\right)\oplus H^1\left(\mathbb{P}^1,f^*E^-\right).$$

By [MirSym, Exercise 27.2.3] together with (5.2.2) and (5.2.4), and with $\{j\}\equiv J-I$,

$$\mathbf{e}(\mathcal{V}_E)\big|_{[\mathbb{P}^1,z_1,\ldots,z_m,f]}=\prod_{i=1}^{a}\prod_{s=0}^{dL_i^+(\overline{IJ})}\left[\lambda_i^+(I)-\frac{s}{d}u_j(I)\right]\prod_{i=1}^{b}\prod_{s=dL_i^-(\overline{IJ})+1}^{-1}\left[\lambda_i^-(I)-\frac{s}{d}u_j(I)\right].\quad(5.4.3)$$

By (5.2.4),

$$\lambda_i^+(I)-\frac{s}{d}u_{J-I}(I)=\begin{cases}\frac{\left[dL_i^+(\overline{IJ})-s\right]\lambda_i^+(I)+s\lambda_i^+(J)}{dL_i^+(\overline{IJ})} & \text{if}\quad L_i^+(\overline{IJ})\neq0,\\ \lambda_i^+(I)=\lambda_i^+(J) & \text{if}\quad L_i^+(\overline{IJ})=s=0,\end{cases}$$

$$\lambda_i^-(I)-\frac{s}{d}u_{J-I}(I)=\frac{\left[dL_i^-(\overline{IJ})-s\right]\lambda_i^-(I)+s\lambda_i^-(J)}{dL_i^-(\overline{IJ})}.$$

5.5 Recursivity for the GW power series

For all $d\in\mathbb{Z}^{>0}$, $I\in\mathscr{V}_M^\tau$, $j\in[N]-I$, let

$$\widetilde{\mathfrak{C}}_{I,j}(d)\equiv\frac{(-1)^d d^{2d-1}}{(d!)^2}\frac{1}{\big[u_j(I)\big]^{2d-1}}\prod_{r\in[N]-(I\cup\{j\})}\frac{\prod\limits_{s=dD_r(\overline{Ij})+1}^{0}\big[u_r(I)-\frac{s}{d}u_j(I)\big]}{\prod\limits_{s=1}^{dD_r(\overline{Ij})}\big[u_r(I)-\frac{s}{d}u_j(I)\big]}\in\mathbb{Q}_\alpha, \tag{5.5.1}$$

$$\begin{aligned}\dot{\mathfrak{C}}_{I,j}(d)&\equiv\widetilde{\mathfrak{C}}_{I,j}(d)\prod_{i=1}^{a}\prod_{s=1}^{dL_i^+(\overline{Ij})}\Big[\lambda_i^+(I)-\frac{s}{d}u_j(I)\Big]\prod_{i=1}^{b}\prod_{s=0}^{-dL_i^-(\overline{Ij})-1}\Big[\lambda_i^-(I)+\frac{s}{d}u_j(I)\Big]\in\mathbb{Q}_\alpha,\\ \ddot{\mathfrak{C}}_{I,j}(d)&\equiv\widetilde{\mathfrak{C}}_{I,j}(d)\prod_{i=1}^{a}\prod_{s=0}^{dL_i^+(\overline{Ij})-1}\Big[\lambda_i^+(I)-\frac{s}{d}u_j(I)\Big]\prod_{i=1}^{b}\prod_{s=1}^{-dL_i^-(\overline{Ij})}\Big[\lambda_i^-(I)+\frac{s}{d}u_j(I)\Big]\in\mathbb{Q}_\alpha.\end{aligned} \tag{5.5.2}$$

Lemma 5.5.1. *If $m\geqslant 3$, $\mathrm{ev}_j:\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})\longrightarrow X_M^\tau$ is the evaluation map at the j-th marked point, $\eta_j\in H^*_{\mathbb{T}^N}(X_M^\tau)$ and $\beta_j\in\mathbb{Z}^{\geqslant 0}$ for $j=2,\ldots,m$, then the power series*

$$\begin{aligned}\dot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q)&\equiv\sum_{\mathbf{d}\in\Lambda}Q^{\mathbf{d}}\mathrm{ev}_{1*}\left[\frac{\mathbf{e}\big(\dot{\mathcal{V}}_E\big)}{\hbar-\psi_1}\prod_{j=2}^{m}\Big(\psi_j^{\beta_j}\mathrm{ev}_j^*\eta_j\Big)\right]\in H^*_{\mathbb{T}^N}(X_M^\tau)[\![\hbar]\!][[\Lambda]]\quad\text{and}\\ \ddot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q)&\equiv\sum_{\mathbf{d}\in\Lambda}Q^{\mathbf{d}}\mathrm{ev}_{1*}\left[\frac{\mathbf{e}\big(\ddot{\mathcal{V}}_E\big)}{\hbar-\psi_1}\prod_{j=2}^{m}\Big(\psi_j^{\beta_j}\mathrm{ev}_j^*\eta_j\Big)\right]\in H^*_{\mathbb{T}^N}(X_M^\tau)[\![\hbar]\!][[\Lambda]]\end{aligned} \tag{5.5.3}$$

are $\dot{\mathfrak{C}}$- and $\ddot{\mathfrak{C}}$-recursive, respectively, with $\dot{\mathfrak{C}}$ and $\ddot{\mathfrak{C}}$ given by (5.5.2).

Proof. This is obtained by applying the Virtual Localization Theorem (5.4.1) on $\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})$, using Section 5.4, and extending the proof of [Z1, Lemma 1.1] from the case of a positive line bundle over $\mathbb{P}^{n-1}$ to that of a split vector bundle $E=E^+\oplus E^-$ as in (1.1.2) over an arbitrary symplectic toric manifold X_M^τ. By (2.3.15), (2.3.1), (5.4.1), and the second equation in (2.3.14), a decorated graph may contribute to $\dot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q)(I)$ and $\ddot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q)(I)$ only if $\mathfrak{p}(\mathrm{dec}(1))=I$. There are thus two types of contributing graphs: the A_I and the B_I graphs, where $I\in\mathscr{V}_M^\tau$. In an A_I graph the first marked point is attached to a vertex v_0 of valence 2, while in a B_I graph the first marked point is attached to a vertex v_0 of valence at least 3. If Γ is a B_I graph and $\mathcal{Z}_\Gamma$ the corresponding component of $\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})^{\mathbb{T}^N}$, then

$$\psi_1^n=0\qquad\forall\,n>\mathrm{val}(\mathrm{v}_0)-3.$$

Thus, Γ contributes a polynomial in $\hbar^{-1}$ to the coefficient of $Q^{\mathbf{d}}$ in $\dot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q)(I)$ and $\ddot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q)(I)$.

In an A_I graph there is a unique vertex v joined to v_0 by an edge. Let $A_{(I,j)}(d_0)$ be the set of all A_I graphs such that $\mathfrak{p}(\mathrm{v})=v(I,j)$ and the edge having v_0 as a vertex is labeled d_0. Thus,

$$A_I=\bigcup_{d_0=1}^{\infty}\bigcup_{j\notin I}A_{(I,j)}(d_0).$$

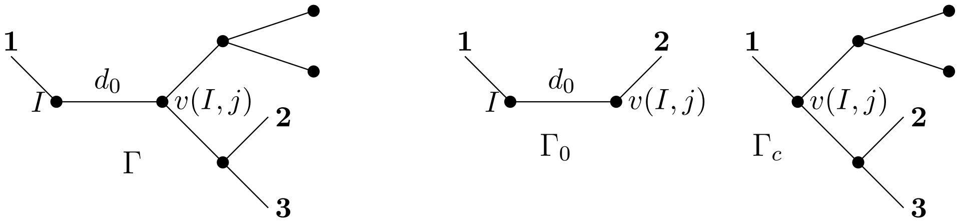

Figure 5.1: A graph of type $A_{(I,j)}(d_0)$ and its two subgraphs

We fix $\Gamma \in A_{(I,j)}(d_0)$ and denote by Γ_0 and Γ_c the two graphs obtained by breaking Γ at v, adding a second marked point to the vertex v in Γ_0 and a first marked point to v in Γ_c, and requiring that marked points $2,\ldots,m$ are in Γ_c; see Figure 5.1.[2] Thus, Γ_0 consists only of the vertices v_0 and v and the marked points 1 and 2 attached to v_0 and v, respectively. With $\mathcal{Z}_\Gamma$ denoting the component in $\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d})^{\mathbb{T}^N}$ corresponding to Γ,

$$\mathcal{Z}_\Gamma \cong \mathcal{Z}_{\Gamma_0} \times \mathcal{Z}_{\Gamma_c};$$

we denote by π_0 and π_c the two projections. Thus,

$$\dot{\mathcal{V}}_E = \pi_0^*\dot{\mathcal{V}}_E \oplus \pi_c^*\dot{\mathcal{V}}_E \quad \text{and} \quad \ddot{\mathcal{V}}_E = \pi_0^*\ddot{\mathcal{V}}_E \oplus \pi_c^*\ddot{\mathcal{V}}_E. \tag{5.5.4}$$

These identities are obtained by considering the short exact sequence of sheaves

$$0 \longrightarrow f^*E^\pm \longrightarrow f_0^*E^\pm \oplus f_c^*E^\pm \longrightarrow E^\pm\big|_p \longrightarrow 0,$$

where $f:\Sigma \longrightarrow X_M^\tau$ is a $\mathbb{T}^N$-fixed stable map whose corresponding graph is Γ, while f_0 and f_c are its restrictions to the components of Σ corresponding to the edge leaving v_0 and the rest of Γ. Let

$$\eta^\beta \equiv \prod_{j=2}^{m}\left(\psi_j^{\beta_j}\mathrm{ev}_j^*\eta_j\right).$$

By (5.5.4),

$$\begin{aligned}
\frac{\mathbf{e}\big(\dot{\mathcal{V}}_E\big)\eta^\beta}{\hbar-\psi_1}\bigg|_{\mathcal{Z}_\Gamma} &= \pi_0^*\left(\frac{\mathbf{e}\big(\dot{\mathcal{V}}_E\big)}{\hbar-\psi_1}\right)\pi_c^*\left(\mathbf{e}\big(\dot{\mathcal{V}}_E\big)\eta^\beta\right),\\
\frac{\mathbf{e}\big(\ddot{\mathcal{V}}_E\big)\eta^\beta}{\hbar-\psi_1}\bigg|_{\mathcal{Z}_\Gamma} &= \pi_0^*\left(\frac{\mathbf{e}\big(\ddot{\mathcal{V}}_E\big)}{\hbar-\psi_1}\right)\pi_c^*\left(\mathbf{e}\big(\ddot{\mathcal{V}}_E\big)\eta^\beta\right).
\end{aligned} \tag{5.5.5}$$

By Proposition 5.4.3, (5.4.2), and (5.2.2),

$$\frac{\mathrm{ev}_1^*\phi_I\big|_{\mathcal{Z}_\Gamma}}{\mathbf{e}(N_\Gamma^{vir})} = \pi_0^*\left(\frac{\mathrm{ev}_1^*\phi_I}{\mathbf{e}(N_{\Gamma_0}^{vir})}\right)\pi_c^*\left(\frac{\mathrm{ev}_1^*\phi_{v(I,j)}}{\mathbf{e}(N_{\Gamma_c}^{vir})}\right)\frac{1}{-\frac{u_j(I)}{d_0}-\pi_c^*\psi_1}. \tag{5.5.6}$$

[2]Figure 5.1 is [Z1, Figure 2] adapted to the toric setting.

By (5.4.3) and (5.2.4), on $\mathcal{Z}_{\Gamma_0}$

$$\begin{aligned}\mathbf{e}\left(\dot{\mathcal{V}}_E\right) &= \prod_{i=1}^{a}\prod_{s=1}^{d_0L_i^+(\overline{Ij})}\left[\lambda_i^+(I)-\frac{s}{d_0}u_j(I)\right]\prod_{i=1}^{b}\prod_{s=0}^{-d_0L_i^-(\overline{Ij})-1}\left[\lambda_i^-(I)+\frac{s}{d_0}u_j(I)\right],\\ \mathbf{e}\left(\ddot{\mathcal{V}}_E\right) &= \prod_{i=1}^{a}\prod_{s=0}^{d_0L_i^+(\overline{Ij})-1}\left[\lambda_i^+(I)-\frac{s}{d_0}u_j(I)\right]\prod_{i=1}^{b}\prod_{s=1}^{-d_0L_i^-(\overline{Ij})}\left[\lambda_i^-(I)+\frac{s}{d_0}u_j(I)\right].\end{aligned} \tag{5.5.7}$$

By Proposition 5.4.3,

$$\mathbf{e}(N_{\Gamma_0}^{vir}) = \frac{(-1)^{d_0}\,(d_0!)^2}{d_0^{2d_0}}\,[u_j(I)]^{2d_0}\prod_{r\in[N]-(I\cup\{j\})}\frac{\prod\limits_{s=0}^{d_0D_r(\overline{Ij})}\left[u_r(I)-\frac{s}{d_0}u_j(I)\right]}{\prod\limits_{s=d_0D_r(\overline{Ij})+1}^{-1}\left[u_r(I)-\frac{s}{d_0}u_j(I)\right]}. \tag{5.5.8}$$

By (5.5.7), (5.5.8), (5.4.2), and (5.5.2),

$$\int_{\mathcal{Z}_{\Gamma_0}}\frac{\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\mathrm{ev}_1^*\phi_I}{(\hbar-\psi_1)\mathbf{e}\left(N_{\Gamma_0}^{vir}\right)} = \frac{\dot{\mathfrak{C}}_{I,j}(d_0)}{\hbar+\frac{u_j(I)}{d_0}} \quad\text{and}\quad \int_{\mathcal{Z}_{\Gamma_0}}\frac{\mathbf{e}\left(\ddot{\mathcal{V}}_E\right)\mathrm{ev}_1^*\phi_I}{(\hbar-\psi_1)\mathbf{e}\left(N_{\Gamma_0}^{vir}\right)} = \frac{\ddot{\mathfrak{C}}_{I,j}(d_0)}{\hbar+\frac{u_j(I)}{d_0}}. \tag{5.5.9}$$

By (5.5.5), (5.5.6), and (5.5.9),

$$\begin{aligned}\int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\dot{\mathcal{V}}_E)\mathrm{ev}_1^*\phi_I\eta^\beta}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma}\frac{1}{\mathbf{e}(N_\Gamma^{vir})} &= \frac{\dot{\mathfrak{C}}_{I,j}(d_0)}{\hbar+\frac{u_j(I)}{d_0}}\int_{\mathcal{Z}_{\Gamma_c}}\frac{\mathbf{e}(\dot{\mathcal{V}}_E)\mathrm{ev}_1^*\phi_{v(I,j)}\eta^\beta}{\hbar-\psi_1}\frac{1}{\mathbf{e}(N_{\Gamma_c}^{vir})}\Big|_{\hbar=-\frac{u_j(I)}{d_0}},\\ \int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\ddot{\mathcal{V}}_E)\mathrm{ev}_1^*\phi_I\eta^\beta}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma}\frac{1}{\mathbf{e}(N_\Gamma^{vir})} &= \frac{\ddot{\mathfrak{C}}_{I,j}(d_0)}{\hbar+\frac{u_j(I)}{d_0}}\int_{\mathcal{Z}_{\Gamma_c}}\frac{\mathbf{e}(\ddot{\mathcal{V}}_E)\mathrm{ev}_1^*\phi_{v(I,j)}\eta^\beta}{\hbar-\psi_1}\frac{1}{\mathbf{e}(N_{\Gamma_c}^{vir})}\Big|_{\hbar=-\frac{u_j(I)}{d_0}}.\end{aligned} \tag{5.5.10}$$

By the first equation in (5.5.10) and the Virtual Localization Theorem (5.4.1), the contribution of the A_I graphs to the coefficient of $Q^{\mathbf{d}}$ in $\dot{\mathcal{Z}}_{\eta,\beta}\big|_I$ is

$$\sum_{d_0\geqslant 1}\sum_{\substack{j\in[N]-I\\ d_0\cdot\deg\overline{Ij}\leq\mathbf{d}}}\frac{\dot{\mathfrak{C}}_{I,j}(d_0)}{\hbar+\frac{u_j(I)}{d_0}}\left[\!\left[\dot{\mathcal{Z}}_{\eta,\beta}(x(v(I,j)),\hbar,Q)\right]\!\right]_{Q;\mathbf{d}-d_0\cdot\deg\overline{Ij}}\Big|_{\hbar=-\frac{u_j(I)}{d_0}}$$

whenever $\mathbf{d}\equiv\mathbf{d}^*$ satisfies the two properties in Definition 5.3.1 (which make evaluation at $\hbar=-\frac{u_j(I)}{d_0}$ meaningful). An analogous statement holds when summing in the second equation in (5.5.10). □

5.6 MPC for the GW power series

Let $\dot{\mathcal{Z}}_1$ and $\ddot{\mathcal{Z}}_1$ be as in (4.0.2) and $\dot{\mathcal{Z}}_{\eta,\beta}$ and $\ddot{\mathcal{Z}}_{\eta,\beta}$ be as in (5.5.3).

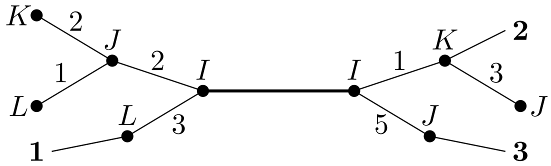

Figure 5.2: A graph representing a fixed locus in $\mathfrak{X}_{\mathbf{d}}(X_M^\tau)$; $I,J,K,L\in\mathscr{V}_M^\tau$, $I\neq J,K,L$.

Lemma 5.6.1. *For all $m\geqslant 3, \eta_j\in H^*_{\mathbb{T}^N}(X_M^\tau), \beta_j\in\mathbb{Z}^{\geqslant 0}$, the pairs $(\ddot{\mathcal{Z}}_1(\hbar,Q), \hbar^{m-2}\dot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q))$ and $(\dot{\mathcal{Z}}_1(\hbar,Q), \hbar^{m-2}\ddot{\mathcal{Z}}_{\eta,\beta}(\hbar,Q))$ satisfy the MPC.*

Lemma 5.6.1 extends [Z1, Lemma 1.2] from the case of a positive line bundle over $\mathbb{P}^{n-1}$ to that of a split vector bundle $E=E^+\oplus E^-$ as in (1.1.2) over an arbitrary symplectic toric manifold X_M^τ. While [Z1, Lemma 1.2] follows from [Z1, Lemma 3.1], Lemma 5.6.1 follows from Lemma 5.6.3 below, which extends [Z1, Lemma 3.1] to the general toric case. The proof of Lemma 5.6.3 uses the Virtual Localization Theorem (5.4.1) instead of the classical one used in the $X_M^\tau=\mathbb{P}^{n-1}$ case and Lemma 5.6.2, which is a general toric version of the first displayed formula in [Z1] after [Z1, (3.32)].

As in [Gi1] and [Z1], we consider the action of $\mathbb{T}^1$ on $V\equiv\mathbb{C}^2$ given by $\xi\cdot(z_0,z_1)\equiv(z_0,\xi^{-1}z_1)$ and the induced action on $\mathbb{P}V$. Let $\hbar$ be the weight of the standard action of $\mathbb{T}^1$ on $\mathbb{C}$. For any $\mathbf{d}\in\Lambda$, let

$$\mathfrak{X}_{\mathbf{d}}(X_M^\tau)\equiv\left\{f\in\overline{\mathfrak{M}}_{0,m}(\mathbb{P}V\times X_M^\tau,(1,\mathbf{d})) : \mathrm{ev}_1(f)\in[1,0]\times X_M^\tau, \mathrm{ev}_2(f)\in[0,1]\times X_M^\tau\right\}.$$

By Proposition 5.4.2, the components of the fixed locus $\mathfrak{X}_{\mathbf{d}}(X_M^\tau)^{\mathbb{T}^1\times\mathbb{T}^N}$ of the $\mathbb{T}^1\times\mathbb{T}^N$-action on $\mathfrak{X}_{\mathbf{d}}(X_M^\tau)$ are indexed by decorated graphs Γ of the following form. Such a graph Γ has a unique edge of positive $\mathbb{P}V$-degree; this special edge corresponds to a degree-one map $f:\mathbb{P}^1\longrightarrow\mathbb{P}V\times[I]$ for some $I\in\mathscr{V}_M^\tau$. Edges to the left (respectively right) of this edge are mapped into $[1,0]\times X_M^\tau$ (respectively $[0,1]\times X_M^\tau$); see Figure 5.2, where we dropped the $\mathbb{P}V$-label of the vertices.[3] Thus, the first marked point is attached to some vertex to the left of the special edge, while the second marked point is attached to some vertex to the right of the special edge.

Let

$$\mathbf{d}_L\equiv\mathbf{d}_L(\Gamma),\quad \mathbf{d}_R\equiv\mathbf{d}_R(\Gamma)\in\Lambda$$

denote the X_M^τ-degrees of the left- and right-hand side (with respect to the special edge) sub-graphs, respectively; thus, $\mathbf{d}=\mathbf{d}_L+\mathbf{d}_R$. Let $\mathcal{Z}_\Gamma$ be the component of $\mathfrak{X}_{\mathbf{d}}(X_M^\tau)^{\mathbb{T}^1\times\mathbb{T}^N}$ corresponding to Γ.

Lemma 5.6.2. *For every $i\in[k]$ and $\mathbf{d}\in\Lambda$, there exists*

$$\Omega_i\in H^2_{\mathbb{T}^1\times\mathbb{T}^N}(\mathfrak{X}_{\mathbf{d}}(X_M^\tau))\quad\text{such that}\quad \Omega_i\big|_{\mathcal{Z}_\Gamma}=x_i(I)+(\mathbf{d}_L(\Gamma))_i\,\hbar$$

for all graphs Γ corresponding to components of $\mathfrak{X}_{\mathbf{d}}(X_M^\tau)^{\mathbb{T}^1\times\mathbb{T}^N}$, with $\mathbf{d}_L(\Gamma)$ and I depending on Γ as above.

[3]Figure 5.2 is [Z1, Figure 3] adapted to the toric setting.

Proof. We follow the proof in [Gi1, Section 11] and [Gi2, Section 2].

Given $s\in\mathbb{Z}^{\geqslant 0}$ and $n\geqslant 1$, let

$$\mathrm{Poly}_s^n \equiv \mathbb{P}\left(\{P\in\mathbb{C}[z_0,z_1] : P \text{ homogeneous of degree } s\}^{\oplus n}\right).$$

We next define a morphism

$$\theta_0 : \overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,s)) \longrightarrow \mathrm{Poly}_s^n.$$

If $[\Sigma,f]$ is an element of $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,s))$, $\Sigma=\Sigma_0\cup\Sigma_1\cup\ldots\cup\Sigma_r$, where Σ_0 is a $\mathbb{P}^1$, $f\big|_{\Sigma_0}$ has degree $(1,s_0)$, Σ_i is connected for all $i\in[r]$, and $f\big|_{\Sigma_i}$ has degree $(0,s_i)$ for all $i\in[r]$. Thus,

$$f(\Sigma_i)\subseteq\{[A_i,B_i]\}\times\mathbb{P}^{n-1} \quad \text{for some} \quad [A_i,B_i]\in\mathbb{P}V \qquad \forall\, i\in[r].$$

Let $\theta_0[\Sigma,f]\equiv[P_1g,\ldots,P_ng]$, where

$$f\big|_{\Sigma_0}\equiv(f_1,f_2), \qquad f_2\circ f_1^{-1}\equiv[P_1,\ldots,P_n]\in\mathrm{Poly}_{s_0}^n, \qquad g\equiv\prod_{i=1}^{r}(A_iz_1-B_iz_0)^{s_i}.$$

Let $\theta\equiv\theta_0\circ\mathrm{fgt}$, where

$$\mathrm{fgt}:\overline{\mathfrak{M}}_{0,m}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,s))\longrightarrow\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,s))$$

is the forgetful morphism. By [Gi1, Section 11, Main Lemma], $\theta\big|_{\mathfrak{X}_s(\mathbb{P}^{n-1})}$ is continuous.

The torus $\mathbb{T}^1\times\mathbb{T}^n$ acts on Poly_s^n by

$$(\xi,t_1,\ldots,t_n)\cdot(P_1[z_0,z_1],\ldots,P_n[z_0,z_1])\equiv(t_1P_1[z_0,\xi z_1],\ldots,t_nP_n[z_0,\xi z_1]).$$

This action naturally lifts to the hyperplane line bundle over Poly_s^n. The map θ_0 is $\mathbb{T}^1\times\mathbb{T}^n$-equivariant and hence so is θ.

Let $\mathcal{L}\longrightarrow X_M^\tau$ be any very ample line bundle. For any $\mathbf{d}\in\Lambda$, let $\mathcal{L}(\mathbf{d})\equiv\langle c_1(\mathcal{L}),\mathbf{d}\rangle$. Consider the canonical lift of the $\mathbb{T}^N$-action on X_M^τ to $\mathcal{L}$ given by Proposition 2.2.5 together with (2.3.3). Thus, there exists n, an injective group homomorphism $\iota_{\mathbb{T}}:\mathbb{T}^N\longrightarrow\mathbb{T}^n$, and an $\iota_{\mathbb{T}}$-equivariant embedding $\iota:X_M^\tau\longrightarrow\mathbb{P}^{n-1}$ such that $\iota^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)=\mathcal{L}$. We consider the $\mathbb{T}^N$-action on $\mathbb{P}^{n-1}$ induced by $\iota_{\mathbb{T}}$. The embedding ι induces a $\mathbb{T}^1\times\mathbb{T}^N$-equivariant embedding

$$\mathfrak{X}_{\mathbf{d}}(X_M^\tau)\xrightarrow{F}\mathfrak{X}_{\mathcal{L}(\mathbf{d})}(\mathbb{P}^{n-1}).$$

The composition

$$\mathfrak{X}_{\mathbf{d}}(X_M^\tau)\xrightarrow{F}\mathfrak{X}_{\mathcal{L}(\mathbf{d})}(\mathbb{P}^{n-1})\xrightarrow{\theta}\mathrm{Poly}_{\mathcal{L}(\mathbf{d})}^n$$

maps $\mathcal{Z}_\Gamma$ onto $[z_0^{\mathcal{L}(\mathbf{d}_R)}z_1^{\mathcal{L}(\mathbf{d}_L)}a_1,\ldots,z_0^{\mathcal{L}(\mathbf{d}_R)}z_1^{\mathcal{L}(\mathbf{d}_L)}a_n]$, where $[a_1,\ldots,a_n]\equiv\iota([I])$.

Let $\Omega\in H^2_{\mathbb{T}^1\times\mathbb{T}^N}(\mathrm{Poly}^n_{\mathcal{L}(\mathbf{d})})$ be the equivariant Euler class of the hyperplane line bundle and

$$\Omega(\mathcal{L})\equiv F^*\theta^*\Omega\in H^2_{\mathbb{T}^1\times\mathbb{T}^N}(\mathfrak{X}_{\mathbf{d}}(X_M^\tau)).$$

It follows that

$$\Omega(\mathcal{L})\big|_{\mathcal{Z}_\Gamma}=\Omega\big|_{[z_0^{\mathcal{L}(\mathbf{d}_R)}z_1^{\mathcal{L}(\mathbf{d}_L)}a_1,\ldots,z_0^{\mathcal{L}(\mathbf{d}_R)}z_1^{\mathcal{L}(\mathbf{d}_L)}a_n]}=\mathbf{e}(\mathcal{L})(I)+\langle c_1(\mathcal{L}),\mathbf{d}_L\rangle\hbar, \tag{5.6.1}$$

where $\mathbf{e}(\mathcal{L})$ is the $\mathbb{T}^N$-equivariant Euler class of $\mathcal{L}$.

By Proposition 2.2.4, there exist very ample line bundles $\mathcal{L}_i$ for all $i \in [k]$ such that $\{c_1(\mathcal{L}_i) : i \in [k]\}$ is a basis for $H^2(X_M^\tau)$; so, using the $\mathbb{T}^N$-action on each $\mathcal{L}_i$ defined by (2.3.3), we find that

$$\mathrm{Span}_{\mathbb{Q}}\{\mathbf{e}(\mathcal{L}_i) : i \in [k]\} = \mathrm{Span}_{\mathbb{Q}}\{x_i : i \in [k]\}.$$

Via Proposition 2.3.3(*b*), this shows that $\{\mathbf{e}(\mathcal{L}_i), \alpha_j : i \in [k], j \in [N]\}$ is a basis for $H^2_{\mathbb{T}^N}(X_M^\tau)$. As in [Gi2], we define a $\mathbb{Q}$-linear map from $H^2_{\mathbb{T}^N}(X_M^\tau)$ to $H^2_{\mathbb{T}^1 \times \mathbb{T}^N}(\mathfrak{X}_{\mathbf{d}}(X_M^\tau))$ by sending $\mathbf{e}(\mathcal{L}_i)$ to $\Omega(\mathcal{L}_i)$ for all $i \in [k]$ and α_j to α_j for all $j \in [N]$. Let $\Omega_i \in H^2_{\mathbb{T}^1 \times \mathbb{T}^N}(\mathfrak{X}_{\mathbf{d}}(X_M^\tau))$ be the image of x_i under this map. The claim now follows from (5.6.1). □

Lemma 5.6.3. *Let* $\eta^\beta \equiv \prod_{j=2}^{m} \left(\psi_j^{\beta_j} \mathrm{ev}_j^* \eta_j\right)$ *in* $H^*_{\mathbb{T}^N}(\overline{\mathfrak{M}}_{0,m}(X_M^\tau, \mathbf{d}))$ *and let*

$$\pi : \overline{\mathfrak{M}}_{0,m}(\mathbb{P}V \times X_M^\tau, (1, \mathbf{d})) \longrightarrow \overline{\mathfrak{M}}_{0,m}(X_M^\tau, \mathbf{d})$$

denote the natural projection. With Φ *as in Definition 5.3.4 and* Ω_i *as in Lemma 5.6.2,*

$$\begin{aligned} (-\hbar)^{m-2}\Phi_{\ddot{\mathcal{Z}}_1, \dot{\mathcal{Z}}_{\eta,\beta}}(\hbar, z, Q) &= \sum_{\mathbf{d}\in\Lambda} Q^{\mathbf{d}} \int_{[\mathfrak{X}_{\mathbf{d}}(X_M^\tau)]^{vir}} \mathrm{e}^{\sum_{i=1}^{k} \Omega_i z_i} \pi^*\left[\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\eta^\beta\right] \prod_{j=3}^{m} \mathrm{ev}_j^* \mathbf{e}(\mathcal{O}_{\mathbb{P}V}(1)), \\ (-\hbar)^{m-2}\Phi_{\dot{\mathcal{Z}}_1, \ddot{\mathcal{Z}}_{\eta,\beta}}(\hbar, z, Q) &= \sum_{\mathbf{d}\in\Lambda} Q^{\mathbf{d}} \int_{[\mathfrak{X}_{\mathbf{d}}(X_M^\tau)]^{vir}} \mathrm{e}^{\sum_{i=1}^{k} \Omega_i z_i} \pi^*\left[\mathbf{e}\left(\ddot{\mathcal{V}}_E\right)\eta^\beta\right] \prod_{j=3}^{m} \mathrm{ev}_j^* \mathbf{e}(\mathcal{O}_{\mathbb{P}V}(1)). \end{aligned} \tag{5.6.2}$$

Proof. We apply the Virtual Localization Theorem (5.4.1) to the right-hand side of each of the two equations in (5.6.2), using Section 5.4 and extending the proof of [Z1, Lemma 3.1] from the case of a positive line bundle over $\mathbb{P}^{n-1}$ to that of a split vector bundle $E = E^+ \oplus E^-$ as in (1.1.2) over an arbitrary symplectic toric manifold X_M^τ. The possible contributing fixed loci graphs are described above. Given such a fixed locus graph Γ, we denote by N_Γ^{vir} the virtual normal bundle to the corresponding component of the fixed locus inside the moduli space. We denote by $\mathcal{A}_I$ the set of all $\mathbb{T}^1 \times \mathbb{T}^N$-fixed loci graphs whose unique edge of positive $\mathbb{P}V$-degree corresponds to a map $\mathbb{P}^1 \longrightarrow \mathbb{P}V \times [I]$, where $I \in \mathscr{V}_M^\tau$. A graph $\Gamma \in \mathcal{A}_I$ breaks into 3 graphs - Γ_L, Γ_R, and Γ_0 - as follows; see also Figure 5.3.[4] The graph Γ_L is obtained by considering all vertices and edges of Γ to the left of the special edge (of positive $\mathbb{P}V$-degree) and adding a marked point labeled 2 at the vertex belonging to the special edge. Given that all vertices in this "left-hand side graph" are labeled $([1, 0], I)$ for some $I \in \mathscr{V}_M^\tau$, it defines a component of $\overline{\mathfrak{M}}_{0,2}(X_M^\tau, \mathbf{d}_L)^{\mathbb{T}^N}$. The graph Γ_R is obtained by considering all vertices of Γ to the right of the special edge and adding a marked point labeled 1 at the vertex belonging to the special edge. Given that all vertices in this "right-hand side graph" are labeled $([0, 1], I)$ for some $I \in \mathscr{V}_M^\tau$, it defines a component of $\overline{\mathfrak{M}}_{0,m}(X_M^\tau, \mathbf{d}_R)^{\mathbb{T}^N}$. Finally, Γ_0 is the special edge with 2 marked points added. They are labeled 1 in the left-hand side and 2 in the right-hand side. Thus,

$$\mathcal{Z}_\Gamma \cong \mathcal{Z}_{\Gamma_L} \times \mathcal{Z}_{\Gamma_0} \times \mathcal{Z}_{\Gamma_R};$$

[4]Figure 5.3 is [Z1, Figure 4] adapted to the toric setting.

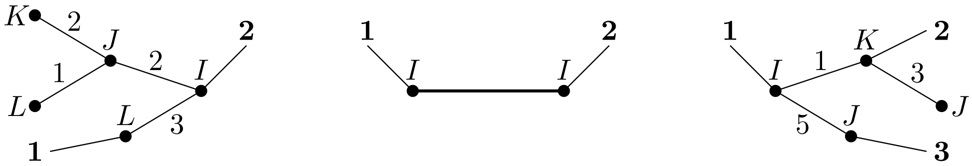

Figure 5.3: The three sub-graphs of the graph in Figure 5.2

we denote by π_L, π_0, and π_R the corresponding projections.

It follows that

$$\begin{aligned} \pi^*\dot{\mathcal{V}}_E &= \pi_L^*\dot{\mathcal{V}}_E \oplus \pi_R^*\dot{\mathcal{V}}_E, \quad \pi^*\ddot{\mathcal{V}}_E = \pi_L^*\ddot{\mathcal{V}}_E \oplus \pi_R^*\ddot{\mathcal{V}}_E, \\ \frac{N_\Gamma^{vir}}{T_{[I]}X_M^\tau} &= \pi_L^*\left(\frac{N_{\Gamma_L}^{vir}}{T_{[I]}X_M^\tau}\right) \oplus \pi_R^*\left(\frac{N_{\Gamma_R}^{vir}}{T_{[I]}X_M^\tau}\right) \oplus \pi_L^*L_2 \otimes \pi_0^*L_1 \oplus \pi_0^*L_2 \otimes \pi_R^*L_1, \end{aligned} \tag{5.6.3}$$

where $L_2 \longrightarrow \mathcal{Z}_{\Gamma_L}, L_1, L_2 \longrightarrow \mathcal{Z}_{\Gamma_0}$, and $L_1 \longrightarrow \mathcal{Z}_{\Gamma_R}$ are the tautological tangent line bundles. The first two equations in (5.6.3) follow similarly to (5.5.4).

By (5.6.3) and (2.3.14),

$$\begin{aligned} \pi^*\left[\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\eta^\beta\right]\prod_{j=3}^m \mathrm{ev}_j^*\left[\mathbf{e}\left(\mathcal{O}_{\mathbb{P}V}(1)\right)\right]\big|_{\mathcal{Z}_\Gamma} &= \pi_L^*\left[\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\right]\pi_R^*\left[\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\eta^\beta(-\hbar)^{m-2}\right], \\ \frac{\mathbf{e}(T_{[I]}X_M^\tau)}{\mathbf{e}\left(N_\Gamma^{vir}\right)} &= \pi_L^*\left[\frac{\mathrm{ev}_2^*\phi_I}{\mathbf{e}\left(N_{\Gamma_L}^{vir}\right)}\right]\pi_R^*\left[\frac{\mathrm{ev}_1^*\phi_I}{\mathbf{e}\left(N_{\Gamma_R}^{vir}\right)}\right]\frac{1}{(\hbar-\pi_L^*\psi_2)\left((-\hbar)-\pi_R^*\psi_1\right)}, \end{aligned} \tag{5.6.4}$$

and the first equation in (5.6.4) with $\dot{\mathcal{V}}_E$ replaced by $\ddot{\mathcal{V}}_E$ also holds. By (5.6.4) and Lemma 5.6.2,

$$\begin{aligned} &\int_{\mathcal{Z}_\Gamma}\frac{\mathrm{e}^{\sum_{i=1}^k\Omega_i z_i}\pi^*\left[\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\eta^\beta\right]\prod_{j=3}^m\mathrm{ev}_j^*\mathbf{e}\left(\mathcal{O}_{\mathbb{P}V}(1)\right)\big|_{\mathcal{Z}_\Gamma}}{\mathbf{e}\left(N_\Gamma^{vir}\right)} = (-\hbar)^{m-2}\frac{\mathrm{e}^{\sum_{i=1}^k x_i(I)z_i}}{\mathbf{e}(T_{[I]}X_M^\tau)} \\ &\times\left\{\mathrm{e}^{\sum_{i=1}^k(\mathbf{d}_L)_i z_i\hbar}\int_{\mathcal{Z}_{\Gamma_L}}\frac{\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\mathrm{ev}_2^*\phi_I}{\hbar-\psi_2}\Big|_{\mathcal{Z}_{\Gamma_L}}\frac{1}{\mathbf{e}\left(N_{\Gamma_L}^{vir}\right)}\right\}\left\{\int_{\mathcal{Z}_{\Gamma_R}}\frac{\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\eta^\beta\mathrm{ev}_1^*\phi_I}{(-\hbar)-\psi_1}\frac{1}{\mathbf{e}\left(N_{\Gamma_R}^{vir}\right)}\right\}; \end{aligned} \tag{5.6.5}$$

(5.6.5) with $\dot{\mathcal{V}}_E$ replaced by $\ddot{\mathcal{V}}_E$ also holds. In the $\mathbf{d}_L = 0$ case, the first curly bracket on the right-hand side of (5.6.5) is defined to be 1. By the Virtual Localization Theorem (5.4.1) and (2.3.1),

$$\begin{aligned} \sum_{\Gamma_L}Q^{\mathbf{d}_L}\left\{\mathrm{e}^{\sum_{i=1}^k(\mathbf{d}_L)_i z_i\hbar}\int_{\mathcal{Z}_{\Gamma_L}}\frac{\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\mathrm{ev}_2^*\phi_I}{\hbar-\psi_2}\Big|_{\mathcal{Z}_{\Gamma_L}}\frac{1}{\mathbf{e}\left(N_{\Gamma_L}^{vir}\right)}\right\} &= \ddot{\mathcal{Z}}_1(\hbar, Q\mathrm{e}^{\hbar z})\big|_I, \\ \sum_{\Gamma_R}Q^{\mathbf{d}_R}\left\{\int_{\mathcal{Z}_{\Gamma_R}}\frac{\mathbf{e}\left(\dot{\mathcal{V}}_E\right)\eta^\beta\mathrm{ev}_1^*\phi_I}{(-\hbar)-\psi_1}\frac{1}{\mathbf{e}\left(N_{\Gamma_R}^{vir}\right)}\right\} &= \dot{\mathcal{Z}}_{\eta,\beta}(-\hbar, Q)\big|_I, \end{aligned} \tag{5.6.6}$$

where the first sum is taken after all graphs Γ_L corresponding to components of $\overline{\mathfrak{M}}_{0,2}(X_M^\tau,\mathbf{d}_L)^{\mathbb{T}^N}$ and with second marked point mapping to $[I]$, while the second is taken after all graphs corresponding to components of $\overline{\mathfrak{M}}_{0,m}(X_M^\tau,\mathbf{d}_R)^{\mathbb{T}^N}$ such that the first marked point is mapped to $[I]$. The equation obtained from (5.6.6) by replacing $\dot{\mathcal{V}}_E$ by $\ddot{\mathcal{V}}_E$, $\dot{\mathcal{Z}}_1$ by $\ddot{\mathcal{Z}}_1$, and $\dot{\mathcal{Z}}_{\eta,\beta}$ by $\ddot{\mathcal{Z}}_{\eta,\beta}$ also holds. The claims follow from (5.6.5) and (5.6.6) (and their $\ddot{\mathcal{V}}_E$ analogues), by summing the left-hand side of (5.6.5) over all graphs $\Gamma\in\mathcal{A}_I$ and all $I\in\mathscr{V}_M^\tau$. □

5.7 Recursivity and MPC for the explicit power series

As in [Gi2], for each $I\in\mathscr{V}_M^\tau$ we define

$$\Delta_I^* \equiv \{\mathbf{d}\in\Lambda : D_j(\mathbf{d})\geqslant 0 \quad \forall\, j\in I\}. \tag{5.7.1}$$

By (4.1.3), (4.1.4), and (2.3.10),

$$[\![\dot{\mathcal{Y}}(x(I),\hbar,q)]\!]_{q;\mathbf{d}}\neq 0 \Longrightarrow \mathbf{d}\in\Delta_I^* \quad\text{and}\quad [\![\ddot{\mathcal{Y}}(x(I),\hbar,q)]\!]_{q;\mathbf{d}}\neq 0 \Longrightarrow \mathbf{d}\in\Delta_I^*. \tag{5.7.2}$$

Lemma 5.7.1. *The power series $\dot{\mathcal{Y}}(x,\hbar,q)$ of (4.1.4) is $\dot{\mathfrak{C}}$-recursive with $\dot{\mathfrak{C}}$ given by (5.5.2). The power series $\ddot{\mathcal{Y}}(x,\hbar,q)$ of (4.1.4) is $\ddot{\mathfrak{C}}$-recursive with $\ddot{\mathfrak{C}}$ given by (5.5.2).*

Proof. The recursivity of $\dot{\mathcal{Y}}$ in the $E=E^+$ case is [Gi2, Proposition 6.3]. The proof of the recursivity of $\dot{\mathcal{Y}}$ in the general case is similar and so is the proof of the recursivity of $\ddot{\mathcal{Y}}$. We prove below the recursivity of $\ddot{\mathcal{Y}}$ extending the proof of (a) in [Z1, Section 2.3] and the proof of [Gi2, Proposition 6.3]. Let $I\in\mathscr{V}_M^\tau$, $j\in[N]-I$, $J\equiv v(I,j)$, $\{\hat{j}\}\equiv I-J$. By (5.7.2), (4.1.4), Remark 5.3.3, and (5.2.2),

$$\ddot{\mathcal{Y}}\left(x(J),-\frac{u_j(I)}{d},q\right)=\sum_{\substack{\mathbf{d}'\in\Delta_J^*\\ D_{j'}(\mathbf{d}')\geqslant -d}} q^{\mathbf{d}'}\frac{\prod\limits_{\substack{r\in[N]\\ D_r(\mathbf{d}')<0}}\prod\limits_{s=D_r(\mathbf{d}')+1}^{0}\big[u_r(J)-\frac{s}{d}u_j(I)\big]}{\prod\limits_{\substack{r\in[N]\\ D_r(\mathbf{d}')\geqslant 0}}\prod\limits_{s=1}^{D_r(\mathbf{d}')}\big[u_r(J)-\frac{s}{d}u_j(I)\big]} \tag{5.7.3}$$
$$\times\prod_{i=1}^{a}\prod_{s=0}^{L_i^+(\mathbf{d}')-1}\Big[\lambda_i^+(J)-\frac{s}{d}u_j(I)\Big]\prod_{i=1}^{b}\prod_{s=1}^{-L_i^-(\mathbf{d}')}\Big[\lambda_i^-(J)+\frac{s}{d}u_j(I)\Big].$$

By (5.7.3), (5.2.5), and (5.2.4),

$$\ddot{\mathcal{Y}}\left(x(J),-\frac{u_j(I)}{d},q\right)=\sum_{\substack{\mathbf{d}'\in\Delta_J^*\\ D_{\hat{j}}(\mathbf{d}')\geqslant -d}} q^{\mathbf{d}'}\frac{\prod\limits_{\substack{r\in[N]\\ D_r(\mathbf{d}')<0}}\prod\limits_{s=D_r(\mathbf{d}')+1+dD_r(\overline{Ij})}^{dD_r(\overline{Ij})}\big[u_r(I)-\frac{s}{d}u_j(I)\big]}{\prod\limits_{\substack{r\in[N]\\ D_r(\mathbf{d}')\geqslant 0}}\prod\limits_{s=1+dD_r(\overline{Ij})}^{D_r(\mathbf{d}')+dD_r(\overline{Ij})}\big[u_r(I)-\frac{s}{d}u_j(I)\big]} \tag{5.7.4}$$
$$\times\prod_{i=1}^{a}\prod_{s=dL_i^+(\overline{Ij})}^{L_i^+(\mathbf{d}')-1+dL_i^+(\overline{Ij})}\Big[\lambda_i^+(I)-\frac{s}{d}u_j(I)\Big]\prod_{i=1}^{b}\prod_{s=1-dL_i^-(\overline{Ij})}^{-L_i^-(\mathbf{d}')-dL_i^-(\overline{Ij})}\Big[\lambda_i^-(I)+\frac{s}{d}u_j(I)\Big].$$

By (5.2.6) and (5.5.1),

$$\widetilde{\mathfrak{C}}_{I,j}(d) = \frac{(-1)^d d^{2d-1}}{(d!)^2} \frac{1}{[u_j(I)]^{2d-1}} \prod_{r\in[N]-\{j,\hat{j}\}} \frac{\prod\limits_{s=1+dD_r(\overline{Ij})}^{0} \big[u_r(I) - \frac{s}{d}u_j(I)\big]}{\prod\limits_{s=1}^{dD_r(\overline{Ij})} \big[u_r(I) - \frac{s}{d}u_j(I)\big]}. \tag{5.7.5}$$

If $d \geqslant 1$, $\mathbf{d}^* \in \Lambda$, $\mathbf{d}' \equiv \mathbf{d}^* - d\cdot\deg\overline{Ij} \in \Lambda$, then,

$$\Big[\mathbf{d}^* \in \Delta_I^* \text{ and } D_j(\mathbf{d}^*) \geqslant d\Big] \quad \Longleftrightarrow \quad \Big[\mathbf{d}' \in \Delta_J^* \text{ and } D_{\hat{j}}(\mathbf{d}') \geqslant -d\Big] \tag{5.7.6}$$

by (5.2.6). By (5.7.2), (4.1.4), (5.7.4), (5.7.5), and (5.5.2),

$$\mathrm{Res}_{z=-\frac{u_j(I)}{d}} \left\{ \frac{1}{\hbar - z} \Big[\!\Big[\ddot{\mathcal{Y}}(x(I), z, q) \Big]\!\Big]_{q;\mathbf{d}^*} \right\} = \frac{\ddot{\mathfrak{C}}_{I,j}(d)}{\hbar + \frac{u_j(I)}{d}} \left[\!\!\left[\ddot{\mathcal{Y}}\left(x(J), -\frac{u_j(I)}{d}, q \right) \right]\!\!\right]_{q;\mathbf{d}^* - d\cdot\deg\overline{Ij}}$$

for all $\mathbf{d}^* \in \Lambda$. Finally, viewing $\frac{1}{\hbar - z} \Big[\!\Big[\ddot{\mathcal{Y}}(x(I), z, q) \Big]\!\Big]_{q;\mathbf{d}^*}$ as a rational function in $\hbar, z$, and α_j and using the Residue Theorem on $\mathbb{P}^1$, we obtain

$$\sum_{\substack{d\geqslant 1\\ d\cdot\deg\overline{Ij}\leq\mathbf{d}^*}} \sum_{j\in[N]-I} \frac{\ddot{\mathfrak{C}}_{I,j}(d)}{\hbar + \frac{u_j(I)}{d}} \left[\!\!\left[\ddot{\mathcal{Y}}\left(x(J), -\frac{u_j(I)}{d}, q \right) \right]\!\!\right]_{q;\mathbf{d}^* - d\cdot\deg\overline{Ij}} = \Big[\!\Big[\ddot{\mathcal{Y}}(x(I), \hbar, q) \Big]\!\Big]_{q;\mathbf{d}^*}$$
$$- \mathrm{Res}_{z=0,\infty} \left\{ \frac{1}{\hbar - z} \Big[\!\Big[\ddot{\mathcal{Y}}\,(x(I), z, q) \Big]\!\Big]_{q;\mathbf{d}^*} \right\},$$

where $\mathrm{Res}_{z=0,\infty}\mathcal{F} \equiv \mathrm{Res}_{z=0}\mathcal{F} + \mathrm{Res}_{z=\infty}\mathcal{F}$. Since

$$\mathrm{Res}_{z=0,\infty} \left\{ \frac{1}{\hbar - z} \Big[\!\Big[\ddot{\mathcal{Y}}\,(x(I), z, q) \Big]\!\Big]_{q;\mathbf{d}^*} \right\} \in \mathbb{Q}_\alpha[\hbar, \hbar^{-1}],$$

this concludes the proof. □

Lemma 5.7.2. *With $\dot{\mathcal{Y}}$ and $\ddot{\mathcal{Y}}$ defined by (4.1.4), $(\dot{\mathcal{Y}}, \ddot{\mathcal{Y}})$ satisfies the MPC.*

We follow the idea of the proof of [Gi2, Proposition 6.2] and begin with some preparations. Let $\mathbf{d} \in \bigcup\limits_{I\in\mathscr{V}_M^\tau} \Delta_I^*$,

$$J \equiv J(\mathbf{d}) \equiv \{ j \in [N] : D_j(\mathbf{d}) \geqslant 0 \}, \quad S \equiv |J| + \sum_{j\in J} D_j(\mathbf{d}).$$

Let A be the $|J| \times S$ matrix giving $\prod\limits_{j\in J} \mathbb{P}^{D_j(\mathbf{d})}$ as in (2.4.2). Denote the coordinates of a point $y \in \mathbb{C}^S$ by

$$\big(y_{j;0}, y_{j;1}, \ldots, y_{j;D_j(\mathbf{d})} \big)_{j\in J}.$$

The pair $(M_J A, \tau)$ is toric in the sense of Definition 2.1.1. It satisfies (ii) in Definition 2.1.1, since

$$\begin{aligned}\mathscr{V}^\tau_{M_J A} = \big\{ &((i_1;p_1),\ldots,(i_k;p_k)) : \\ &\{i_1,\ldots,i_k\}\in\mathscr{V}^\tau_M,\ \{i_1,\ldots,i_k\}\subseteq J,\ 0\leqslant p_r\leqslant D_{i_r}(\mathbf{d}) \quad \forall\, r\in[k]\big\}\end{aligned} \tag{5.7.7}$$

by the second statement in Lemma 2.1.4(b). We identify $\mathbb{C}^S$ with $\bigoplus_{j\in J} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D_j(\mathbf{d})))$ via

$$\big(y_{j;0}, y_{j;1}, \ldots, y_{j;D_j(\mathbf{d})}\big)_{j\in J} \longrightarrow \left(\sum_{r=0}^{D_j(\mathbf{d})} y_{j;r} z_0^{D_j(\mathbf{d})-r} z_1^r\right)_{j\in J}$$

and set $X_{\mathbf{d}} \equiv X^\tau_{M_J A}$. The torus $\mathbb{T}^1\times\mathbb{T}^{|J|}$ acts on $\bigoplus_{j\in J} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D_j(\mathbf{d}))$ by

$$\Big(\xi, (t_j)_{j\in J}\Big)\cdot\big(P_j(z_0,z_1)\big)_{j\in J} \equiv \big(t_j P_j(z_0, \xi z_1)\big)_{j\in J}\,, \tag{5.7.8}$$

while the torus $\mathbb{T}^{|J|}$ acts on $\bigoplus_{j\in J} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D_j(\mathbf{d})))$ by restricting this action via

$$\mathbb{T}^{|J|}\ni t \hookrightarrow (1,t)\in\mathbb{T}^1\times\mathbb{T}^{|J|};$$

these actions descend to actions on $X_{\mathbf{d}}$.

Lemma 5.7.3. *(a) The fixed points of the $\mathbb{T}^1\times\mathbb{T}^{|J|}$-action on $X_{\mathbf{d}}$ are*

$$[I,\mathbf{p}] \equiv \Big[\big(P_j(z_0,z_1)\big)_{j\in J}\Big], \tag{5.7.9}$$

where $I\in\mathscr{V}^\tau_M$, $I\subseteq J$, $\mathbf{p}=(p_i)_{i\in I}\in\mathbb{Z}^k$, $0\leqslant p_i\leqslant D_i(\mathbf{d})$ for all $i\in I$, and

$$P_j(z_0,z_1) \equiv \begin{cases} z_0^{D_j(\mathbf{d})-p_j} z_1^{p_j}, & \text{if } j\in I;\\ 0, & \text{otherwise.}\end{cases}$$

(b) Let $I\in\mathscr{V}^\tau_M$ and $\mathbf{p}=(p_i)_{i\in I}\in\mathbb{Z}^k$. Then

$$0\leqslant p_i\leqslant D_i(\mathbf{d}) \quad \forall\, i\in I \qquad \Longleftrightarrow \qquad \mathbf{p}M_I^{-1}, \mathbf{d}-\mathbf{p}M_I^{-1}\in\Delta^*_I.$$

Proof. Let $[(P_j(z_0,z_1)_{j\in J})]$ be any fixed point of the $\mathbb{T}^1\times\mathbb{T}^{|J|}$-action on $X_{\mathbf{d}}$ and $(\xi_0,\xi_1)\in\mathbb{C}^2$ be such that $P_j(\xi_0,\xi_1)\neq 0$ whenever $P_j\neq 0$. By Lemma 2.1.4(i) and (5.7.7), $(P_j(\xi_0,\xi_1))_{j\in J}\in\widetilde{X}^\tau_{M_J}$. Since $[(P_j(\xi_0,\xi_1))_{j\in J}]$ is a $\mathbb{T}^{|J|}$-fixed point in $X^\tau_{M_J}$, there exists $I\in\mathscr{V}^\tau_M$ with $I\subseteq J$ such that $P_j\neq 0$ if and only if $j\in I$; see Corollary 2.3.2(a). This concludes the proof of (a). Part (b) follows from (5.7.1) and the identity $(D_i(\mathbf{r}))_{i\in I}\equiv\mathbf{r}M_I$ for $\mathbf{r}=\mathbf{p}M_I^{-1}$; see the second equation in (2.2.4). □

We consider the $\mathbb{T}^1\times\mathbb{T}^N$-action on $X_{\mathbf{d}}$ obtained by composing the projection $\mathbb{T}^1\times\mathbb{T}^N\longrightarrow \mathbb{T}^1\times\mathbb{T}^{|J|}$ induced by $J\hookrightarrow[N]$ with the action (5.7.8) of $\mathbb{T}^1\times\mathbb{T}^{|J|}$ on $X_{\mathbf{d}}$. We denote by $\widetilde{\mathbf{e}}(T_{[I,\mathbf{p}]}X_{\mathbf{d}})$ the $\mathbb{T}^1\times\mathbb{T}^N$-equivariant Euler class of $T_{[I,\mathbf{p}]}X_{\mathbf{d}}$ and by

$$\cdot(I,\mathbf{p}):H^*_{\mathbb{T}^1\times\mathbb{T}^N}(X_{\mathbf{d}})\longrightarrow H^*_{\mathbb{T}^1\times\mathbb{T}^N}$$

the restriction map induced by the inclusion $[I,\mathbf{p}]\hookrightarrow X_{\mathbf{d}}$, where $[I,\mathbf{p}]$ is the $\mathbb{T}^1\times\mathbb{T}^N$-fixed point defined by (5.7.9). Let $\hbar$ denote the weight of the standard action of $\mathbb{T}^1$ on $\mathbb{C}$.

Lemma 5.7.4. *There exist classes* $(\mathbf{x}_i)_{i\in[k]},(\mathbf{u}_r)_{r\in[N]},(\boldsymbol{\lambda}_i^+)_{i\in[a]},(\boldsymbol{\lambda}_i^-)_{i\in[b]}\in H^*_{\mathbb{T}^1\times\mathbb{T}^N}(X_{\mathbf{d}})$ *such that*

$$\mathbf{u}_r=\sum_{i=1}^k m_{ir}\mathbf{x}_i-\alpha_r\qquad\forall\, r\in[N],\tag{5.7.10}$$

and such that for all $(I,\mathbf{d}')$ *with* $I\in\mathscr{V}_M^\tau$, $I\subseteq J$, $\mathbf{d}',\mathbf{d}-\mathbf{d}'\in\Delta_I^*$, *and all* $[I,\mathbf{p}]$ *as in (5.7.9),*

$$(\mathbf{x}_1(I,\mathbf{d}'M_I),\ldots,\mathbf{x}_k(I,\mathbf{d}'M_I))=(x_1(I),\ldots,x_k(I))+\hbar\,\mathbf{d},'\tag{5.7.11}$$

$$\widetilde{\mathbf{e}}(T_{[I,\mathbf{p}]}X_{\mathbf{d}})=\prod_{j\in J-I}\prod_{0\leqslant s\leqslant D_j(\mathbf{d})}[\mathbf{u}_j(I,\mathbf{p})-s\,\hbar]\times\prod_{j\in I}\prod_{\substack{0\leqslant s\leqslant D_j(\mathbf{d})\\ s\neq p_j}}[\mathbf{u}_j(I,\mathbf{p})-s\,\hbar],\tag{5.7.12}$$

$$\boldsymbol{\lambda}_i^\pm(I,\mathbf{d}'M_I)=\lambda_i^\pm(I)+\hbar\,L_i^\pm(\mathbf{d}')\qquad\forall\, i\in[a]\quad(\forall\, i\in[b]).\tag{5.7.13}$$

Proof. We define the classes $\widetilde{x}_1,\ldots,\widetilde{x}_k$ and $u_{j;s}$ in $H^*_{\mathbb{T}^S}(X_{\mathbf{d}})$ with $j\in J$ and $0\leqslant s\leqslant D_j(\mathbf{d})$ by (2.3.7) with (M,τ) replaced by (M_JA,τ). By (2.3.9),

$$u_{j;s}=\sum_{i=1}^k m_{ij}\widetilde{x}_i-\alpha_{j;s},\tag{5.7.14}$$

where $\alpha_{j;s}\equiv\pi_{j;s}^*c_1(\mathcal{O}_{\mathbb{P}^\infty}(1))$ and $\pi_{j;s}:(\mathbb{P}^\infty)^S\longrightarrow\mathbb{P}^\infty$ is the projection onto the $(j;s)$ component. By Corollary 2.3.2(a), (5.7.7), and (2.3.14), the $\mathbb{T}^S$-fixed points in $X_{\mathbf{d}}$ are the points $[I,\mathbf{p}]$ and

$$\mathbf{e}^{\mathbb{T}^S}(T_{[I,\mathbf{p}]}X_{\mathbf{d}})=\prod_{j\in J-I}\prod_{0\leqslant s\leqslant D_j(\mathbf{d})}\Big[u_{j;s}\big|_{[I,\mathbf{p}]}\Big]\times\prod_{j\in I}\prod_{\substack{0\leqslant s\leqslant D_j(\mathbf{d})\\ s\neq p_j}}\Big[u_{j;s}\big|_{[I,\mathbf{p}]}\Big],\tag{5.7.15}$$

where $\mathbf{e}^{\mathbb{T}^S}(T_{[I,\mathbf{p}]}X_{\mathbf{d}})$ denotes the $\mathbb{T}^S$-equivariant Euler class of $T_{[I,\mathbf{p}]}X_{\mathbf{d}}$ and

$$\Big|_{[I,\mathbf{p}]}:H^*_{\mathbb{T}^S}(X_{\mathbf{d}})\longrightarrow H^*_{\mathbb{T}^S}$$

the restriction homomorphism induced by $[I,\mathbf{p}]\hookrightarrow X_{\mathbf{d}}$. The map

$$F:(\mathbb{C}^\infty-\{0\})^{N+1}\longrightarrow(\mathbb{C}^\infty-\{0\})^S,\quad F(e_0,e_1,\ldots,e_N)\equiv\Big(e_j,e_j\cdot e_0,e_j\cdot e_0^2,\ldots,e_j\cdot e_0^{D_j(\mathbf{d})}\Big)_{j\in J},$$

where

$$(z_1,z_2,\ldots)^d \equiv (z_1^d,z_2^d,\ldots) \qquad \forall\, d\geqslant 1,\ (z_1,z_2,\ldots)\in\mathbb{C}^\infty-\{0\}, \quad \text{and}$$
$$(z_1,z_2,\ldots)\cdot(y_1,y_2,\ldots) \equiv (z_iy_j)_{(i,j)\in\mathbb{Z}^{>0}\times\mathbb{Z}^{>0}} \qquad \forall\,(z_1,z_2,\ldots),(y_1,y_2,\ldots)\in\mathbb{C}^\infty-\{0\}$$

is equivariant with respect to the homomorphism

$$f\colon \mathbb{T}^1\times\mathbb{T}^N\longrightarrow\mathbb{T}^S, \quad f(\xi,t_1,\ldots,t_N)\equiv\big(t_j,t_j\xi,t_j\xi^2,\ldots,t_j\xi^{D_j(\mathbf{d})}\big)_{j\in J}.$$

It induces a map $\overline{F}\colon(\mathbb{C}^\infty-\{0\})^{N+1}\times_{\mathbb{T}^1\times\mathbb{T}^N}X_{\mathbf{d}}\longrightarrow(\mathbb{C}^\infty-\{0\})^S\times_{\mathbb{T}^S}X_{\mathbf{d}}$,

$$\overline{F}\,[e_0,e_1,\ldots,e_N,[(P_j)_{j\in J}]]\equiv[F(e_0,e_1,\ldots,e_N),[(P_j)_{j\in J}]]$$
$$\forall\,(e_0,e_1,\ldots,e_N)\in(\mathbb{C}^\infty-0)^{N+1},\ [(P_j)_{j\in J}]\in X_{\mathbf{d}},$$

and thus a homomorphism $\overline{F}^*\colon H^*_{\mathbb{T}^S}(X_{\mathbf{d}})\longrightarrow H^*_{\mathbb{T}^1\times\mathbb{T}^N}(X_{\mathbf{d}})$. It follows that

$$\overline{F}^*\alpha_{j;s}=\alpha_j+s\hbar \qquad \forall\,(j;s) \quad \text{with} \quad j\in J,\ 0\leqslant s\leqslant D_j(\mathbf{d}). \tag{5.7.16}$$

We define $\mathbf{x}_i$ and $\mathbf{u}_r$ as the $\mathbb{T}^1\times\mathbb{T}^N$-equivariant Euler classes of the line bundles

$$\widetilde{X}^\tau_{M_JA}\times\mathbb{C}/\sim_i\longrightarrow X_{\mathbf{d}} \quad \text{and} \quad \widetilde{X}^\tau_{M_JA}\times\mathbb{C}/\sim_r\longrightarrow X_{\mathbf{d}},$$

where

$$\begin{array}{l}((P_j)_{j\in J},c)\sim_i\big((t^{M_j}P_j)_{j\in J},t_ic\big)\\ ((P_j)_{j\in J},c)\sim_r\big((t^{M_j}P_j)_{j\in J},t^{M_r}c\big)\end{array} \qquad \forall\,t\in\mathbb{T}^k,\ ((P_j)_{j\in J},c)\in\widetilde{X}^\tau_{M_JA}\times\mathbb{C} \tag{5.7.17}$$

with respect to the lifts of the $\mathbb{T}^1\times\mathbb{T}^N$-action on $X_{\mathbf{d}}$ given by

$$\begin{array}{l}(\xi,t_1,\ldots,t_N)\cdot[(P_j(z_0,z_1))_{j\in J},c]\equiv[(t_jP_j(z_0,\xi z_1))_{j\in J},c] \quad \text{and}\\ (\xi,t_1,\ldots,t_N)\cdot[(P_j(z_0,z_1))_{j\in J},c]\equiv[(t_jP_j(z_0,\xi z_1))_{j\in J},t_rc]\end{array} \tag{5.7.18}$$

respectively. It follows that

$$\mathbf{x}_i=\overline{F}^*\widetilde{x}_i \tag{5.7.19}$$

and $\mathbf{u}_j$ satisfy (5.7.10). The latter follows similarly to the proof of (2.3.9) using equations analogous to (2.3.5) and (2.3.6) with $\mathbb{T}^N$ replaced by $\mathbb{T}^1\times\mathbb{T}^N$. Equation (5.7.11) follows from (5.7.19), Proposition 2.3.3(a), and (5.7.16). Equation (5.7.21) follows from (5.7.10), (5.7.11), and (2.3.9). Equation (5.7.12) follows from (5.7.15) together with (5.7.14), (5.7.16), (5.7.19), and (5.7.10). Finally, define

$$\boldsymbol{\lambda}_i^+\equiv\sum_{r=1}^k\ell^+_{ri}\mathbf{x}_r \quad \text{and} \quad \boldsymbol{\lambda}_i^-\equiv\sum_{r=1}^k\ell^-_{ri}\mathbf{x}_r, \tag{5.7.20}$$

with ℓ^+_{ri},ℓ^-_{ri} as in (3.1.3). Equations (5.7.13) then follow from (5.7.20), (5.7.11), and (4.1.1). □

With $\mathbf{u}_j$ and $(I,\mathbf{d}')$ as in Lemma 5.7.4,

$$\mathbf{u}_j\,(I,\mathbf{d}'M_I) = u_j(I)+\hbar\,D_j(\mathbf{d}') \qquad \forall\, j\in[N], \tag{5.7.21}$$

by (5.7.10), (5.7.11), and (2.3.9).

Lemma 5.7.5. *There exists a vector bundle $V_{\mathbf{d}}\longrightarrow X_{\mathbf{d}}$ and a lift of the $\mathbb{T}^1\times\mathbb{T}^N$-action to $V_{\mathbf{d}}$ such that the $\mathbb{T}^1\times\mathbb{T}^N$-equivariant Euler class $\widetilde{\mathbf{e}}(V_{\mathbf{d}})$ satisfies*

$$\widetilde{\mathbf{e}}(V_{\mathbf{d}})(I,\mathbf{p}) = \prod_{j=1}^{N}\prod_{s=1}^{-D_j(\mathbf{d})-1}\big[\mathbf{u}_j(I,\mathbf{p})+s\,\hbar\big]$$

for all $\mathbb{T}^1\times\mathbb{T}^N$-fixed points $[I,\mathbf{p}]$ defined by (5.7.9) and with $\mathbf{u}_j\in H^*_{\mathbb{T}^1\times\mathbb{T}^N}(X_{\mathbf{d}})$ as in Lemma 5.7.4.

Proof. Let

$$\widetilde{V}_{\mathbf{d}} \equiv \left\{(P_j)_{j\in[N]-J}\in\bigoplus_{j\in[N]-J}H^0\big(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-D_j(\mathbf{d})-1)\big): P_j(1,0)=0 \quad \forall\, j\in[N]-J\right\},$$

$$V_{\mathbf{d}} \equiv \frac{\widetilde{X}^{\tau}_{M_JA}\times\widetilde{V}_{\mathbf{d}}}{\sim}\longrightarrow X_{\mathbf{d}},\quad \Big((P_j)_{j\in J},(P_j)_{j\in[N]-J}\Big)\sim\Big(\big(t^{M_j}P_j\big)_{j\in J},\big(t^{M_j}P_j\big)_{j\in[N]-J}\Big)\quad \forall\, t\in\mathbb{T}^k.$$

Since $\widetilde{X}^{\tau}_{M_JA}\longrightarrow X_{\mathbf{d}}$ is a principal bundle, $V_{\mathbf{d}}\longrightarrow X_{\mathbf{d}}$ is a holomorphic vector bundle. The $\mathbb{T}^1\times\mathbb{T}^N$-action on $X_{\mathbf{d}}$ lifts to $V_{\mathbf{d}}$ via

$$(\xi,t_1,\ldots,t_N)\cdot\Big[(P_j(z_0,z_1))_{j\in J},(P_j(z_0,z_1))_{j\in[N]-J}\Big]\equiv\Big[(t_jP_j(z_0,\xi z_1))_{j\in J},(t_jP_j(z_0,\xi z_1))_{j\in[N]-J}\Big].$$

The lemma now follows from the definition of $\mathbf{u}_j$ in (5.7.17) and (5.7.18). □

By the Localization Theorem (2.3.2), Lemma 5.7.3, Lemma 5.7.5, and (5.7.12),

$$\int_{X_{\mathbf{d}}} f\,\widetilde{\mathbf{e}}(V_{\mathbf{d}}) = \sum_{\substack{I\in\mathscr{V}^{\tau}_M\\ \mathbf{d}',\mathbf{d}-\mathbf{d}'\in\Delta^*_I}}\frac{f(I,\mathbf{d}'M_I)\prod\limits_{j=1}^{N}\prod\limits_{s=D_j(\mathbf{d})+1}^{-1}[\mathbf{u}_j(I,\mathbf{d}'M_I)-s\hbar]}{\prod\limits_{j\in J-I}\prod\limits_{0\leqslant s\leqslant D_j(\mathbf{d})}[\mathbf{u}_j(I,\mathbf{d}'M_I)-s\hbar]\prod\limits_{j\in I}\prod\limits_{\substack{0\leqslant s\leqslant D_j(\mathbf{d})\\ s\neq D_j(\mathbf{d}')}}[\mathbf{u}_j(I,\mathbf{d}'M_I)-s\hbar]}, \tag{5.7.22}$$

for all $f\in H^*_{\mathbb{T}^1\times\mathbb{T}^N}(X_{\mathbf{d}})$.

Proof of Lemma 5.7.2. By Definition 5.3.4, (5.7.2), (4.1.4), (5.7.22), (5.7.11), (5.7.21), and

(5.7.13),

$$
\begin{aligned}
\Phi_{\overset{\blacktriangledown}{\mathcal{Y}},\overset{\blacktriangledown\blacktriangledown}{\mathcal{Y}}}(\hbar,z,Q) &= \sum_{I\in\mathscr{V}_M^\tau}\sum_{\mathbf{d}\in\Delta_I^*} Q^{\mathbf{d}} \Bigg\{ \sum_{\substack{\mathbf{d}',\mathbf{d}''\in\Delta_I^* \\ \mathbf{d}'+\mathbf{d}''=\mathbf{d}}} \frac{\mathrm{e}^{(x(I)+\hbar\mathbf{d}')\cdot z}}{\prod\limits_{j\in[N]-I} u_j(I)} \frac{\prod\limits_{\substack{j\in[N]\\ D_j(\mathbf{d})<0}} u_j(I) \prod\limits_{\substack{j\in[N]\\ D_j(\mathbf{d})<0}} \prod\limits_{s=D_j(\mathbf{d}')+1}^{-D_j(\mathbf{d}'')-1} [u_j(I)+s\hbar]}{\prod\limits_{D_j(\mathbf{d})\geqslant 0} \prod\limits_{\substack{-D_j(\mathbf{d}'')\leqslant s\leqslant D_j(\mathbf{d}') \\ s\neq 0}} [u_j(I)+s\hbar]} \\
&\qquad \times \prod_{i=1}^{a} \prod_{s=-L_i^+(\mathbf{d}'')+1}^{L_i^+(\mathbf{d}')} \big[\lambda_i^+(I)+s\hbar\big] \prod_{i=1}^{b} \prod_{s=L_i^-(\mathbf{d}')+1}^{-L_i^-(\mathbf{d}'')} \big[\lambda_i^-(I)+s\hbar\big] \Bigg\} \\
&= \sum_{\mathbf{d}\in\bigcup\limits_{I\in\mathscr{V}_M^\tau}\Delta_I^*} Q^{\mathbf{d}} \int_{X_{\mathbf{d}}} \widetilde{\mathbf{e}}(V_{\mathbf{d}})\mathrm{e}^{\mathbf{x}\cdot z} \prod_{i=1}^{a} \prod_{s=-L_i^+(\mathbf{d})+1}^{0} \big[\boldsymbol{\lambda}_i^+ + s\hbar\big] \prod_{i=1}^{b} \prod_{s=1}^{-L_i^-(\mathbf{d})} \big[\boldsymbol{\lambda}_i^- + s\hbar\big].
\end{aligned}
$$

The last expression is in $\mathbb{Q}[\alpha,\hbar][[z,\Lambda]]$. □

Appendix A

Derivation of (5.1.2) from [LLY3]

[LLY3]	our notation
m	k
e^{t_j}	q_j
$\mathcal{R}$	$\mathbb{C}(\alpha_1,\ldots,\alpha_N)[\hbar]$
$\mathbb{C}[\mathcal{T}^*]$	$\mathbb{Q}[\alpha_1,\ldots,\alpha_N]$
α	$\hbar$
T	$\mathbb{T}^N$
e_T	$\mathbf{e}$
$c_1(L_d)$	$-\psi_1 \in H^2\left(\overline{\mathfrak{M}}_{0,1}(X_M^\tau,\mathbf{d})\right)$
ρ	forgetful morphism $\overline{\mathfrak{M}}_{0,1}(X,\mathbf{d}) \longrightarrow \overline{\mathfrak{M}}_{0,0}(X,\mathbf{d})$
e_d^X	$\mathrm{ev}_1 : \overline{\mathfrak{M}}_{0,1}(X,\mathbf{d}) \longrightarrow X$
$LT_{0,1}(d,X)$	$\left[\overline{\mathfrak{M}}_{0,1}(X,\mathbf{d})\right]^{vir}$
V_d	$\mathcal{V}_E \longrightarrow \overline{\mathfrak{M}}_{0,0}(X,\mathbf{d})$
$U_d=\rho^*V_d$	$\mathcal{V}_E \longrightarrow \overline{\mathfrak{M}}_{0,1}(X,\mathbf{d})$
$(D_a)_{a\in[N]}$	$(u_j)_{j\in[N]}$

In [LLY3, Section 3.2], we take $b_T \equiv e_T$ (that is, $\mathbf{e}$), $X = X_M^\tau$, and $V \equiv E$. By [LLY3, Section 3.2],

$$A^{V,b_T}(t) = A(t) = e^{-H\cdot t/\alpha}\left[\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)} + \sum_{d\in\Lambda-0} A_d e^{d\cdot t}\right], \qquad A_d = \mathrm{e}_*^X\left(\frac{\rho^* b_T(V_d)\cap LT_{0,1}(d,X)}{e_G(F_0/M_d(X))}\right),$$

where $\{H_a\}\subset H^2_{\mathbb{T}^N}(X_M^\tau)$ is a basis whose restriction to $H^2(X_M^\tau)$ is a basis of first Chern classes of ample line bundles; see [LLY3, Section 3,viii]. By [LLY3, Lemma 3.5],

$$e_G(F_0/M_d(X)) = \alpha(\alpha - c_1(L_d)).$$

Thus, in our notation,

$$A(t) = \mathrm{e}^{-H\cdot t/\hbar}\left\{\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)} + \sum_{\mathbf{d}\in\Lambda-0} \mathrm{e}^{\mathbf{d}\cdot t}\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}_E)}{\hbar(\hbar+\psi_1)}\right]\right\}, \tag{A.0.1}$$

where $\mathrm{ev}_1 : \overline{\mathfrak{M}}_{0,1}(X_M^\tau, \mathbf{d}) \longrightarrow X_M^\tau$ is the evaluation map at the marked point. By (4.0.2), (A.0.1), and the string relation [MirSym, Section 26.3],

$$A(t) = \mathrm{e}^{-H\cdot t/\hbar}\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)}\dot{\mathcal{Z}}_1(-\hbar, e^t). \tag{A.0.2}$$

By (4.1.4), Remark 5.1.1, and (4.1.1), (5.1.2) is independent of the choice of a $\mathbb{Q}[\alpha]$-basis for $H^2_{\mathbb{T}^N}(X_M^\tau)$ and so it is not necessary to assume that the restrictions of x_i to $H^2(X_M^\tau)$ are Chern classes of ample line bundles. Thus, we may take $H=(x_1, \ldots, x_k)$ in [LLY3]. By [LLY3, (5.2)] and [LLY3, Theorem 4.9],

$$B(t) = \mathrm{e}^{-H\cdot t/\hbar}\frac{\mathbf{e}(E^+)}{\mathbf{e}(E^-)}\dot{\mathcal{Y}}\left(x, -\hbar, \mathrm{e}^t\right) \tag{A.0.3}$$

in [LLY3, Theorem 4.7]. In the notation of the proof of [LLY3, Theorem 4.7] correlated with Remark 5.1.1,

$$\begin{gathered} C=\dot{I}_{\mathbf{0}}(q), \quad C'=-\dot{I}_{\mathbf{0}}(q)\left(G(q)+\sum_{j=1}^N \alpha_j g_j(q)\right), \quad C''=-\dot{I}_{\mathbf{0}}(q)\cdot(f_1(q), \ldots, f_k(q)), \\ \mathrm{e}^{f/\alpha}=\mathrm{e}^{-\log C-\frac{C'}{C\alpha}}=\frac{1}{\dot{I}_{\mathbf{0}}(q)}\mathrm{e}^{\frac{1}{\hbar}\left[G(q)+\sum\limits_{j=1}^N \alpha_j g_j(q)\right]}, \quad g=-\frac{C''}{C}=(f_1(q), \ldots, f_k(q)). \end{gathered} \tag{A.0.4}$$

Finally, by [LLY3, Section 5.2] and [LLY3, Corollary 4.11], the hypothesis of [LLY3, Theorem 4.7] are satisfied with $A(t)$ and $B(t)$ as in (A.0.2) and (A.0.3) if $\nu_E(\mathbf{d}) \geqslant 0$ for all $\mathbf{d} \in \Lambda$, since $\mathbf{e}(E^+)$ and $\mathbf{e}(E^-)$ are non-zero whenever restricted to any $\mathbb{T}^N$-fixed point; see Proposition 2.3.3(a). Thus, (5.1.2) follows from [LLY3, Theorem 4.7], (A.0.2), (A.0.3), and (A.0.4).

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