MAT 562: Symplectic Geometry

Problem Set 1

Due by 09/11, in class

(if you have not passed the orals yet)

Two of the exercises from MS's Section 3.1 and/or the following. You do not need to copy the statements of problems (just indicate clearly what problems you are doing).

Problem A (counts as 2 exercises)

The $\mathbb{C}^*$-action on $\mathbb{C}^n$ by the coordinate multiplication restricts to a $\mathbb{C}^*$-action on $\mathbb{C}^n-\{0\}$ and S^1-actions on $\mathbb{C}^n$ and the unit sphere $S^{2n-1}\subset\mathbb{C}^n$. Show that

(a) the quotient topologies on $\mathbb{C}P^{n-1}$ given by $(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ and S^{2n-1}/S^1 are the same (i.e. the map $S^{2n-1}/S^1\longrightarrow(\mathbb{C}^n-\{0\})/\mathbb{C}^*$ induced by inclusions is a homeomorphism);

(b) $\mathbb{C}P^{n-1}$ is a compact topological $2(n-1)$-manifold that admits a complex structure so that the quotient projections

$$q\colon \mathbb{C}^n-\{0\}\longrightarrow \mathbb{C}P^{n-1}=(\mathbb{C}^n-\{0\})/\mathbb{C}^* \quad\text{and}\quad p\colon S^{2n-1}\longrightarrow \mathbb{C}P^{n-1}=S^{2n-1}/S^1$$

are a holomorphic submersion and a smooth submersion, respectively;

(c) the S^1-action on $\mathbb{C}^n$ preserves the standard symplectic form $\omega_{\mathbb{C}^n}$ on $\mathbb{C}^n$;

(d) the orbits of the restriction of this action to S^{2n-1} are compact connected one-dimensional submanifolds of S^{2n-1};

(e) for each $z\in S^{2n-1}$ the $\omega_{\mathbb{C}^n}$-symplectic complement of T_zS^{2n-1},

$$\big(T_zS^{2n-1}\big)^{\omega_{\mathbb{C}^n}} \equiv \big\{v\in T_z\mathbb{C}^n\colon \omega_{\mathbb{C}^n}(v,w)=0\ \forall\, w\in T_zS^{2n-1}\big\},$$

is the tangent space to the S^1-orbit at z;

(f) there is a unique 2-form $\omega_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ such that $p^*\omega_{\mathbb{C}P^{n-1}}=\omega_{\mathbb{C}^n}|_{TS^{2n-1}}$, and this form $\omega_{\mathbb{C}P^{n-1}}$ is symplectic.

Problem B (counts as 1 exercise)

Let Y be a smooth manifold, λ_{T^*Y} be the canonical 1-form on T^*Y, and $\omega_{T^*Y}\equiv-\mathrm{d}\lambda_{T^*Y}$ be the canonical symplectic form on T^*Y.

(a) Suppose $y\in Y$. Show that there is a canonical decomposition $T_y(T^*Y)=T_yY\oplus T_y^*Y$ and

$$\omega_{T^*Y}\big|_y(v,w)=\begin{cases}0, & \text{if } v,w\in T_yY \text{ or } v,w\in T_y^*Y;\\ w(v), & v\in T_yY \text{ and } w\in T_y^*Y.\end{cases}$$

Suppose instead that $\alpha\in\Omega^1(Y)$. Show that

(b) $\alpha^*\lambda_{T^*Y}=\alpha$;

(c) the map $\phi_\alpha: T^*Y\longrightarrow T^*Y$, $\phi_\alpha(\theta)=\alpha_{\pi(\theta)}-\theta$, is a smooth involution (i.e. $\phi_\alpha\circ\phi_\alpha=\mathrm{id}_{T^*M}$) satisfying $\phi_\alpha^*\lambda_{T^*Y}=\pi^*\alpha-\lambda_{T^*Y}$, where $\pi\colon T^*Y\longrightarrow Y$ is the bundle projection;

(d) the involution ϕ_α above is anti-symplectic (i.e. $\phi_\alpha^*\omega_{T^*Y}=-\omega_{T^*Y}$) if and only if $\mathrm{d}\alpha=0$.