MAT 545: Complex Geometry
Fall 2008

Notes on Connections

1 Connections in real vector bundles

1.1 Connections and splittings

Suppose M is a smooth manifold and $\pi\colon E \longrightarrow M$ is a vector bundle. Trivializations of M induce a bundle inclusion $\pi^*E \longrightarrow TE$ so that the sequence of vector bundles over E

$$0 \longrightarrow \pi^*E \longrightarrow TE \xrightarrow{d\pi} \pi^*TM \longrightarrow 0 \tag{1.1}$$

is exact. For each $f \in C^\infty(M)$, define

$$m_f\colon E \longrightarrow E \qquad \text{by} \qquad m_f(v) = f\big(\pi(x)\big)\cdot v \quad \forall\, v \in E. \tag{1.2}$$

We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi^*E \arrow[r] \arrow[d, "\pi^*m_f"] & TE \arrow[r, "d\pi"] \arrow[d, "dm_f"] & \pi^*TM \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \pi^*E \arrow[r] & m_f^*TE \arrow[r, "d\pi"] & \pi^*TM \arrow[r] & 0
\end{tikzcd} \tag{1.3}$$

of bundle maps over E.

A `connection in E` is an $\mathbb{R}$-linear map

$$\begin{aligned} \nabla\colon \Gamma(M;E) &\longrightarrow \Gamma(M;T^*M\otimes E) \qquad \text{s.t.} \\ \nabla(f\xi) = df\otimes\xi + f\nabla\xi \quad &\forall\, f\in C^\infty(M),\ \xi\in\Gamma(M;E). \end{aligned} \tag{1.4}$$

The Leibnitz property implies that any two connections in E differ by a one-form on M. In other words, if ∇ and $\tilde\nabla$ are connections in E there exists

$$\begin{aligned} \theta \in \Gamma\big(M;T^*M\otimes \mathrm{Hom}_{\mathbb{R}}(E,E)\big) \qquad &\text{s.t.} \\ \tilde\nabla_v\xi = \nabla_v\xi + \big\{\theta(v)\big\}\xi \qquad &\forall\, \xi\in\Gamma(M;E),\ v\in T_xM,\ x\in M. \end{aligned} \tag{1.5}$$

A connection ∇ in E is necessarily a local differential operator, i.e. the value of $\nabla\xi$ at a point $x\in M$ depends only on the restriction of ξ to any neighborhood $\mathcal{U}$ of x. If f is a smooth function on M supported in $\mathcal{U}$ and such that $f(x)=1$, then

$$\nabla\xi\big|_x = \nabla\big(f\xi\big)\big|_x - df\big|_x\otimes\xi(x) \tag{1.6}$$

by (1.4). The right-hand side of (1.6) depends only on $\xi|_{\mathcal{U}}$.

In fact, a connection ∇ in E is a first-order differential operator. Suppose $\mathcal{U}$ is an open subset of M and $\xi_1,\ldots,\xi_n \in \Gamma(\mathcal{U};E)$ is a frame for E on $\mathcal{U}$, i.e.

$$\xi_1(x),\ldots,\xi_n(x)\in E_x$$

is a basis for E_x for all $x\in\mathcal{U}$. By definition of ∇, there exist

$$\theta_l^k\in\Gamma(M;T^*M) \qquad \text{s.t.} \qquad \nabla\xi_l=\sum_{k=1}^{k=n}\xi_k\theta_l^k\equiv\sum_{k=1}^{k=n}\theta_l^k\otimes\xi_k \quad \forall\, l=1,\ldots,n.$$

We will call

$$\theta\equiv\big(\theta_l^k\big)_{k,l=1,\ldots,n}\in\Gamma\big(\Sigma;T^*M\otimes\mathrm{Mat}_n\mathbb{R}\big)$$

the connection one-form of ∇ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi=\sum_{l=1}^{l=n}f^l\xi_l\in\Gamma(\mathcal{U};E),$$

by (1.4) we have

$$\nabla\xi=\sum_{k=1}^{k=n}\xi_k\Big(df^k+\sum_{l=1}^{l=n}\theta_l^kf^l\Big), \qquad \text{i.e.} \qquad \nabla\big(\underline{\xi}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\big\{d+\theta\big\}\underline{f}^t, \tag{1.7}$$
$$\text{where} \qquad \underline{\xi}=(\xi_1,\ldots,\xi_n), \quad \underline{f}=(f^1,\ldots,f^n).$$

Thus, ∇ is a first-order differential operator. It is immediate from (1.4) that the symbol of ∇ is given by

$$\sigma_\nabla\colon T^*M\longrightarrow\mathrm{Hom}\big(E,T^*M\otimes E\big), \qquad \big\{\sigma_\nabla(\eta)\big\}(f)=\eta\otimes f.$$

Since $M\subset E$ as the zero section, there is a natural splitting

$$TE|_M\approx TM\oplus E \tag{1.8}$$

of the exact sequence (1.1) restricted to M. If $x\in M$ and $\xi\in\Gamma(M;E)$ is such that $\xi(x)=0$, then

$$\nabla\xi\big|_x=\pi_2|_x\circ d\xi|_x, \tag{1.9}$$

where $\pi_2|_x : T_xE\longrightarrow E_x$ is the projection onto the second component in (1.8). This observation follows from (1.5), as well as from (1.7).

Lemma 1.1 *Suppose M is a smooth manifold and $\pi\colon E\longrightarrow M$ is a vector bundle. A connection ∇ in E induces a splitting*

$$TE\approx\pi^*TM\oplus\pi^*E \tag{1.10}$$

of the exact sequence (1.1) extending the splitting (1.8) such that

$$\nabla\xi\big|_x=\pi_2|_x\circ d\xi|_x \qquad \forall\,\xi\in\Gamma(M;E),\ x\in M, \tag{1.11}$$

where $\pi_2|_x\colon T_xE\longrightarrow E_x$ is the projection onto the second component in (1.10), and

$$dm_t\approx\pi^*\mathrm{id}\oplus\pi^*m_t \qquad \forall\, t\in\mathbb{R}, \tag{1.12}$$

i.e. the splitting is consistent with the commutative diagram (1.3).

Proof: For each $x\in M$ and $v\in E_x$, choose $\xi\in\Gamma(M;E)$ such that $\xi(x)=v$ and let

$$T_vE^h = \mathrm{Im}\,\{d\xi-\nabla\xi\}\big|_x \subset T_vE.$$

Since $\pi\circ\xi=\mathrm{id}_M$,

$$d\pi|_v\circ\big\{d\xi-\nabla\xi\big\} = \mathrm{id}_{T_xM} \qquad\Longrightarrow\qquad T_vE\approx T_vE^h\oplus E_x\approx T_xM\oplus E_x.$$

If $v=0$, then by (1.9)

$$T_vE^h = T_vM.$$

If $v\neq 0$, $\zeta\in\Gamma(M;E)$ is another section such that $\zeta(x)=v$, and $\mathcal{U}$ is sufficiently small, then $\zeta=f\xi$ for some $f\in C^\infty(\mathcal{U})$ with $f(x)=1$ and thus

$$\begin{aligned}\{d\zeta-\nabla\zeta\}\big|_x &= \{d(f\xi)-\nabla(f\xi)\}\big|_x = \big\{df|_x\otimes\xi(x)+f(x)d\xi|_x\big\} - \big\{df|_x\otimes\xi(x)+f(x)\nabla\xi|_x\big\}\\ &= d\xi-\nabla\xi.\end{aligned}$$

The second equality above is obtained by considering a trivialization of E near x. Thus, T_vE^h is independent of the choice of ξ in either case and we obtain a well-defined splitting (1.10) of (1.1) that satisfies (1.11) and extends (1.8).

It remains to verify (1.12). Since $\pi\circ m_t=\pi$, $d\pi\circ dm_t=d\pi$, i.e. the first component of dm_t vanishes on TE and is the identity on π^*TM. On the other hand, if $\xi\in\Gamma(M;E)$ and $x\in M$, then

$$\begin{aligned}T_{t\xi(x)}E^h &\equiv \big\{d(m_t\circ\xi)-\nabla(t\xi)\big\}\big|_x = \big\{dm_t\circ d\xi-m_t\nabla\xi\big\}\big|_x = dm_t\circ\big\{d\xi-\nabla\xi\big\}\big|_x\\ &\equiv dm_t\big(T_{t\xi(x)}E^h\big).\end{aligned}$$

The last equality on the first line follows from (1.3). These two observations imply (1.12).

1.2 Metric-compatible connections

Suppose $E\longrightarrow M$ is a smooth vector bundle. Let g be a metric on E, i.e.

$$g\in\Gamma(M;E^*\otimes E^*) \qquad \text{s.t.} \qquad g(v,w)=g(w,v),\ \ g(v,v)>0 \quad \forall\, v,w\in E_x,\ v\neq 0,\ x\in M.$$

A connection ∇ in E is g-compatible if

$$d\big(g(\xi,\zeta)\big) = g(\nabla\xi,\zeta)+g(\xi,\nabla\zeta)\in\Gamma(M;T^*M) \qquad \forall\,\xi,\zeta\in\Gamma(M;E).$$

Suppose $\mathcal{U}$ is an open subset of M and $\xi_1,\ldots,\xi_n\in\Gamma(\mathcal{U};E)$ is a frame for E on $\mathcal{U}$. For $i,j=1,\ldots,n$, let

$$g_{ij} = g(\xi_i,\xi_j)\in C^\infty(\mathcal{U}).$$

If ∇ is a connection in E and θ_{kl} is the connection one-form for ∇ with respect to the frame $\{\xi_k\}_k$, then ∇ is g-compatible on $\mathcal{U}$ if and only if

$$\sum_{k=1}^{k=n}\big(g_{ik}\theta_j^k+g_{jk}\theta_i^k\big) = dg_{ij} \qquad \forall\, i,j=1,2,\ldots,n. \tag{1.13}$$

1.3 Torsion-free connections

If M is a smooth manifold, a connection ∇ in TM is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

If $(x_1,\ldots,x_n)\colon \mathcal{U} \longrightarrow \mathbb{R}^n$ is a coordinate chart on M, let

$$\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n} \in \Gamma(\mathcal{U};TM)$$

be the corresponding frame for TM on $\mathcal{U}$. If ∇ is a connection, the corresponding connection one-form θ can be written as

$$\theta_j^k = \sum_{i=1}^{i=n} \Gamma_{ij}^k dx^i, \qquad \text{where} \qquad \nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j} = \sum_{k=1}^{k=n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

The connection ∇ is torsion-free on $TM|_{\mathcal{U}}$ if and only if

$$\Gamma_{ij}^k = \Gamma_{ji}^k \qquad \forall\, i,j,k=1,\ldots,n. \tag{1.14}$$

Lemma 1.2 *If (M,g) is a Riemannian manifold, there exists a unique torsion-free g-compatible connection ∇ in TM.*

Proof: (1) Suppose ∇ and $\tilde{\nabla}$ are torsion-free g-compatible connections in TM. By (1.5), there exists

$$\begin{gathered}\theta \in \Gamma\big(M;T^*M\otimes \mathrm{Hom}_{\mathbb{R}}(TM,TM)\big) \qquad \text{s.t.}\\ \tilde{\nabla}_X Y - \nabla_X Y = \big\{\theta(X)\big\}Y \qquad \forall\, Y\in\Gamma(M;TM),\ X\in T_xM,\ x\in M.\end{gathered}$$

Since ∇ and $\tilde{\nabla}$ are torsion-free,

$$\big\{\theta(X)\big\}Y = \big\{\theta(Y)\big\}X \qquad \forall\, X,Y\in T_xM,\ x\in M. \tag{1.15}$$

Since ∇ and $\tilde{\nabla}$ are g-compatible,

$$\begin{cases} g\big(\{\theta(X)\}Y,Z\big)+g\big(Y,\{\theta(X)\}Z\big)=0\\ g\big(\{\theta(Y)\}X,Z\big)+g\big(X,\{\theta(Y)\}Z\big)=0\\ g\big(\{\theta(Z)\}X,Y\big)+g\big(X,\{\theta(Z)\}Y\big)=0\end{cases} \qquad \forall\, X,Y,Z\in T_xM,\ x\in M. \tag{1.16}$$

Adding the first two equations in (1.16), subtracting the third, and using (1.15) and the symmetry of g, we obtain

$$2g\big(\{\theta(X)\}Y,Z\big)=0 \quad \forall\, X,Y,Z\in T_xM,\ x\in M \qquad \Longrightarrow \qquad \theta\equiv 0.$$

Thus, $\tilde{\nabla} = \nabla$.

(2) Let $(x_1, \ldots, x_n) : \mathcal{U} \longrightarrow \mathbb{R}^n$ be a coordinate chart on M. With notation as in the paragraph preceding Lemma 1.2, ∇ is g-compatible on $TM|_{\mathcal{U}}$ if and only if

$$\sum_{l=1}^{l=n} \big(g_{il}\Gamma_{kj}^l + g_{jl}\Gamma_{ki}^l\big) = \partial_{x_k} g_{ij}; \tag{1.17}$$

see (1.13). Define a connection ∇ in $TM|_{\mathcal{U}}$ by

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{l=n} g^{kl}\big(\partial_{x_i} g_{jl} + \partial_{x_j} g_{il} - \partial_{x_l} g_{ij}\big) \qquad \forall\, i, j, k = 1, \ldots, n,$$

where g^{ij} is the (i,j)-entry of the inverse of the matrix $(g_{ij})_{i,j=1,\ldots,n}$. By direct computation, Γ_{ij}^k satisfies (1.14) and (1.17). Therefore, ∇ is a torsion-free g-compatible connection on $TM|_{\mathcal{U}}$. In this way, we can define a torsion-free g-compatible connection on every coordinate chart. By the uniqueness property, these connections agree on the overlaps.

2 Complex structures

2.1 Complex linear connections

Suppose M is a smooth manifold and $\pi : (E, \mathfrak{i}) \longrightarrow M$ is a complex vector bundle. Similarly to Subsection 1.1, there is an exact sequence of vector bundles over E

$$0 \longrightarrow \pi^*E \longrightarrow TE \xrightarrow{d\pi} \pi^*TM \longrightarrow 0 \tag{2.1}$$

is exact. If $f \in C^\infty(M;\mathbb{C})$ and $m_f : E \longrightarrow E$ is defined as in (1.2), we then have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*E & \longrightarrow & TE & \xrightarrow{d\pi} & \pi^*TM & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \pi^* m_f} & & \big\downarrow{\scriptstyle dm_f} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \pi^*E & \longrightarrow & m_f^*TE & \xrightarrow{d\pi} & \pi^*TM & \longrightarrow & 0 \end{array} \tag{2.2}$$

of bundle maps over E.

Suppose

$$\nabla : \Gamma(M;E) \longrightarrow \Gamma(M; T^*M \otimes_{\mathbb{R}} E)$$

is a $\mathbb{C}$-linear connection (with respect to the complex structure in E on both sides). If $\mathcal{U}$ is an open subset of M and $\xi_1, \ldots, \xi_n \in \Gamma(\mathcal{U}; E)$ is a $\mathbb{C}$-frame for E on $\mathcal{U}$, then there exist

$$\theta_l^k \in \Gamma(M; T^*M) \qquad \text{s.t.} \qquad \nabla \xi_l = \sum_{k=1}^{k=n} \xi_k \theta_l^k \equiv \sum_{k=1}^{k=n} \theta_l^k \otimes \xi_k \quad \forall\, l = 1, \ldots, n.$$

We will call

$$\theta \equiv \big(\theta_l^k\big)_{k,l=1,\ldots,n} \in \Gamma\big(\Sigma; T^*M \otimes_{\mathbb{R}} \mathrm{Mat}_n\mathbb{C}\big)$$

the complex connection one-form of ∇ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi = \sum_{l=1}^{l=n} f^l \xi_l \in \Gamma(\mathcal{U}; E),$$

by (1.4) and $\mathbb{C}$-linearity of ∇ we have

$$\nabla \xi = \sum_{k=1}^{k=n} \xi_k \Big(df^k + \sum_{l=1}^{l=n} \theta_l^k f^l \Big), \qquad \text{i.e.} \qquad \nabla\big(\underline{\xi} \cdot \underline{f}^t\big) = \underline{\xi} \cdot \big\{d + \theta\big\} \underline{f}^t, \tag{2.3}$$

$$\text{where} \qquad \underline{\xi} = (\xi_1, \ldots, \xi_n), \quad \underline{f} = (f^1, \ldots, f^n). \tag{2.4}$$

Let h be a hermitian metric on E, i.e.

$$h \in \Gamma\big(M; \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{C}} \bar{E}, \mathbb{C})\big) \quad \text{s.t.} \quad h(v,w) = \overline{g(w,v)}, \quad h(v,v) > 0 \quad \forall\, v,w \in E_x,\ v \neq 0,\ x \in M.$$

A $\mathbb{C}$-linear connection ∇ in E is h-compatible if

$$d\big(h(\xi,\zeta)\big) = h(\nabla\xi,\zeta) + h(\xi,\nabla\zeta) \in \Gamma(M; T^*M \otimes_{\mathbb{R}} \mathbb{C}) \qquad \forall\, \xi,\zeta \in \Gamma(M;E).$$

With notation as in the previous paragraph, let

$$h_{ij} = h(\xi_i, \xi_j) \in C^\infty(\mathcal{U}; \mathbb{R}) \qquad \forall\, i,j = 1, \ldots, n.$$

Then ∇ is h-compatible on $\mathcal{U}$ if and only if

$$\sum_{k=1}^{k=n} \big(h_{ik} \bar{\theta}_j^k + \bar{h}_{jk} \theta_i^k \big) = dh_{ij} \qquad \forall\, i,j = 1,2,\ldots,n. \tag{2.5}$$

2.2 Generalized $\bar{\partial}$-operators

If $(M, \mathfrak{j})$ is an almost complex manifold, let

$$T^*M^{1,0} \equiv \big\{\eta \in T^*M \otimes_{\mathbb{R}} \mathbb{C} \colon \eta \circ \mathfrak{j} = \mathfrak{i}\,\eta\big\},$$
$$T^*M^{0,1} \equiv \big\{\eta \in T^*M \otimes_{\mathbb{R}} \mathbb{C} \colon \eta \circ \mathfrak{j} = -\mathfrak{i}\,\eta\big\}$$

be the bundle of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear 1-forms on M. If $(M, \mathfrak{j})$ and (E, J) are smooth almost complex manifolds and $u \colon M \longrightarrow E$ is a smooth function, define

$$\bar{\partial}_{J,\mathfrak{j}} u \in \Gamma\big(M; T^*M^{0,1} \otimes_{\mathbb{C}} u^*TE\big) \qquad \text{by} \qquad \bar{\partial}_{J,\mathfrak{j}} u = \frac{1}{2}\big(du + J \circ du \circ \mathfrak{j}\big).$$

A smooth map $u \colon (M, \mathfrak{j}) \longrightarrow (E, J)$ will be called $(J, \mathfrak{j})$-`holomorphic` if $\bar{\partial}_{J,\mathfrak{j}} u = 0$.

Definition 2.1 *Suppose $(M, \mathfrak{j})$ is an almost complex manifold and $\pi \colon (E, \mathfrak{i}) \longrightarrow M$ is a complex vector bundle. A $\bar{\partial}$-`operator on` $(E, \mathfrak{i})$ is a $\mathbb{C}$-linear map*

$$\bar{\partial} \colon \Gamma(M; E) \longrightarrow \Gamma(M; T^*M^{0,1} \otimes_{\mathbb{C}} E)$$

such that

$$\bar{\partial}\big(f\xi\big) = (\bar{\partial} f) \otimes \xi + f(\bar{\partial}\xi) \qquad \forall\, f \in C^\infty(M),\ \xi \in \Gamma(M;E), \tag{2.6}$$

where $\bar{\partial} f = \bar{\partial}_{\mathfrak{i},\mathfrak{j}} f$ is the usual $\bar{\partial}$-operator on complex-valued functions.

Similarly to Subsection 1.1, a $\bar{\partial}$-operator on $(E,\mathfrak{i})$ is necessarily a first-order differential operator. If $\mathcal{U}$ is an open subset of M and $\xi_1,\ldots,\xi_n\in\Gamma(\mathcal{U};E)$ is a $\mathbb{C}$-frame for E on $\mathcal{U}$, then there exist

$$\theta_l^k\in\Gamma(\mathcal{U};T^*M^{0,1})\qquad\text{s.t.}\qquad\bar{\partial}\xi_l=\sum_{k=1}^{k=n}\xi_k\theta_l^k\equiv\sum_{k=1}^{k=n}\theta_l^k\otimes\xi_k\quad\forall\,l=1,\ldots,n.$$

We will call

$$\theta\equiv\big(\theta_l^k\big)_{k,l=1,\ldots,n}\in\Gamma\big(\mathcal{U};T^*M^{0,1}\otimes_{\mathbb{C}}\mathrm{Mat}_n\mathbb{C}\big)$$

the connection one-form of $\bar{\partial}$ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi=\sum_{l=1}^{l=n}f^l\xi_l\in\Gamma(\mathcal{U};E),$$

by (2.6) we have

$$\bar{\partial}\xi=\sum_{k=1}^{k=n}\xi_k\Big(\bar{\partial}f^k+\sum_{l=1}^{l=n}\theta_l^kf^l\Big),\qquad\text{i.e.}\qquad\bar{\partial}\big(\underline{\xi}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\big\{\bar{\partial}+\theta\big\}\underline{f}^t,\tag{2.7}$$

where $\underline{\xi}$ and $\underline{f}$ are as in (2.4). It is immediate from (2.6) that the symbol of $\bar{\partial}$ is given by

$$\sigma_{\bar{\partial}}\colon T^*M\longrightarrow\mathrm{Hom}\big(E,T^*M^{0,1}\otimes_{\mathbb{C}}E\big),\qquad\big\{\sigma_{\bar{\partial}}(\eta)\big\}(f)=\frac{1}{2}\big(\eta+\mathfrak{i}\,\eta\circ\mathfrak{j}\big)\otimes f=\eta^{0,1}\otimes f.$$

In particular, $\bar{\partial}$ is an elliptic operator (i.e. $\sigma_{\bar{\partial}}(\eta)$ is an isomorphism for $\eta\neq0$) if $(M,\mathfrak{j})$ is a Riemann surface.

Lemma 2.2 *Suppose $(M,\mathfrak{j})$ is an almost complex manifold and $\pi\colon(E,\mathfrak{i})\longrightarrow M$ is a complex vector bundle. If*

$$\bar{\partial}\colon\Gamma(M;E)\longrightarrow\Gamma(M;T^*M^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar{\partial}$-operator on $(E,\mathfrak{i})$, there exists a unique almost complex structure $J=J_{\bar{\partial}}$ on (the total space of) E such that π is a $(\mathfrak{j},J)$-holomorphic map, the restriction of J to the vertical tangent bundle $TE^v\approx\pi^*E$ agrees with $\mathfrak{i}$, and

$$\bar{\partial}_{J,\mathfrak{j}}\xi=0\in\Gamma(\mathcal{U};T^*M^{0,1}\otimes_{\mathbb{C}}\xi^*TE)\qquad\Longleftrightarrow\qquad\bar{\partial}\xi=0\in\Gamma(\mathcal{U};T^*M^{0,1}\otimes_{\mathbb{C}}E)\tag{2.8}$$

for every open subset $\mathcal{U}$ of Σ and $\xi\in\Gamma(\mathcal{U};E)$.

Proof: (1) With notation as above, define

$$\varphi\colon\mathcal{U}\times\mathbb{C}^n\longrightarrow E|_{\mathcal{U}}\qquad\text{by}\qquad\varphi(x,c^1,\ldots,c^n)=\underline{\xi}(x)\cdot\underline{c}^t\equiv\sum_{k=1}^{k=n}c^k\xi_k(x)\in E_x.$$

The map φ is a trivialization of E over $\mathcal{U}$. If $J_{\bar{\partial}}$ is an almost complex structure on E with the desired properties, let $\tilde{J}$ be the almost complex structure on $\mathcal{U}\times\mathbb{C}^n$ given by

$$\tilde{J}|_{(x,\underline{c})}=\big\{d\varphi|_{(x,\underline{c})}\big\}^{-1}\circ J_{\bar{\partial}}\big|_{\varphi(x,\underline{c})}\circ d\varphi|_{(x,\underline{c})}\qquad\forall\,(x,\underline{c})\in\mathcal{U}\times\mathbb{C}^n.\tag{2.9}$$

Since $J_{\bar\partial}$ restricts to $\mathfrak{i}$ on TE^v,

$$\tilde J|_{(x,\underline{c})}w = \mathfrak{i}w \in T_{\underline{c}}\mathbb{C}^n \subset T_{(x,\underline{c})}(\mathcal{U}\times\mathbb{C}^n) \qquad \forall\ w\in T_{\underline{c}}\mathbb{C}^n. \tag{2.10}$$

Since the projection map π is $(\mathfrak{j},J_{\bar\partial})$-holomorphic, there exists

$$\begin{aligned} &\tilde J_{2,1}\in\Gamma\big(\mathcal{U};\mathrm{Hom}(\pi_{\mathcal{U}}^*T\mathcal{U},\pi_{\mathbb{C}^n}^*T\mathbb{C}^n)\big) \qquad \text{s.t.}\\ &\tilde J|_{(x,\underline{c})}w = \mathfrak{j}w+\tilde J_{2,1}w \qquad \forall\ w\in T_x\mathcal{U}\subset T_{(x,\underline{c})}(\mathcal{U}\times\mathbb{C}^n). \end{aligned} \tag{2.11}$$

If $\xi\in\Gamma(\mathcal{U};E)$, let

$$\tilde\xi \equiv \varphi^{-1}\circ\xi \equiv \big(\mathrm{id}_{\mathcal{U}},\underline{f}\big), \qquad \text{where} \qquad \underline{f}\in C^\infty(\mathcal{U};\mathbb{C}^n).$$

By (2.9)-(2.11),

$$\begin{aligned} 2\,\bar\partial_{J,\mathfrak{j}}\xi\big|_x = d\varphi\big|_{\tilde\xi(x)}\circ 2\bar\partial_{\tilde J,\mathfrak{j}}\tilde\xi\big|_x &= d\varphi\big|_{\tilde\xi(x)}\circ\big\{\big(\mathrm{Id}_{T_x\mathcal{U}},d\underline{f}|_x\big)+\tilde J|_{\tilde\xi(x)}\circ\big(\mathrm{Id}_{T_x\mathcal{U}},d\underline{f}|_x\big)\circ\mathfrak{j}|_x\big\}\\ &= d\varphi\big|_{\tilde\xi(x)}\circ\big(0,2\,\bar\partial f|_x+\tilde J_{2,1}|_{\tilde\xi(x)}\circ\mathfrak{j}|_x\big). \end{aligned} \tag{2.12}$$

On the other hand, by (2.7),

$$\begin{aligned} \bar\partial\xi|_x = \bar\partial\big(\underline{\xi}\cdot f^t\big)\big|_x &= \underline{\xi}(x)\cdot\big\{\bar\partial+\theta\big\}f^t\big|_x\\ &= \varphi\big(\bar\partial f|_x+\theta_x\cdot f(x)^t\big). \end{aligned} \tag{2.13}$$

By (2.12) and (2.13), the property (2.8) is satisfied for all $\xi\in\Gamma(\mathcal{U};E)$ if and only if

$$\tilde J_{2,1}|_{(x,\underline{c})} = 2\big(\theta_x\cdot\underline{c}^t\big)\circ(-\mathfrak{j}|_x) = 2\mathfrak{i}\,\theta_x\cdot\underline{c}^t \qquad \forall\ (x,\underline{c})\in\mathcal{U}\times\mathbb{C}^n.$$

In summary, the almost complex structure $J=J_{\bar\partial}$ on E has the three desired properties if and only if for any trivialization of E over an open subset $\mathcal{U}$ of Σ

$$\begin{aligned} &\tilde J\big|_{(x,\underline{c})}\big(w_1,w_2\big) = \big(\mathfrak{j}w_1,\mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big)\\ &\forall\ (x,\underline{c})\in\mathcal{U}\times\mathbb{C}^n,\ (w_1,w_2)\in T_x\mathcal{U}\oplus T_{\underline{c}}\mathbb{C}^n = T_{(x,\underline{c})}(\mathcal{U}\times\mathbb{C}^n), \end{aligned} \tag{2.14}$$

where $\tilde J$ is the almost complex structure on $\mathcal{U}\times\mathbb{C}^n$ induced by J via the trivialization and θ is the connection-one form corresponding to $\bar\partial$ with respect to the frame inducing the trivialization.

(2) By (2.14), there exists at most one almost complex structure J satisfying the three properties. Conversely, (2.14) determines such an almost complex structure on E. Since

$$\begin{aligned} \tilde J\big|^2_{(x,\underline{c})}\big(w_1,w_2\big) = \tilde J\big|_{(x,\underline{c})}\big(\mathfrak{j}w_1,\mathfrak{i}w_2+2\theta_x(w_1)\cdot\underline{c}^t\big) &= \big(\mathfrak{j}^2w_1,\mathfrak{i}\big(\mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big)+2\mathfrak{i}\theta_x(\mathfrak{j}w_1)\cdot\underline{c}^t\big)\\ &= -(w_1,w_2), \end{aligned}$$

$\tilde J$ is indeed an almost complex structure for $\bar\partial$-operator on $(E,\mathfrak{i})$. The almost complex structure induced by $\tilde J$ on $E|_{\mathcal{U}}$ must satisfy the three properties by part (a). By the uniqueness property, the almost complex structures on E induced by the different trivializations must agree on the overlaps. Therefore, they define an almost complex structure $J=J_{\bar\partial}$ on the total space of E with the desired properties.

2.3 Connections and $\bar\partial$-operators

Suppose $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle, and

$$\bar\partial\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^{0,1}\Sigma\otimes E)$$

is a $\bar\partial$-operator on $(E,\mathfrak{i})$. A $\mathbb{C}$-linear connection ∇ in $(E,\mathfrak{i})$ is $\bar\partial$-compatible if

$$\bar\partial\xi=\bar\partial_\nabla\xi\equiv\frac{1}{2}\big(\nabla\xi+\mathfrak{i}\nabla\xi\circ\mathfrak{j}\big)\qquad\forall\,\xi\in\Gamma(M;\Sigma).\tag{2.15}$$

Lemma 2.3 *Suppose $(M,\mathfrak{j})$ is an almost complex manifold, $\pi:(E,\mathfrak{i})\longrightarrow M$ is a complex vector bundle,*

$$\bar\partial\colon\Gamma(M;E)\longrightarrow\Gamma(M;T^*M^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar\partial$-operator on $(E,\mathfrak{i})$, and $J_{\bar\partial}$ is the complex structure in the vector bundle $TE\longrightarrow E$ provided by Lemma 2.2. A $\mathbb{C}$-linear connection ∇ in $(E,\mathfrak{i})$ is $\bar\partial$-compatible if and only if the splitting (1.10) determined by ∇ respects the complex structures.

Proof: Since $J_{\bar\partial}=\pi^*\mathfrak{i}$ on $\pi^*E\subset TE$ by definition $J_{\bar\partial}$, by the construction of the splitting (1.10) it is sufficient to check that

$$J_{\bar\partial}|_v\circ\big\{d\xi-\nabla\xi\big\}\big|_x=\big\{d\xi-\nabla\xi\big\}\big|_x\circ\mathfrak{j}_x\colon T_xM\longrightarrow T_vE$$

for all $x\in M$, $v\in E_x$, and $\xi\in\Gamma(M;E)$ such that $\xi(x)=v$. This identity is equivalent to

$$\bar\partial_{J_{\bar\partial},\mathfrak{j}}\xi=\bar\partial_\nabla\xi\qquad\forall\,\xi\in\Gamma(M;E).\tag{2.16}$$

On the other hand, by the proof of Lemma 2.2,

$$\bar\partial_{J_{\bar\partial},\mathfrak{j}}\xi=\bar\partial\xi\qquad\forall\,\xi\in\Gamma(M;E);\tag{2.17}$$

see (2.12)-(2.14). The lemma follows immediately from (2.16) and (2.17).

2.4 Holomorphic vector bundles

Let $(\Sigma,\mathfrak{j})$ be a complex manifold. A holomorphic vector bundle $(E,\mathfrak{i})$ on $(\Sigma,\mathfrak{j})$ is a complex vector bundle with a collection of trivializations that overlap holomorphically.

A collection of holomorphically overlapping trivializations of (E,i) determines a holomorphic structure J on the total space of E and a $\bar\partial$-operator

$$\bar\partial\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^{0,1}\Sigma\otimes E).$$

The latter is defined as follows. If $\xi_1,\ldots,\xi_n$ is a holomorphic complex frame for E over an open subset $\mathcal{U}$ of M, then

$$\bar\partial\sum_{k=1}^{k=n}f^k\xi_k=\sum_{k=1}^{k=n}\bar\partial f^k\otimes\xi_k\qquad\forall\,f^1,\ldots,f^k\in C^\infty(\mathcal{U};\mathbb{C}).$$

In particular, for all $\xi\in\Gamma(M;E)$

$$\bar\partial_{J,\mathfrak{j}}\xi=0\qquad\Longleftrightarrow\qquad\bar\partial\xi=0.$$

Thus, $J=J_{\bar\partial}$; see Lemma 2.2.

Lemma 2.4 *Suppose* $(\Sigma,\mathfrak{j})$ *is a Riemann surface and* $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ *is a complex vector bundle. If*

$$\bar{\partial}\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^{0,1}\Sigma\otimes E)$$

is a $\bar{\partial}$*-operator on* $(E,\mathfrak{i})$*, the almost complex structure* $J=J_{\bar{\partial}}$ *on* E *is integrable. With this complex structure,* $\pi\colon E\longrightarrow\Sigma$ *is a holomorphic vector bundle and* $\bar{\partial}$ *is the corresponding* $\bar{\partial}$*-operator.*

Proof: By (2.8), it is sufficient to show that there exists a $(J,\mathfrak{j})$-holomorphic local section through every point $v\in E$, i.e. there exist a neighborhood $\mathcal{U}$ of $x\equiv\pi(v)$ in Σ and $\xi\in\Gamma(\mathcal{U};E)$ such that

$$\xi(x)=v \qquad \text{and} \qquad \bar{\partial}_{J,\mathfrak{j}}\xi=0.$$

By Lemma 2.2 and (2.13), this is equivalent to showing that the equation

$$\Big\{\bar{\partial}+\theta\Big\}f^t=0, \qquad f(x)=v, \qquad f\in C^\infty(\mathcal{U};\mathbb{C}^n), \tag{2.18}$$

has a solution for every $v\in\mathbb{C}^n$. We can assume that $\mathcal{U}$ is a small disk contained in S^2. Let

$$\eta\colon S^2\longrightarrow[0,1]$$

be a smooth function supported in $\mathcal{U}$ and such that $\eta\equiv1$ on a neighborhood of x. Then,

$$\eta\theta\in\Gamma(S^2;T^{0,1}S^2\otimes\mathrm{Mat}_n\mathbb{C}).$$

Choose $p>2$. The operator

$$\Theta: L^p_1(S^2;\mathbb{C}^n)\longrightarrow L^p\big(S^2;T^{0,1}S^2\otimes\mathbb{C}^n\big)\oplus\mathbb{C}^n, \qquad \Theta(f)=\big(\bar{\partial}_{\mathfrak{i},\mathfrak{j}}f,f(x)\big),$$

is surjective. If η has sufficiently small support, so is

$$\Theta_\eta: L^p_1(S^2;\mathbb{C}^n)\longrightarrow L^p\big(S^2;T^{0,1}S^2\otimes\mathbb{C}^n\big)\oplus\mathbb{C}^n, \qquad \Theta_\eta(f)=\big(\{\bar{\partial}_{\mathfrak{i},\mathfrak{j}}+\eta\theta\}f,f(x)\big).$$

Then, the restriction of $\Theta_\eta^{-1}(0,v)$ to a neighborhood of x on which $\eta\equiv1$ is a solution of (2.18). By elliptic regularity, $\Theta_\eta^{-1}(0,v)\in C^\infty(S^2;\mathbb{C}^n)$.