MAT 545: Complex Geometry

Problem Set 6

Written Solutions due by Tuesday, 11/26, 1pm

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *please write up concise solutions to problems worth 10 points.*

Problem 1 (10 pts)

(a) For each $z \in \mathbb{C}^n$, let $\mathcal{O}_z$ be the ring of germs at z of holomorphic functions on $\mathbb{C}^n$. If $f, g\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ are holomorphic functions and $p \in \mathbb{C}^n$ are such that $f(p)=0$, let

$$\mathrm{ord}_{f^{-1}(0),p}\, g = \max\big\{a \in \mathbb{Z}\colon \; [g]/[f^a] \in \mathcal{O}_p\big\}.$$

Show that for any $p \in f^{-1}(0)$ such that $[f] \in \mathcal{O}_p$ is irreducible, there exists a neighborhood $U_p(f,g)$ of p in $\mathbb{C}^n$ with the property such that

$$\mathrm{ord}_{f^{-1}(0),z}\, g = \mathrm{ord}_{f^{-1}(0),p}\, g \qquad \forall\, z \in f^{-1}(0) \cap U_p(f,g).$$

(b) Let M be a complex manifold and $V \subset M$ be an irreducible analytic hypersurface; thus, $V^* \subset M$ is connected. Suppose $s \equiv \big\{s_\alpha^+, s_\alpha^- \in \mathcal{O}(U_\alpha)\big\}_{\alpha \in \mathcal{A}}$ is a meromorphic section of a holomorphic line bundle $L \longrightarrow M$. Show that the number

$$\mathrm{ord}_{V,p}\, s \equiv \mathrm{ord}_{V,p}\, s_\alpha^+ - \mathrm{ord}_{V,p}\, s_\alpha^-$$

is independent of the choice of $\alpha \in \mathcal{A}$ and $p \in V^* \cap U_\alpha$.

Problem 2 (5 pts)

Let Σ be a compact connected Riemann surface (complex one-dimensional manifold) and $p, q \in \Sigma$ be any two distinct points. Show that
(a) if $\Sigma = \mathbb{P}^1$, then $[p]=[q]$.
(b) if $[p]=[q]$, then $\Sigma = \mathbb{P}^1$ (up to bi-holomorphism).

Problem 3 (10 pts)

Let Σ be a compact connected Riemann surface and $V \longrightarrow \Sigma$ be a holomorphic line bundle.
(a) Give a necessary and sufficient condition on V so that there exists a holomorphic line bundle $L \longrightarrow \Sigma$ such that $L^{\otimes 2} = V$.
(b) If this condition holds, how many "square roots" L does V have?
(c) If M is a complex surface ($\dim_{\mathbb{C}} M = 2$) and $\Sigma \subset M$ is a smooth canonical divisor with normal bundle N, show that $N^{\otimes 2} = \mathcal{K}_\Sigma$.
Note: a pair (Σ, L) such that $L^{\otimes 2} = \mathcal{K}_\Sigma$ is called a *spin curve*.

Problem 4 (5 pts)

Let M be a complex manifold. Show that every C^∞ complex line bundle $L\longrightarrow M$ admits
(a) at most one holomorphic structure if and only if $H^{0,1}_{\bar\partial}(M)=0$.
(b) at least one holomorphic structure if and only if $H^{0,2}_{\bar\partial}(M)=0$.

Problem 5 (10 pts)

Show that
(a) there exists a short exact sequence of sheaves on $\mathbb{P}^n$:

$$0\longrightarrow \mathcal{O}_{\mathbb{P}^n}\longrightarrow (n+1)\mathcal{O}_{\mathbb{P}^n}(1)\longrightarrow \mathcal{O}(T\mathbb{P}^n)\longrightarrow 0\,;$$

(b) $H^q_{\bar\partial}(\mathbb{P}^n;\mathcal{O}(T\mathbb{P}^n))=0$ for all $q>0$;
(c) $H^q_{\bar\partial}(\Sigma;\mathcal{O}(u^*T\mathbb{P}^n))=0$ for all $q>0$ for every compact connected Riemann surface Σ of genus g and holomorphic map $u\colon\Sigma\longrightarrow\mathbb{P}^n$ of degree $d\ge 2g-1$;
(d) the homomorphism

$$H^0_{\bar\partial}(\Sigma;\mathcal{O}(u^*T\mathbb{P}^n))\longrightarrow \bigoplus_{i=1}^{i=k}a_iT_{u(z_i)}\mathbb{P}^n,\qquad s\longrightarrow \big(\nabla^{j-1}_{e_i}s\big)_{1\le j\le a_i,1\le i\le k}\,,$$

is surjective for every (Σ,u) as in (c), every choice of distinct points $z_1,\ldots,z_k\in\Sigma$, $e_i\in T_{z_i}\Sigma$ with $e_i\neq 0$, and $a_1,\ldots,a_k\in\mathbb{Z}^+$ with $d\ge 2g-1+\sum_{i=1}^{i=k}a_i$.
Note: $\nabla^j_{e_i}s$ denotes the j-th vertical derivative of the section s with respect to a connection ∇ evaluated at e_j; thus,

$$\nabla^0_{e_j}(s)=s(z_j),\qquad \nabla^1_{e_j}(s)=\{\nabla s\}|_{z_j}(e_j),\qquad \nabla^2_{e_j}(s)=\{\nabla(\nabla s)\}|_{z_j}(e_j,e_j).$$

Problem 6 (10 pts)

Let $\gamma\longrightarrow\mathbb{P}^1$ be the tautological line bundle, $\mathcal{O}_{\mathbb{P}^1}(-1)$, and $E\longrightarrow\mathbb{P}^1$ be any holomorphic vector bundle. Show that
(a) $H^1_{\bar\partial}(\mathbb{P}^1;\gamma)=0$;
(b) $H^0_{\bar\partial}(\mathbb{P}^1;E\otimes\gamma^{*a})\neq 0$ for for some $a\in\mathbb{Z}$;
(c) if $a_E=\min\{a\in\mathbb{Z}:\ H^0_{\bar\partial}(\mathbb{P}^1;E\otimes\gamma^{*a})\neq 0\}$, then $E\otimes\gamma^{*a_E}$ admits a nowhere zero holomorphic section and E contains a holomorphic subbundle isomorphic to γ^{a_E};
(d) if $F\equiv E/\gamma^{a_E}\approx\gamma^{a_1}\oplus\ldots\oplus\gamma^{a_k}$, then $a_E\le a_i$ for all $i=1,\ldots,k$;
(e) $H^1_{\bar\partial}(\mathbb{P}^1;F^*\otimes\gamma^{a_E})=0$ and the exact sequences of holomorphic vector bundles

$$0\longrightarrow F^*\otimes\gamma^{a_E}\longrightarrow E^*\otimes\gamma^{a_E}\longrightarrow\tau_1\longrightarrow 0,\qquad 0\longrightarrow\gamma^{a_E}\longrightarrow E\longrightarrow F\longrightarrow 0$$

split;
(f) E is isomorphic to a unique vector bundle

$$\bigoplus_{i=0}^{i=k}\mathcal{O}_{\mathbb{P}^1}(b_i)\equiv\bigoplus_{i=0}^{i=k}\gamma^{*b_i}$$

with $b_0\ge b_1\ge\ldots\ge b_k$.
Note: the last statement is *Grothendieck's theorem.*