MAT 545: Complex Geometry

Problem Set 4

Written Solutions due by Thursday, 10/31, 1pm

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *please write up concise solutions to problems worth 10 points.*

Problem 1 (5 pts)

Let (V,J) be a complex vector space of complex dimension m and g a Riemannian metric on V compatible with J. Show that

$$*\colon \Lambda^{p,q}V \longrightarrow \Lambda^{m-p,m-q}V \subset \Lambda^{2m-p-q}_{\mathbb{C}}(V\otimes_{\mathbb{R}}\mathbb{C}),$$

where $*$ is the Hodge operator on (V,g) with the orientation induced by J and extended $\mathbb{C}$-anti-linearly to $V\otimes_{\mathbb{R}}\mathbb{C}$.

Problem 2 (5 pts)

Let M and N be complex manifolds with hermitian metrics h_M and h_N. Show that

$$\Delta_{M\times N} = \Delta_M \otimes 1 + 1 \otimes \Delta_N\,,$$

where Δ_M, Δ_N, and $\Delta_{M\times N}$ are the Laplacians on M, N, and $M\times N$ with respect to the metrics h_M, h_N, and $h_M\otimes 1+1\otimes h_N$.

Problem 3 (5 pts)

Let $E\longrightarrow M$ be a holomorphic vector bundle with a hermitian inner-product over a compact complex manifold with a hermitian metric and Δ_E the corresponding Laplacian on E. Show that
(a) all eigenvalues of Δ_E are non-negative;
(b) eigenfunctions corresponding to distinct eigenvalues of Δ_E are orthogonal;
(c) eigenspaces of Δ_E are finite-dimensional;
(d) the set of eigenvalues of Δ_E has no limit point.

Problem 4 (10 pts)

With notation as in Problem 3, show that
(a) Δ_E has a positive eigenvalue;
(b) Δ_E has infinitely many positive eigenvalues;
(c) the linear span of eigenfunctions of Δ_E is L^2-dense in $\Gamma(M;\Lambda^*_{\mathbb{C}}(T^*M\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}E)$;
(d) the linear span of eigenfunctions of Δ_E is L^∞-dense in $\Gamma(M;\Lambda^*_{\mathbb{C}}(T^*M\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}E)$.
This problems supplies the missing ingredient for the proof of Kunneth formula in the textbook: the existence of a complete set of eigenfunctions of a laplacian.
Hint for (a): Let G be the inverse of Δ_E on $\mathcal{H}^{*,*}(M,E)^\perp$. Show that $\lambda_1\in\mathbb{R}^+$ given by

$$1/\lambda_1 = \sup_{\alpha\in\mathcal{H}^{*,*}(M,E)^\perp,\|\alpha\|=1} \|G\alpha\|$$

is the smallest positive eigenvalue of Δ_E.