MAT 545: Complex Geometry

Problem Set 2

Written Solutions due by Tuesday, 9/24, 1pm

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *please write up concise solutions to problems worth 10 points.*

Problem 1 (10 pts)

Let J be an almost complex structure on a smooth manifold M. For vector fields X and Y on M, let

$$N_J(X,Y) = \frac{1}{2}\Big([X,Y] + J[X,JY] + J[JX,Y] - [JX,JY]\Big).$$

(a) Show that N_J is in fact a tensor on M (called *Nijenhuis tensor*).
(b) Show that the following are equivalent:

1. $N_J \equiv 0$;
2. the Lie bracket on vector fields on M (extended over $\mathbb{C}$ to commute with $\mathfrak{i}$) restricts to
$$\Gamma(M;TM^{1,0}) \times \Gamma(M;TM^{1,0}) \longrightarrow \Gamma(M;TM^{1,0});$$
3. $\bar\partial^2 = 0$ on $C^\infty(M;\mathbb{C})$;
4. $\bar\partial^2 = 0$ on $\Omega^*(M;\mathbb{C})$;
5. $d = \partial + \bar\partial$ on $\Omega^*(M;\mathbb{C})$.

Note: these are equivalent to the integrability of J by the Newlander-Nirenberg theorem.

Problem 2 (5 pts)

With $k,l\in\mathbb{Z}^+$, let

$$\Delta^k = \big\{(z_1,\ldots,z_k)\in\mathbb{C}^k : |z_i|<1\ \forall\, i\big\}, \quad \Delta^{*l} = \big\{(z_1,\ldots,z_l)\in\mathbb{C}^l : 0<|z_i|<1\ \forall\, i\big\}.$$

Show that $H^{p,q}_{\bar\partial}(\Delta^k \times \Delta^{*l}) = 0$ for all $q>0$.

Problem 3 (5 pts)

Let (M,J) be an almost complex manifold.
(a) If h is a positive-definite Hermitian form on M, show that $g = \mathrm{Re}\, h$ is a Riemannian metric on M compatible with J and $\omega = -\frac{1}{2}\mathrm{Im}\, h$ is a 2-form on M which is compatible with J and positive, i.e. $\omega(v,Jv)>0$ for all $v\in TM - M$.
(b) Show that a J-compatible Riemannian metric on M determines a hermitian form on M, as does a positive J-compatible 2-form ω on M. *no local coordinates please*

Problem 4 (5 pts)

Show that all linearly embedded $\mathbb{C}P^k$ in $\mathbb{C}P^n$ have the same volume with respect to the Fubini-Study metric on $\mathbb{C}P^n$. Determine what this volume is.

Problem 5 (10 pts)

Let M be an orientable Riemannian manifold and $X \subset M$ a compact oriented submanifold.
(a) Show that the volume form on X is the restriction of a differential form on M.
(b) Show that however it may not be possible to find a non-vanishing differential form on M with the desired property.
Note: This problem is intended to correct the statement at the bottom of p31; you can get the entire 10 points by giving a counterexample to (a).

Problem 6 (10 pts)

The sets $\check{H}^0$ and $\check{H}^1$ can be defined for sheaves of non-abelian groups as well The main example of interest is the sheaf $\mathcal{S}$ of germs of smooth (or continuous) functions to a Lie group G (a smooth manifold and a group so that the group operations are smooth; examples include $O(k)$, $SO(k)$, $\mathcal{U}(k)$, $S\mathcal{U}(k)$). If $\underline{\mathcal{U}}=\{\mathcal{U}_\alpha\}$ is an open cover, $f\in\check{C}^0(\underline{\mathcal{U}};\mathcal{S})$, and $g\in\check{C}^1(\underline{\mathcal{U}};\mathcal{S})$, define

$$\partial_0 f \in \check{C}^1(\underline{\mathcal{U}};\mathcal{S}) \qquad \text{by} \qquad (\partial_0 f)_{\alpha_0\alpha_1} = f_{\alpha_0}\big|_{\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}} \cdot f_{\alpha_1}^{-1}\big|_{\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}},$$

$$\partial_1 g \in \check{C}^2(\underline{\mathcal{U}};\mathcal{S}) \quad \text{by} \quad (\partial_1 g)_{\alpha_0\alpha_1\alpha_2} = g_{\alpha_1\alpha_2}\big|_{\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}\cap\mathcal{U}_{\alpha_2}} \cdot g_{\alpha_0\alpha_2}^{-1}\big|_{\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}\cap\mathcal{U}_{\alpha_2}} \cdot g_{\alpha_0\alpha_1}\big|_{\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}\cap\mathcal{U}_{\alpha_2}},$$

where for all $\alpha_0,\alpha_1,\alpha_2\in\mathcal{A}$, $f\in\check{C}^0(\underline{\mathcal{U}};\mathcal{S})$, $g\in\check{C}^1(\underline{\mathcal{U}};\mathcal{S})$, and $h\in\check{C}^2(\underline{\mathcal{U}};\mathcal{S})$,

$$f_{\alpha_0} \in \Gamma(\mathcal{U}_{\alpha_0};\mathcal{S}), \qquad g_{\alpha_0\alpha_1} \in \Gamma(\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1};\mathcal{S}), \qquad h_{\alpha_0\alpha_1\alpha_2} \in \Gamma(\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}\cap\mathcal{U}_{\alpha_2};\mathcal{S}).$$

Define an action of $\check{C}^0(\underline{\mathcal{U}};\mathcal{S})$ on $\check{C}^1(\underline{\mathcal{U}};\mathcal{S})$ by

$$\{f*g\}_{\alpha_0\alpha_1} = f_{\alpha_0}\big|_{\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}} \cdot g_{\alpha_0\alpha_1} \cdot f_{\alpha_1}^{-1}\big|_{\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1}} \in \Gamma(\mathcal{U}_{\alpha_0}\cap\mathcal{U}_{\alpha_1};\mathcal{S}).$$

(a) Show that under this action $\check{C}^0(\underline{\mathcal{U}};\mathcal{S})$ maps $\ker\partial_1$ into itself.
(b) Show that for every Čech 1-cocycle g (i.e. $g\in\ker\partial_1$) for an open cover $\underline{\mathcal{U}}=\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$,

$$g_{\alpha\alpha} = e|_{\mathcal{U}_\alpha}, \quad g_{\alpha\beta}g_{\beta\alpha} = e|_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta}, \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = e|_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta\cap\mathcal{U}_\gamma}, \qquad \forall\,\alpha,\beta,\gamma\in\mathcal{A},$$

where e is the "zero" (or "identity") section of $\mathcal{S}$ (i.e. $e(m)$ is the identity element of the group $\mathcal{S}_m$ for every $m\in M$).
By part (a), we can define

$$\check{H}^0(\underline{\mathcal{U}};\mathcal{S}) = \ker\partial_0 \qquad \text{and} \qquad \check{H}^1(\underline{\mathcal{U}};\mathcal{S}) = \ker\partial_1/\check{C}^0(\underline{\mathcal{U}};\mathcal{S}).$$

The first set is a group being the kernel of a group homomorphism. If $\underline{\mathcal{U}}'=\{\mathcal{U}'_\alpha\}_{\alpha\in\mathcal{A}'}$ is a refinement of $\underline{\mathcal{U}}=\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$, any refining map $\mu\colon\mathcal{A}'\longrightarrow\mathcal{A}$ induces group homomorphisms

$$\mu_p^*\colon \check{C}^p(\underline{\mathcal{U}};\mathcal{S}) \longrightarrow \check{C}^p(\underline{\mathcal{U}}';\mathcal{S}),$$

which commute with ∂_0, ∂_1, and the action of $\check{C}^0(\cdot;\mathcal{S})$ on $\check{C}^1(\cdot;\mathcal{S})$. Thus, μ induces a group homomorphism and a map

$$R^0_{\mathcal{U}',\mathcal{U}}\colon \check{H}^0(\underline{\mathcal{U}};\mathcal{S}) \longrightarrow \check{H}^0(\underline{\mathcal{U}}';\mathcal{S}) \qquad \text{and} \qquad R^1_{\mathcal{U}',\mathcal{U}}\colon \check{H}^1(\underline{\mathcal{U}};\mathcal{S}) \longrightarrow \check{H}^1(\underline{\mathcal{U}}';\mathcal{S}).$$

(c) Show that these maps are independent of the choice of μ.
Thus, we can again define $\check{H}^0(M;\mathcal{S})$ and $\check{H}^1(M;\mathcal{S})$ by taking the direct limit of all $\check{H}^0(\underline{\mathcal{U}};\mathcal{S})$ and $\check{H}^1(\underline{\mathcal{U}};\mathcal{S})$ over open covers of M. The first set is a group, while the second need not be (unless $\mathcal{S}$ is a sheaf of abelian groups). These sets will be denoted by $\check{H}^0(M;G)$ and $\check{H}^1(M;G)$ if $\mathcal{S}$ is the sheaf of germs of smooth (or continuous) functions into a Lie group G. As in the abelian case, $\check{H}^0(M;\mathcal{S})$ is the space of global sections of $\mathcal{S}$.
(d) Show that there is a natural correspondence

$$\big\{\text{isomorphism classes of rank-}k\text{ complex vector bundles over } M\big\} \longleftrightarrow \check{H}^1\big(M;\mathcal{U}(k)\big).$$

Note: Do not forget that $\check{H}^1(M;\mathcal{S})$ is a *direct limit*.