Suggested Problems for Week 6

You may hand in solutions to at most 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.


Problem J

Let $K$ and $L$ be simplicial complexes and $g: |K| \to |L|$ be a continuous map. For each $v \in \text{Ver}(K)$, let

$$
\tau_v = \{ w \in \text{Ver}(L): g(\text{St}(v, K)) \subset \text{St}(w, L) \}.
$$

For each simplex $\sigma = \{v_0, \ldots, v_p\} \in K$, let

$$
\tau_\sigma = \tau_{v_0} \cup \ldots \cup \tau_{v_p} \subset \text{Ver}(L).
$$

(1) Suppose $x \in \text{Int}\sigma$ for some $\sigma \in K$ and $g(x) \in \text{Int}\tau$ for some $\tau \in L$. Show that $\tau_\sigma \subset \tau$ and that this inclusion may be strict even if $\sigma$ is a vertex and $\tau_v \neq \emptyset$ for every $v \in \text{Ver}(K)$.

(2) Let $\sigma \in K$ and $\sigma'$ be a face of $\sigma$. Show that $\tau_\sigma \in L$ and $\tau_{\sigma'}$ is a face of $\tau_\sigma$.

(3) Let $\hat{g}: \text{Ver}(K) \to \text{Ver}(L)$ be a simplicial approximation to $g$, i.e.

$$
g(\text{St}(v, K)) \subset \text{St}(\hat{g}(v), L) \quad \forall \ v \in \text{Ver}(K).
$$

Show that $\hat{g}$ is carried by $\sigma \mapsto \tau_\sigma$, i.e. $\hat{g}(\sigma) \subset \tau_\sigma$ for every $\sigma \in K$.

(4) By definition, $g$ admits a simplicial approximation if and only if $\tau_v \neq \emptyset$ for every $v \in \text{Ver}(K)$. Conclude that the homomorphism

$$
g_* \equiv \hat{g}: H_\ast(K) \to H_\ast(L)
$$

is independent of the choice of the simplicial approximation $\hat{g}$ to $g$, if one exists.

This gives a more systematic perspective on the proof of Lemma 14.1 in Munkres.