Problem I

Let $K$ be a simplicial complex of dimension $n$ (the maximal dimension of the simplicies) and $K^m$ be its $m$-th skeleton (the union of all simplicies of dimension at most $m$). Show that $K - K^m$ is a union of open subsets $U_{m+1}, \ldots, U_n$ such that all possible intersections

$$U_{i_1} \cap \ldots \cap U_{i_k} \subset K, \quad i_1, \ldots, i_k \in \{m+1, \ldots, n\}, \quad k \in \mathbb{Z}^+,$$

are disjoint unions of contractible open subsets of $K$. 

From Munkres: 15.1, 15.2, 17.1-3, 18.1-3