MAT 541: Algebraic Topology
Comments on Problem A

For a set J, let

$$\mathbb{R}^J = \{f\colon J\longrightarrow\mathbb{R}\} = \{(r_i)_{i\in J}\colon r_i\in\mathbb{R}\}, \quad \mathbb{R}^J_c = \{f\in\mathbb{R}^J\colon |\{i\in J\colon f(i)\neq 0\}|<\infty\}.$$

For J infinite countable (especially $J=\mathbb{Z}^+$), these sets are often denoted $\mathbb{R}^\omega$ and $\mathbb{R}^\infty$. If J is finite, then $\mathbb{R}^J_c=\mathbb{R}^J$. For $i\in J$, let

$$e_i\in\mathbb{R}^J_c\subset\mathbb{R}^J, \qquad e_i(j)=\begin{cases}1, & \text{if } j=i;\\ 0, & \text{if } j\neq i;\end{cases}$$

be the i-th coordinate vector.

Let $\mathcal{S}_J$ denote the simplicial complex consisting of all finite subsets of J. An abstract simplicial complex $\mathcal{S}$ is a subcomplex of $\mathcal{S}_J$ with $J=\mathrm{Ver}(\mathcal{S})$; its canonical geometric realization $|\mathcal{S}|$ is a geometric subcomplex of

$$|\mathcal{S}_J|\subset\mathbb{R}^J_c\subset\mathbb{R}^J.$$

If J is finite, the simplicial topologies on $|\mathcal{S}_J|$ and $|\mathcal{S}|$ agree with the subspace topologies with respect to the canonical topology on the finite-dimensional vector space $\mathbb{R}^J$. From now on, J is thus assumed to be infinite and to contain $\mathbb{Z}^+$ as a subset.

There are several natural topologies on $\mathbb{R}^J$:

(T1) product topology, which has open sets of the form

$$\prod_{i\in J}U_i, \qquad U_i\subset\mathbb{R}\text{ open}, \ \big|\{i\in J\colon U_i\neq\mathbb{R}\}\big|<\infty;$$

(T2) uniform topology, which has open sets of the form

$$\prod_{i\in J}(r_i-\delta, r_i+\delta), \qquad \delta\in\mathbb{R};$$

(T3) box topology, which has open sets of the form

$$\prod_{i\in J}(a_i,b_i), \qquad a_i,b_i\in\mathbb{R};$$

(T4) coherent topology, which has open sets U so that $U\cap V$ is open in V for every finite-dimensional subspace $V\subset\mathbb{R}^J$.

The vector space operations

$$\mathbb{R}^J\times\mathbb{R}^J\longrightarrow\mathbb{R}^J, \ (v,w)\longrightarrow v+w, \qquad \mathbb{R}\times\mathbb{R}^J\longrightarrow\mathbb{R}^J, \ (r,v)\longrightarrow rw,$$

are continuous with respect to any of the four topologies above.

The product topology is strictly coarser than the uniform topology. The latter is strictly coarser than the box and coherent topologies, while the last two topologies are not comparable:

$$\text{(T1)} \subsetneq \text{(T2)} \subsetneq \text{(T3)}, \text{(T4)}, \qquad \text{(T3)} \not\subset \text{(T4)}, \qquad \text{(T3)} \not\supset \text{(T4)}.$$

Every set $U \subset \mathbb{R}^J$ open in the product topology is open in the uniform topology; every set $U \subset \mathbb{R}^J$ open in the uniform topology is open in the box and coherent topologies. The subset

$$A_1 \equiv \big\{ e_1 + \frac{1}{2}(e_i - e_1) \colon i \in \mathbb{Z}^+ \big\} \subset |\mathcal{S}_J| \tag{1}$$

is closed in $\mathbb{R}^J$ with respect to the uniform topology, but its closure in $\mathbb{R}^J$ with respect to the product topology contains $e_1/2$. However, the restrictions of the two topologies to $|\mathcal{S}_J|$ are the same. The subset

$$A_2 \equiv \big\{ e_1 + \frac{1}{i}(e_i - e_1) \colon i \in \mathbb{Z}^+ \big\} \subset |\mathcal{S}_J| \tag{2}$$

is closed in the box and coherent topologies, but the closure of this set in the uniform topology contains e_1. The subset

$$U = \prod_{i=1}^{\infty} \big(-1/i, 1/i \big) \times \prod_{i \in J - \mathbb{Z}^+} \mathbb{R} \subset \mathbb{R}^{\mathbb{Z}^+} \times \mathbb{R}^{J - \mathbb{Z}^+}$$

is open in the box topology, but not in the coherent topology because its intersection with the one-dimensional linear subspace V spanned by the vector

$$f \colon J \longrightarrow \mathbb{R}, \qquad f(i) = 1 \qquad \forall\, i \in J,$$

is $\{\mathbf{0}\}$ (which is not open in V).

We next describe a subset $B \subset \mathbb{R}^J$ closed in the coherent topology, but not in the box topology. Let $\aleph$ be the collection of *infinite* subsets of $\mathbb{Z}^+$. For each $\sigma \in \aleph$, define

$$f_\sigma \colon J \longrightarrow \mathbb{R}, \qquad f_\sigma(i) = \begin{cases} 1/\min\{j \in \sigma \colon j > i\}, & \text{if } i \in \sigma; \\ 0, & \text{if } i \in J - \sigma. \end{cases}$$

We take

$$B = \big\{ f_\sigma \colon \sigma \in \aleph \big\} \subset \mathbb{R}^J. \tag{3}$$

As shown at the end of this note, the linear span of any finite subset B_0 of B contains finitely many elements of B:

$$\big| \mathrm{Span}(B_0) \cap B \big| < \infty \qquad \text{if } B_0 \subset B,\ |B_0| < \infty. \tag{4}$$

Thus, $B \cap V$ is a finite set for every finite-dimensional subspace $V \subset \mathbb{R}^J$ and in particular closed in V. It follows that $V \subset \mathbb{R}^J$ is closed in the coherent topology. Every neighborhood

$$W = \prod_{i \in J}^{\infty} (-\delta_i, \delta_i) \subset \mathbb{R}^J$$

of $\mathbf{0}$ in the box topology contains $f_\sigma \in B$ with

$$\sigma = \big\{ i_1 < i_2 < \ldots \big\} \subset \mathbb{Z}^+ \qquad \text{s.t.} \quad i_{r+1} > 1/\delta_r \ \forall\, r \in \mathbb{Z}^+.$$

Thus, B is not closed in the box topology.

There are even more natural topologies on $\mathbb{R}^J_c$:

(T_c1) the restriction of the product topology on $\mathbb{R}^J$;

(T_c2) the restriction of the uniform topology on $\mathbb{R}^J$, which is equivalent to the metric topology

$$d_\infty(f,g)=\max_{i\in J}\big|f(i)-g(i)\big|;$$

(T_c3) the restriction of the box topology on $\mathbb{R}^J$;

(T_c4) the restriction of the coherent topology on $\mathbb{R}^J$, which has open sets $U\subset\mathbb{R}^J_c$ so that $U\cap\mathbb{R}^{J_0}$ is open in $\mathbb{R}^{J_0}$ for every finite subset $J_0\subset J$;

(T_c5) various other metric topologies such as

$$d_p(f,g)=\bigg(\sum_{i\in J}^{\infty}\big|f(i)-g(i)\big|^p\bigg)^{1/p},\quad p\in[1,\infty).$$

The vector space operations

$$\mathbb{R}^J_c\times\mathbb{R}^J_c\longrightarrow\mathbb{R}^J_c,\quad (v,w)\longrightarrow v+w,\qquad \mathbb{R}\times\mathbb{R}^J_c\longrightarrow\mathbb{R}^J_c,\quad (r,v)\longrightarrow rw,$$

are continuous with respect to these topologies. The simplicial topology on $|\mathcal{S}|$ is the restriction of the coherent topology.

For $p,q\in[1,\infty)$ with $p<q$, let $r_i\in(0,1)$ be a sequence such that

$$\sum_{i=1}^{\infty}r_i^p=\infty,\qquad \sum_{i=1}^{\infty}r_i^q<\infty\,.$$

Define

$$x_k=\sum_{i=1}^{k}r_ie_i\Big/\bigg(\sum_{i=1}^{k}r_i^p\bigg)^{1/p}\in\mathbb{R}^\infty_c. \tag{5}$$

The set $\{x_k\}$ is closed in the d_p-metric topology and in the box and coherent topologies, but has $\mathbf{0}$ as a limit point in the d_q-metric topology and in the uniform topology. On the other hand, a set $U\subset\mathbb{R}^J_c$ open in the d_q-metric topology is also open in the d_p-metric topology and in the coherent topology. A set $U\subset\mathbb{R}^J_c$ open in the uniform topology is also open in the d_p-metric topology. A set $U\subset\mathbb{R}^J_c$ open in the box topology is also open in the coherent topology. Thus,

$$\begin{gathered}(\mathrm{T}_c1)\subsetneq(\mathrm{T}_c2)\subsetneq(\mathrm{T}_c5)_q\subsetneq(\mathrm{T}_c5)_p\subsetneq(\mathrm{T}_c4)\quad\forall\,q<p,\\ (\mathrm{T}_c5)_p\not\supset(\mathrm{T}_c3)\quad\forall\,p,\qquad (\mathrm{T}_c2)\subsetneq(\mathrm{T}_c3)\subset(\mathrm{T}_c4)\,;\end{gathered}$$

the penultimate inclusion above follows from the sentence containing (2). By this sentence, the restriction of the topology (T_c2) to $|\mathcal{S}_J|$ is still strictly coarser than the restriction of (T_c3). On the other hand, the restrictions of the topologies (T_c1), (T_c2), and $(\mathrm{T}_c5)_p$ to $|\mathcal{S}_J|$ are the same.

It remains to compare the box topology on $\mathbb{R}^J_c$ with the d_p-metric topology and with the coherent topology. Let

$$B=\big\{x\in\mathbb{R}^J_c:d_p(\mathbf{0},x)=\frac{1}{2}\big\},\quad W=\prod_{i\in J}(-\delta_i,\delta_i)\,.$$

If $W\cap\mathbb{R}^J_c\subset B$, then

$$\big|\{i\in J\colon k\delta_i\geq 1\}\big|\leq k^p \qquad \forall\, k\in\mathbb{Z}^+\,. \tag{6}$$

If W is a neighborhood of $\mathbf{0}$ in $|\mathcal{S}_J|$ in the box topology, then $\delta_i>0$ for all $i\in J$ and (6) implies that J is countable. Thus,

$$(\mathrm{T}_c5)_p \not\subset (\mathrm{T}_c3) \subsetneq (\mathrm{T}_c4) \qquad \text{if } J \text{ is uncountable.}$$

We next note that the restrictions of the last two topologies to $|\mathcal{S}_J|$ satisfy the same property. For a finite subset $S\subset J-\{1\}$, define $f_S\in|\mathcal{S}_J|$ by

$$f_S\colon J\longrightarrow\mathbb{R}, \qquad f_S(i)=\begin{cases}1-|S|/2^{|S|}, & \text{if } i=1;\\ 1/2^{|S|}, & \text{if } i\in S;\\ 0, & \text{otherwise.}\end{cases}$$

The intersection of the subset

$$A_J\equiv\big\{f_S\colon S\subset J-\{1\},\ 1\leq|S|<\infty\big\}$$

with every finite-dimensional linear subspace $V\subset\mathbb{R}^J$ is finite. Thus, A_J is closed in the coherent topology. If the neighborhood

$$W\equiv\big(1-\delta_1,1+\delta_1\big)\times\prod_{i\in J-\{1\}}(-\delta_i,\delta_i)\subset\mathbb{R}^{\{1\}}\times\mathbb{R}^{J-\{1\}}$$

of e_1 is disjoint from A_J, then

$$\min\big\{\delta_i\colon i\in S\big\}\leq\frac{1}{2^{|S|}} \qquad \forall\ \ S\subset J-\{1\}\ \text{ s.t. } \frac{|S|}{2^{|S|}}<\delta_1\,.$$

This is impossible if J is uncountable.

Suppose $A\subset\mathbb{R}^\infty-\{0\}$ is closed with respect to the coherent topology. We define a neighborhood

$$W=\prod_{i=1}^{\infty}(-\delta_i,\delta_i)\subset\mathbb{R}^\omega$$

of $\mathbf{0}$ in the box topology containing no element of A as follows. Suppose $i\in\mathbb{Z}^+$ and we have chosen $\delta_1,\ldots,\delta_{i-1}\in\mathbb{R}^+$ so that

$$[-\delta_1,\delta_1]\times\ldots\times[-\delta_{i-1},\delta_{i-1}]\cap\big(A\cap\mathbb{R}^{i-1}\big)=\emptyset.$$

Since $A\cap\mathbb{R}^i$ is closed in $\mathbb{R}^i$, there exists $\delta_i\in\mathbb{R}^+$ such that

$$[-\delta_1,\delta_1]\times\ldots\times[-\delta_i,\delta_i]\cap\big(A\cap\mathbb{R}^i\big)=\emptyset.$$

Since $A\subset\mathbb{R}^i$ for some i, it follows that $W\cap A=\emptyset$. Thus, A is closed in the box topology on $\mathbb{R}^\infty$. This establishes that

$$(\mathrm{T}_c5)_p \subsetneq (\mathrm{T}_c3) = (\mathrm{T}_c4) \qquad \text{if } J \text{ is countable.}$$

The same statements hold for the restrictions of these topologies to $|\mathcal{S}_J|$.

One conclusion of the above is that the simplicial topologies on $|\mathcal{S}_J|$ and $|\mathcal{S}|$ agree with the subspace topologies with respect to $\mathbb{R}^J_{\mathrm{box}}$ if J is countable. The simplicial topology on $|\mathcal{S}|$ also agrees with the subspace topology with respect to $\mathbb{R}^J_{\mathrm{box}}$ if the set

$$\big\{S\in\mathcal{S}\colon v\in S\big\}$$

is (at most) countable for every $v\in\mathrm{Ver}(\mathcal{S})$ for the following reason. If $x\in|\mathcal{S}|$, then the set J_p of the vertices of the subcomplex $\mathcal{S}_p$ forming the closed star $\overline{\mathrm{St}}_{\mathcal{S}}(p)$ of the open simplex containing p is countable. The corresponding open star $\mathrm{St}_{\mathcal{S}}(p)$ is an open neighborhood of p in $|\mathcal{S}|$ with respect to the simplicial topologies on $|\mathcal{S}|$ and $|\mathcal{S}_p|$. By the countable J case, every $U\subset\mathrm{St}_{\mathcal{S}}(p)$ open in the simplicial topology is also open in the subspace topology induced from $\mathbb{R}^{J_p}_{\mathrm{box}}$ and thus from $\mathbb{R}^J_{\mathrm{box}}$. Therefore, the simplicial topology on $|\mathcal{S}|$ agrees with the subspace topology with respect to $\mathbb{R}^J_{\mathrm{box}}$.

It remains to establish (4). For each $k\in\mathbb{Z}^{\ge0}$, define

$$\mathbb{Z}^+_k=\mathbb{Z}^+-\{1,\ldots,k\},\qquad \pi_k\colon\mathbb{R}^J\longrightarrow\mathbb{R}^{\mathbb{Z}^+_k},\quad \pi_k\big((r_i)_{i\in J}\big)=(r_i)_{i\in\mathbb{Z}^+_k}.$$

If in addition $m\in\mathbb{Z}^{\ge0}$, define

$$\begin{aligned}\pi_{k+m}\colon\mathbb{R}^{\mathbb{Z}^+_k}&\longrightarrow\mathbb{R}^{\mathbb{Z}^+_{k+m}}, &\qquad \pi_{k+m}\big((r_i)_{i\in\mathbb{Z}^+_k}\big)&=(r_i)_{i\in\mathbb{Z}^+_{k+m}},\\ \pi_{k;m}\colon\mathbb{R}^{\mathbb{Z}^+_k}&\longrightarrow\mathbb{R}^m, &\qquad \pi_{k;m}\big((r_i)_{i\in\mathbb{Z}^+_k}\big)&=\big(r_{k+1},\ldots,r_{k+m}\big).\end{aligned}$$

We first note that if $B_0\subset\mathbb{R}^{\mathbb{Z}^+_k}$ is a finite collection of linearly independent elements, then so is $\pi_{k;m}(B_0)\subset\mathbb{R}^m$ for all $m\in\mathbb{Z}^+$ sufficiently large. For each $m\in\mathbb{Z}^+$, let

$$\mathbb{R}^{B_0}_m=\Big\{(c_f)_{f\in B_0}\colon\sum_{f\in B_0}c_ff(i)=0\ \forall\,i=k+1,\ldots,k+m\Big\}.$$

Thus, $\mathbb{R}^{B_0}_1\supset\mathbb{R}^{B_0}_2\supset\ldots\ldots$ are linear subspaces of $\mathbb{R}^{B_0}$. If some $(c_f)_{f\in B_0}$ belongs to all of them,

$$\sum_{f\in B_0}c_ff(i)=0\qquad\forall\ i\in\mathbb{Z}^+_k\,.$$

Since the elements of B_0 are linearly independent, it follows that $c_f=0$ for every $f\in B_0$ and so $\mathbb{R}^{B_0}_m=\{0\}$ for all m sufficiently large.

We now show by induction on $|B_0|$ that

$$\big|\mathrm{Span}(B_0)\cap\pi_k(B)\big|<\infty\qquad\text{if } B_0\subset\pi_k(B),\ |B_0|<\infty,\ k\in\mathbb{Z}^{\ge0}\,;\tag{7}$$

the $k=0$ case of this claim is equivalent to (4). If $\pi_k(f_{\sigma_1})$ and $\pi_k(f_{\sigma_2})$ with $\sigma_1,\sigma_2\in\aleph$ are linearly dependent, then

$$\sigma_1\cap\mathbb{Z}^+_k=\sigma_2\cap\mathbb{Z}^+_k,\qquad\pi_k(f_{\sigma_1})=\pi_k(f_{\sigma_2}).$$

Thus, (7) is true if $|B_0|=1$. Suppose $\ell\in\mathbb{Z}^+$, (7) is true whenever $|B_0|\le\ell$ (no matter what k is), $k\in\mathbb{Z}^+$, and $B_0\subset\pi_k(B)$ is a subset of linearly independent elements of cardinality $\ell+1$. Let $m\in\mathbb{Z}^+$

be such that the elements $\pi_{k;m}(f)\in\mathbb{R}^m$ with $f\in B$ are linearly independent.

Suppose first that the elements $\pi_{k+m}(f)$ with $f\in B_0$ are linearly dependent and thus

$$\mathrm{Span}\big(\{\pi_{k+m}(f)\colon f\in B_0\}\big) = \mathrm{Span}\big(\{\pi_{k+m}(f)\colon f\in B_0'\}\big)$$

for some $B_0'\subset B_0$ with $|B_0'|=\ell$. From the inductive assumption, we then obtain

$$\big|\mathrm{Span}\big(\{\pi_{k+m}(f)\colon f\in B_0\}\big)\cap\pi_{k+m}(B)\big| < \infty. \tag{8}$$

Since each element of $\pi_{k+m}(B)$ has finitely many preimages in B, (8) implies (7).

Suppose instead that the elements $\pi_{k+m}(f)$ with $f\in B_0$ are linearly independent. Let $m'\in\mathbb{Z}^+$ be such that the elements $\pi_{k+m;m'}(f)$ with $f\in B_0$ are linearly independent. This implies that for every $f\in\mathrm{Span}(B_0)\cap\pi_k(B)$ nonzero there exists

$$j\in\mathbb{Z}^+ \qquad \text{s.t.} \quad k+m< j \le k+m+m', \quad f(j)\neq 0.$$

From the definition of B, it then follows that

$$f(i)\in\{0\}\sqcup\big\{1/j\colon k< j\le k+m+m'\big\} \quad \forall\ f\in\mathrm{Span}(B_0)\cap\pi_k(B),\ k<i\le k+m,$$
$$\big|\pi_{k;m}\big(\mathrm{Span}(B_0)\cap\pi_k(B)\big)\big| < \infty.$$

Since the elements $\pi_{k;m}(f)$ of $\mathbb{R}^m$ with $f\in B_0$ are linearly independent, it follows

$$\big|\mathrm{Span}(B_0)\cap\pi_k(B)\big| = \big|\pi_{k;m}\big(\mathrm{Span}(B_0)\cap\pi_k(B)\big)\big| < \infty.$$

This completes the inductive step.

This note is based on discussions with Xujia Chen, Ying Honh Tham, and Hang Yuan in Fall 2016 who had discovered that the original formulation of Problem A was wrong. If you see any problems with any of the above statements and/or have suggestions for streamlining any of the arguments in this note (in particular that the box topology on $\mathbb{R}^\omega$ is not finer than the coherent topology), please let me know.