MAT 541: Algebraic Topology
Suggested Problems for Week 11

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 47.1, 47.3, 47.5, 47.7, 48.1-3, 49.1-3, 49.8, 66.4-6, 57.1

Problem R

Let R be a commutative ring with 1 and M be a module over R. A directed set is a set S with a partial order $\prec$ such that for any pair of elements $i,j\in S$ there exists $k\in S$ with $i,j\le k$. A directed system of R-modules over a directed set $(S,\prec)$ is an assignment of an R-module M_i for each $i\in S$ and an R-module homomorphism $\rho_{ji}\colon M_i\longrightarrow M_j$ for all $i,j\in S$ with $i\preceq j$ such that

$$\rho_{ii}=\mathrm{id}_{M_i}\ \ \forall\, i\in S, \qquad \rho_{ki}=\rho_{kj}\circ\rho_{ji}\ \ \forall\, i,j,k\in S,\ i\preceq j\preceq k.$$

An inverse system of R-modules over a directed set $(S,\prec)$ is an assignment of an R-module M_i for each $i\in S$ and an R-module homomorphism $\varrho_{ij}\colon M_j\longrightarrow M_i$ for all $i,j\in S$ with $i\preceq j$ such that

$$\varrho_{ii}=\mathrm{id}_{M_i}\ \ \forall\, i\in S, \qquad \varrho_{ik}=\varrho_{ij}\circ\varrho_{jk}\ \ \forall\, i,j,k\in S,\ i\preceq j\preceq k.$$

The direct limit of a directed system and the inverse limit of an inverse system are

$$\varinjlim M_i \equiv \Big(\bigsqcup_i M_i\Big)\Big/\!\sim, \quad A_i\ni x_i\sim x_j\in A_j \quad \text{if } \exists\, k\in S \text{ s.t. } i,j\preceq k,\ \rho_{ki}(x_i)=\rho_{kj}(x_j),$$

$$\varprojlim M_i \equiv \big\{(x_i)_{i\in S}\in\prod_i M_i\colon x_i=\varrho_{ij}(x_j)\ \forall\, i,j\in S,\ i\prec j\big\},$$

respectively. For example, the collection of compact subsets K of a topological spaces forms a directed set. The associated homology and cohomology groups form a direct system and an inverse system under the homomorphisms

$$\rho_{K'K}\colon H_*(K;R)\longrightarrow H_*(K';R) \qquad \text{and} \qquad \varrho_{KK'}\colon H^*(K';M)\longrightarrow H^*(K;M)$$

induced by the inclusions $K\subset K'$.

(a) Show that the direct and inverse limits are R-modules.

(b) Construct a ring isomorphism $\varinjlim H_*(K;R)\longrightarrow H_*(X;R)$.

(c) Construct a ring homomorphism $H^*(X;R)\longrightarrow \varprojlim H^*(K;R)$.

(d) Take $X=\mathbb{RP}^\infty$ and $R=\mathbb{Z}_2$. Construct an isomorphism $\varprojlim H^*(K;R)\longrightarrow \varprojlim H^*(\mathbb{RP}^n;R)$ and conclude that the above homomorphism is injective, but not surjective in this case.

Problem S

Let R be a commutative ring with 1. Let X,Y be Hausdorff topological spaces and $X\vee Y$ be their wedge product at points that are deformation retracts of neighborhoods in their spaces.

(a) Construct an isomorphism

$$H^p\big(X\vee Y;R\big)\longrightarrow H^p(X;R)\oplus H^p(Y;R) \qquad \forall\, p\in\mathbb{Z}^+$$

and show that it respects the cup product.

(b) Use this to establish the claims of Example 4 and Exercise 5 in Section 49 without any simplicial complexes.