MAT 541: Algebraic Topology

Suggested Problems for Week 10

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 41.1, 41.4, 45.3, 46.1, 44.1, 44.3-5, 43.4, 43.5, 44.5

Problem P

Suppose R is a commutative ring with unity 1, M is an R-module, $(\mathcal{C}_*, \partial)$ is a chain complex over R, and

$$(\mathcal{C}^*, \delta) \equiv \big(\mathrm{Hom}(\mathcal{C}_*, M), \partial^*\big)$$

is its dual cochain complex. Since $\delta = \partial^*$, the natural pairing of $\mathcal{C}^p$ with $\mathcal{C}_p$ induces a homomorphism

$$\kappa_p \colon H^p\big(\mathcal{C}_*, \partial; M\big) \longrightarrow \mathrm{Hom}\big(H_p(\mathcal{C}_*, \partial); M\big).$$

Show that

(a) if R is a principal ideal domain (PID) and A is a submodule of a free R-module B, then A is also free;

(b) if R is a PID and $\mathcal{C}_*$ is a free R-module, then κ_p is onto;

(c) if R is a PID, $\mathcal{C}_*$ is a free R-module, and $H_{p-1}(\mathcal{C}_*, \partial)$ is also a free R-module, then κ_p is injective.

Give an example when R is an integral domain and κ_p is not onto.

Hint for the main questions: see proofs of Lemmas 11.1/11.2, Corollary 23.2 and Lemma 45.7, and Theorem 45.8 in Munkres.

Problem Q

Suppose R is a commutative ring with unity 1, M is an R-module, and X is a non-empty topological space. Let $\mu \in M - 0$. For each $p \in \mathbb{Z}^{\geq 0}$, denote by $\mu_p \in S^p(X; M)$ the cochain such that

$$\mu_p\big(\sigma \colon \Delta^p \longrightarrow X\big) = \mu$$

for every singular p-simplex σ in X. Show that

(a) μ_p is a cocycle if and only if $p \in 2\mathbb{Z}^{\geq 0}$;

(b) μ_p is a coboundary if and only if $p \in 2\mathbb{Z}^{+}$.

Thus, μ_p determines a nonzero element of $H^*(X; M)$ if and only if $p = 0$.