MAT 531: Topology&Geometry, II
Spring 2006

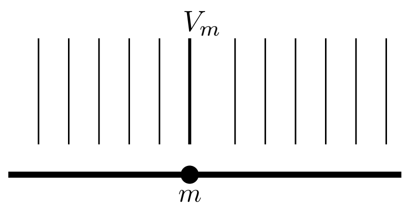
Notes on Vector Bundles

1 Definitions and Examples

A *(smooth) real vector bundle* V of rank k over a smooth manifold M is a smoothly varying family of k-dimensional real vector spaces which is locally trivial. More formally, a real vector bundle is a triple (M, V, π), where M and V are smooth manifolds and

$$\pi\colon V \longrightarrow M$$

is a smooth map. For each $m \in M$, the fiber $V_m \equiv \pi^{-1}(m)$ of V over m is a real k-dimensional vector space:



$$\begin{tikzcd} V \arrow[d, "\pi"] \\ M \end{tikzcd}$$

The vector-space structures vary smoothly with m. This means that the scalar multiplication map

$$\mathbb{R} \times V \longrightarrow V, \qquad (c, v) \longrightarrow c \cdot v,$$

and the addition map

$$V \times_M V \equiv \big\{(v_1, v_2) \in V \times V : \pi(v_1) = \pi(v_2) \in M\big\} \longrightarrow V, \qquad (v_1, v_2) \longrightarrow v_1 + v_2,$$

are smooth. Note that we can add $v_1, v_2 \in V$ only if they lie in the same fiber over M, i.e.

$$\pi(v_1) = \pi(v_2) \qquad \Longleftrightarrow \qquad (v_1, v_2) \in V \times_M V.$$

The space $V \times_M V$ is a smooth submanifold of $V \times V$, as follows immediately from the Implicit Function Theorem or can be seen directly. The local triviality condition means that for every point $m \in M$ there exist a neighborhood $\mathcal{U}$ of m in M and a diffeomorphism

$$h\colon V|_{\mathcal{U}} \equiv \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{R}^k,$$

such that h takes every fiber of π to the corresponding fiber of the projection map $\pi_1 : \mathcal{U} \times \mathbb{R}^k \longrightarrow \mathcal{U}$, i.e. $\pi_1 \circ h = \pi$ on $V|_{\mathcal{U}}$ so that the diagram

$$\begin{tikzcd} V|_{\mathcal{U}} \arrow[r, "h", "\approx"'] \arrow[d, "\pi"'] & \mathcal{U} \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ \mathcal{U} & \end{tikzcd}$$

commutes, and the restriction of $h_{\mathcal{U}}$ to each fiber is linear:

$$h(c_1v_1+c_2v_2) = c_1h(v_1) + c_2h(v_2) \in x \times \mathbb{R}^k \qquad \forall\, c_1,c_2\in\mathbb{R},\ v_1,v_2\in V_x,\ x\in\mathcal{U}.$$

These conditions imply that the restriction of h to each fiber V_x of π is an isomorphism of vector spaces. In summary, V locally (and not just pointwise) looks like bundles of $\mathbb{R}^k$'s over open sets in M glued together. This is in a sense analogous to an n-manifold being open subsets of $\mathbb{R}^n$ glued together in a nice way. Here is a formal definition.

Definition 1 *A* `real vector bundle of rank` k *is a tuple* $(M,V,\pi,\cdot,+)$ *such that*

(1) M *and* V *are smooth manifolds and* $\pi\colon V\longrightarrow M$ *is a smooth map;*

(2) $\cdot\colon \mathbb{R}\times V\longrightarrow V$ *is a map s.t.* $\pi(c\cdot v)=\pi(v)$ *for all* $(c,v)\in\mathbb{R}\times V$*;*

(3) $+\colon V\times_M V\longrightarrow V$ *is a map s.t.* $\pi(v_1+v_2)=\pi(v_1)=\pi(v_2)$ *for all* $(v_1,v_2)\in V\times_M V$*;*

(4) for every $m\in M$ *there exist a neighborhood* $\mathcal{U}$ *of* m *in* M *and a diffeomorphism* $h\colon V|_{\mathcal{U}}\longrightarrow \mathcal{U}\times\mathbb{R}^k$ *such that*

(4a) $\pi_1\circ h=\pi$ *on* $V|_{\mathcal{U}}$ *and*

(4b) the map $h|_{V_x}\colon V_x\longrightarrow x\times\mathbb{R}^k$ *is an isomorphism of vector spaces for all* $x\in\mathcal{U}$*.*

Remark: Condition (4) implies that the vector space structures in the fibers of π vary smoothly over M, i.e. the maps $\cdot$ and $+$ in (2) and (3) of Definition 1 are smooth.

The spaces M and V are called the base and the total space of the vector bundle (M,V,π). It is customary to call $\pi\colon V\longrightarrow M$ a vector bundle and V a vector bundle over M. Note that if M is an n-manifold and $V\longrightarrow M$ is a real vector bundle of rank k, then V is an $(n+k)$-manifold. Its local coordinate charts are obtained by restricting the trivialization maps h for V, as above, to small coordinate charts in M.

Example 1 If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1\colon M\times\mathbb{R}^k \longrightarrow M$$

is a real vector bundle of rank k over M. It is called `the trivial rank-`k `real vector bundle over` M.

Example 2 Let M be the circle S^1; it can be written as the quotient

$$S^1 = I/\sim, \qquad \text{where} \quad I=[0,1], \quad 0\sim 1.$$

Let V be the infinite Mobius band:

$$V = (I\times\mathbb{R})\big/\sim, \qquad \text{where} \quad (0,v)\sim(1,-v) \ \ \forall\, v\in\mathbb{R}.$$

Then, the projection $\pi\colon V\longrightarrow S^1$ onto the first coordinate is well-defined and is a real `line bundle` (i.e. rank-one bundle) over S^1.

Example 3 The `real projective space of dimension` n, denoted $\mathbb{R}P^n$, is the space of real one-dimensional subspaces of $\mathbb{R}^{n+1}$ (or lines through the origin in $\mathbb{R}^{n+1}$) in the natural quotient topology. In other words, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a nonzero vector in

$\mathbb{R}^{n+1}$, i.e. an element of $\mathbb{R}^{n+1}-0$. Two such vectors determine the same one-dimensional subspace in $\mathbb{R}^{n+1}$ and the same element of $\mathbb{R}P^n$ if and only if they differ by a non-zero scalar. Thus, as sets

$$\mathbb{R}P^n = \big(\mathbb{R}^{n+1}-0\big)/\mathbb{R}^* \equiv \big(\mathbb{R}^{n+1}-0\big)/\sim, \qquad \text{where}$$
$$\mathbb{R}^* = \mathbb{R}-0, \quad c\cdot v = cv \in \mathbb{R}^{n+1}-0 \;\; \forall\, c\in\mathbb{R}^*,\, v\in\mathbb{R}^{n+1}-0, \quad v\sim cv \;\; \forall\, c\in\mathbb{R}^*,\, v\in\mathbb{R}^{n+1}-0.$$

Alternatively, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a unit vector in $\mathbb{R}^{n+1}$, i.e. an element of S^n. Two such vectors determine the same element of $\mathbb{R}P^n$ if and only if they differ by a non-zero scalar, which in this case must necessarily be ±1. Thus, as sets

$$\mathbb{R}P^n = S^n/\mathbb{Z}_2 \equiv S^n/\sim, \qquad \text{where}$$
$$\mathbb{Z}_2 = \{\pm1\}, \quad c\cdot v = cv \in S^n \;\; \forall\, c\in\mathbb{Z}_2,\, v\in S^n, \quad v\sim cv \;\; \forall\, c\in\mathbb{Z}_2,\, v\in S^2.$$

Thus, as sets,

$$\mathbb{R}P^n = \big(\mathbb{R}^{n+1}-0\big)/\mathbb{R}^* = S^n/\mathbb{Z}_2.$$

It follows that $\mathbb{R}P^n$ has two natural quotient topologies; these two topologies are the same, however. The space $\mathbb{R}P^n$ has a natural smooth structure, induced from that of $\mathbb{R}^{n+1}-0$ and S^n. Let

$$\gamma_n = \big\{(\ell,v)\in\mathbb{R}P^n\times\mathbb{R}^{n+1} : v\in\ell\big\}.$$

The projection $\pi\colon\gamma_n\longrightarrow\mathbb{R}P^n$ defines a smooth real line bundle. The fiber over a point $\ell\in\mathbb{R}P^n$ is the *one-dimensional subspace* ℓ *of* $\mathbb{R}^{n+1}$! For this reason, γ_n is called `the tautological line bundle over` $\mathbb{R}P^n$. Note that $\mathbb{R}P^1=S^1$ and $\gamma_1\longrightarrow S^1$ is the infinite Mobius band of Example 2. *Please check all statements made here.*

Example 4 If M is a smooth n-manifold, the tangent and cotangent bundles of M, TM and T^*M, are real vector bundles of rank n over M.

Definition 2 *A* `complex vector bundle of rank` k *is a tuple* $(M,V,\pi,\cdot,+)$ *such that*

(1) M *and* V *are smooth manifolds and* $\pi\colon V\longrightarrow M$ *is a smooth map;*
(2) $\cdot\colon\mathbb{C}\times V\longrightarrow V$ *is a map s.t.* $\pi(c\cdot v)=\pi(v)$ *for all* $(c,v)\in\mathbb{C}\times V$*;*
(3) $+\colon V\times_M V\longrightarrow V$ *is a map s.t.* $\pi(v_1+v_2)=\pi(v_1)=\pi(v_2)$ *for all* $(v_1,v_2)\in V\times_M V$*;*
(4) for every $m\in M$ *there exists a neighborhood* $\mathcal{U}$ *of* m *in* M *and a diffeomorphism* $h\colon V|_{\mathcal{U}}\longrightarrow\mathcal{U}\times\mathbb{C}^k$ *such that*
(4a) $\pi_1\circ h=\pi$ *on* $V|_{\mathcal{U}}$ *and*
(4b) the map $h|_{V_x}\colon V_x\longrightarrow x\times\mathbb{C}^k$ *is an isomorphism of complex vector spaces for all* $x\in\mathcal{U}$*.*

Similarly to a real vector bundle, a complex vector bundle over M locally looks like bundles of $\mathbb{C}^k$'s over open sets in M glued together. If M is an n-manifold and $V\longrightarrow M$ is a complex vector bundle of rank k, then V is an $(n+2k)$-manifold. A complex vector bundle of rank k is also a real vector bundle of rank $2k$, but a real vector bundle of rank $2k$ need not in general admit a complex structure.

Example 5 If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1\colon M\times\mathbb{C}^k\longrightarrow M$$

is a complex vector bundle of rank k over M. It is called `the trivial rank-`k `complex vector bundle over` M.

Example 6 The `complex projective space of dimension` n, denoted $\mathbb{C}P^n$, is the space of complex one-dimensional subspaces of $\mathbb{C}^{n+1}$ (or lines through the origin in $\mathbb{C}^{n+1}$) in the natural quotient topology. Similarly to the real case,

$$\mathbb{C}P^n = \big(\mathbb{C}^{n+1}-0\big)/\mathbb{C}^* = S^{2n+1}/S^1, \qquad \text{where}$$
$$S^1 = \big\{c\in\mathbb{C}^* : |c|=1\big\}, \qquad S^{2n+1} = \big\{v\in\mathbb{C}^{n+1}-0 : |v|=1\big\},$$
$$c\cdot v = cv \in \mathbb{C}^{n+1}-0 \quad \forall\, c\in\mathbb{C}^*,\, v\in\mathbb{C}^{n+1}-0.$$

The two quotient topologies on $\mathbb{C}P^n$ arising from these quotients are again the same. The space $\mathbb{C}P^n$ has a natural complex structure, induced from that of $\mathbb{C}^{n+1}-0$. Let

$$\gamma_n = \big\{(\ell, v)\in\mathbb{C}P^n\times\mathbb{C}^{n+1} : v\in\ell\big\}.$$

The projection $\pi\colon \gamma_n \longrightarrow \mathbb{C}P^n$ defines a smooth complex line bundle. The fiber over a point $\ell\in\mathbb{C}P^n$ is the *one-dimensional complex subspace* ℓ *of* $\mathbb{C}^{n+1}$. For this reason, γ_n is called `the tautological line bundle over` $\mathbb{C}P^n$.
Please check all statements made here.

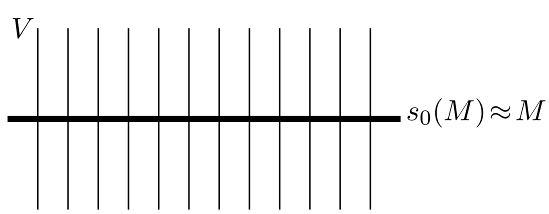
Example 7 If M is a complex n-manifold, the tangent and cotangent bundles of M, TM and T^*M, are complex vector bundles of rank n over M.

2 Vector Bundle Sections and Homomorphisms

If $\pi : V \longrightarrow M$ is a vector bundle (real or complex), `a section of` π `or` V is a smooth map $s\colon M \longrightarrow V$ such that $\pi\circ s = id_M$, i.e. $s(x)\in V_x$ for all $x\in M$. If s is a section, then $s(M)$ is an embedded submanifold of V: the injectivity of s and ds is immediate from $\pi\circ s = id_M$, while the embedding property follows from the continuity of π. Every fiber V_x of V is a vector space and thus has a distinguished element, the zero-vector in V_x, which we denote by 0_x. It follows that every vector bundle admits a section:

$$s_0(x) = (x, 0_x) \in V_x.$$

This map is smooth, since on a trivialization of V over an open subset $\mathcal{U}$ of M it is given by the inclusion of $\mathcal{U}$ as $\mathcal{U}\times 0$ into $\mathcal{U}\times\mathbb{R}^k$ or $\mathcal{U}\times\mathbb{C}^k$. Thus, M can be thought of as sitting inside of V as the zero section; it is a deformation retract of V:



Definition 3 *(1) Suppose* $\pi\colon V\longrightarrow M$ *and* $\pi'\colon V'\longrightarrow N$ *are real (or complex) vector bundles. A smooth map* $\tilde f\colon V\longrightarrow V'$ *is a* `vector bundle homomorphism` *if* $\tilde f$ *descends to a map* $f\colon M\longrightarrow N$*, i.e. the diagram*

$$\begin{tikzcd} V \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & V' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes, and the restriction $\tilde{f}\colon V_x \longrightarrow V_{f(x)}$ is linear (or $\mathbb{C}$-linear, respectively) for all $x \in M$.
(2) If $\pi\colon V \longrightarrow M$ and $\pi'\colon V' \longrightarrow M$ are vector bundles, a vector bundle homomorphism $\tilde{f}\colon V \longrightarrow V'$ is an `isomorphism of vector bundles` *if $\pi' \circ \tilde{f} = \pi$, i.e. the diagram*

$$\begin{tikzcd} V \arrow[rr, "\tilde{f}"] \arrow[dr, "\pi"'] & & V' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes, and $\tilde{f}$ is a diffeomorphism (or equivalently, its restriction to each fiber is an isomorphism of V). If such an isomorphism exists, then V and V' are said to be `isomorphic vector bundles`.

Note that the two conditions above on $\tilde{f}$ are equivalent due to the local structures of V and V'.

Lemma 4 *The real line bundle $V \longrightarrow S^1$ given by the infinite Mobius band of Example 2 is not isomorphic to the trivial line bundle $S^1 \times \mathbb{R} \longrightarrow S^1$.*

Proof: In fact, (V, S^1) is not even homeomorphic to $(S^1 \times \mathbb{R}, S^1)$. Since

$$S^1 \times \mathbb{R} - S^1 \equiv S^1 \times \mathbb{R} - S^1 \times 0 = S^1 \times \mathbb{R}^- \sqcup S^1 \times \mathbb{R}^+,$$

the space $S^1 \times \mathbb{R} - S^1$ is not connected. On the other hand, $V - S^1$ is connected. If MB is the standard Mobius Band and $S^1 \subset MB$ is the central circle, $MB - S^1$ is a deformation retract of $V - S^1$. On the other hand, the boundary of MB has only one connected component (this is the primary feature of MB) and is a deformation retract of $MB - S^1$. Thus, $V - S^1$ is connected as well.

Lemma 5 *If $\pi : V \longrightarrow M$ is a real (or complex) vector bundle of rank k, V is isomorphic to the trivial real (or complex) vector bundle of rank k over M if and only if V admits k sections $s_1, \ldots, s_k$ such that the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent (over $\mathbb{C}$, respectively) in V_x for all $x \in M$.*

Proof: We consider the real case; the proof in the complex case is nearly identical.
(1) Suppose $h\colon M \times \mathbb{R}^k \longrightarrow V$ is an isomorphism of vector bundles over M. Let $e_1, \ldots, e_k$ be the standard coordinate vectors in $\mathbb{R}^k$. Define sections $s_1, \ldots, s_k$ of V over M by

$$s_l(x) = h\big(x, e_l\big) \qquad \forall\, l = 1, \ldots, k,\ x \in M.$$

Since the maps $x \longrightarrow (x, e_l)$ are sections of $M \times \mathbb{R}^k$ over M and h is a bundle homomorphism, the maps s_l are sections of V. Since the vectors (x, e_l) are linearly independent in $x \times \mathbb{R}^k$ and h is an isomorphism on every fiber, the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent in V_x for all $x \in M$, as needed.

(2) Suppose $s_1,\ldots,s_k$ are sections of V such that the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x for all $x\in M$. Define the map

$$h\colon M\times\mathbb{R}^k \longrightarrow V \qquad \text{by} \qquad h(x,c_1,\ldots,c_k)=c_1s_1(x)+\ldots+c_ks_k(x)\in V_x.$$

Since the sections $s_1,\ldots,s_k$ and the vector space operations on V are smooth, the map h is smooth. It is immediate that $\pi(h(x,c))=x$ and the restriction of h to $x\times\mathbb{R}^k$ is linear; thus, h is a vector bundle homomorphism. Since the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x, the homomorphism h is injective and thus an isomorphism on every fiber. We conclude that h is an isomorphism between vector bundles over M.

3 Transition Data for Vector Bundles

Suppose $\pi: V\longrightarrow M$ is a real vector bundle of rank k. By Definition 1, there exists a collection $\{(\mathcal{U}_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for V such that $\bigcup_{\alpha\in\mathcal{A}}\mathcal{U}_\alpha=M$. Since $(\mathcal{U}_\alpha,h_\alpha)$ is a trivialization for V,

$$h_\alpha\colon V|_{\mathcal{U}_\alpha}\longrightarrow \mathcal{U}_\alpha\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_\alpha=\pi$ and the restriction $h_\alpha\colon V_x\longrightarrow x\times\mathbb{R}^k$ is linear for all $x\in\mathcal{U}_\alpha$. Thus, for all $\alpha,\beta\in\mathcal{A}$,

$$h_{\alpha\beta}\equiv h_\alpha\circ h_\beta^{-1}\colon \big(\mathcal{U}_\alpha\cap U_\beta\big)\times\mathbb{R}^k\longrightarrow\big(\mathcal{U}_\alpha\cap U_\beta\big)\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_{\alpha\beta}=\pi_1$, i.e. $h_{\alpha\beta}$ maps $x\times\mathbb{R}^k$ to $x\times\mathbb{R}^k$, and the restriction of $h_{\alpha\beta}$ to $x\times\mathbb{R}^k$ defines an isomorphism of $x\times\mathbb{R}^k$ with itself. Such an isomorphism must be given by

$$(x,v)\longrightarrow\big(x,g_{\alpha\beta}(x)v\big)\qquad\forall\,v\in\mathbb{R}^k,$$

for a unique element $g_{\alpha\beta}(x)\in\mathrm{GL}_k\mathbb{R}$ (the general linear group of $\mathbb{R}^k$). The map $h_{\alpha\beta}$ is then given by

$$h_{\alpha\beta}(x,v)=\big(x,g_{\alpha\beta}(x)v\big)\qquad\forall\,x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta,\ v\in\mathbb{R}^k,$$

and is completely determined by the map $g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}$ (and $g_{\alpha\beta}$ is determined by $h_{\alpha\beta}$). Since $h_{\alpha\beta}$ is smooth, so is $g_{\alpha\beta}$.

By the previous paragraph, starting with a real rank-k vector bundle $\pi: V\longrightarrow M$, we can obtain an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth transition maps

$$\big\{g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}.$$

Let I_k denote the identity element in $\mathrm{GL}_k\mathbb{R}$. These transition maps satisfy:

(i) $g_{\alpha\alpha}\equiv\mathrm{I}_k$, since $h_{\alpha\alpha}\equiv h_\alpha\circ h_\alpha^{-1}=\mathrm{id}$;

(ii) $g_{\alpha\beta}g_{\beta\alpha}\equiv\mathrm{I}_k$, since $h_{\alpha\beta}h_{\beta\alpha}=\mathrm{id}$;

(iii) $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\equiv\mathrm{I}_k$, since $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}=\mathrm{id}$.

The last condition is known as the (Čech) cocycle condition; we will encounter it again in Chapter 5. It is sometimes written as

$$g_{\alpha_1\alpha_2}g_{\alpha_0\alpha_2}^{-1}g_{\alpha_0\alpha_1}\equiv\mathrm{I}_k\qquad\forall\,\alpha_0,\alpha_1,\alpha_2\in\mathcal{A}.$$

In light of (ii), the two versions of the cocycle condition are equivalent.

Conversely, given an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\big\{g_{\alpha\beta}\colon \mathcal{U}_\alpha\cap\mathcal{U}_\beta \longrightarrow \mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfy (i), (ii), and (iii), we can assemble a rank-k vector bundle $\pi'\colon V'\longrightarrow M$ as follows. Let

$$V' = \Big(\bigsqcup_{\alpha\in\mathcal{A}} \alpha\times\mathcal{U}_\alpha\times\mathbb{R}^k\Big)\Big/\sim, \qquad \text{where}$$
$$(\beta,x,v)\sim\big(\alpha,x,g_{\alpha\beta}(x)v\big) \quad \forall\ \alpha,\beta\in\mathcal{A},\ x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta,\ v\in\mathbb{R}^k.$$

The relation $\sim$ is reflexive by (i), symmetric by (ii), and transitive by (iii) and (ii). Thus, $\sim$ is an equivalence relation, and V' carries the quotient topology. Let

$$q\colon \bigsqcup_{\alpha\in\mathcal{A}} \alpha\times\mathcal{U}_\alpha\times\mathbb{R}^k \longrightarrow V' \qquad \text{and} \qquad \pi'\colon V'\longrightarrow M, \quad [\alpha,x,v]\longrightarrow x,$$

be the quotient map and the natural projection map (which is well-defined). If $\beta\in\mathcal{A}$ and W is a subset of $\mathcal{U}_\beta\times\mathbb{R}^k$, then

$$q^{-1}\big(q(\beta\times W)\big) = \bigsqcup_{\alpha\in\mathcal{A}} \alpha\times h_{\alpha\beta}(W), \qquad \text{where}$$
$$h_{\alpha\beta}\colon \big(\mathcal{U}_\alpha\cap U_\beta\big)\times\mathbb{R}^k \longrightarrow \big(\mathcal{U}_\alpha\cap U_\beta\big)\times\mathbb{R}^k, \qquad h_{\alpha\beta}(x,v) = \big(x,g_{\alpha\beta}(x)v\big).$$

In particular, if $\beta\times W$ is an open subset of $\beta\times\mathcal{U}_\beta\times\mathbb{R}^k$, then $q^{-1}\big(q(\beta\times W)\big)$ is an open subset of $\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times\mathcal{U}_\alpha\times\mathbb{R}^k$. Thus, q is an open continuous map. Since its restriction

$$q_\alpha \equiv q|_{\alpha\times\mathcal{U}_\alpha\times\mathbb{R}^k}$$

is injective, $(q_\alpha(\alpha\times\mathcal{U}_\alpha\times\mathbb{R}^k), q_\alpha^{-1})$ is a coordinate chart on V'[1]. The overlap maps between these charts are the maps $h_{\alpha\beta}$[2]. Thus, V' carries a smooth structure. The projection map $\pi'\colon V'\longrightarrow M$ is continuous with respect to this smooth structure, since it induces projection maps on the charts. Since

$$\pi_1 = \pi'\circ q_\alpha\colon \alpha\times\mathcal{U}_\alpha\times\mathbb{R}^k \longrightarrow \mathcal{U}_\alpha\subset M,$$

the diffeomorphism q_α induces a vector space structure in V'_x for each $x\in\mathcal{U}_\alpha$ such that the restriction of q_α to each fiber is a vector space isomorphism. Since the restriction of the overlap map $h_{\alpha\beta}$ to $x\times\mathbb{R}^k$, with $x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta$, is a vector space isomorphism, the vector space structures defined on V'_x via the maps q_α and q_β are the same. We conclude that $\pi'\colon V'\longrightarrow M$ is a real vector bundle of rank k.

If $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ and $\big\{g_{\alpha\beta}\colon \mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$ are transition data arising from a vector bundle $\pi\colon V\longrightarrow M$, then the vector bundle V' constructed in the previous paragraph is isomorphic to V. Let $\{(\mathcal{U}_\alpha,h_\alpha)\}$ be the trivializations as above, giving rise to the transition functions $g_{\alpha\beta}$. We define

$$\tilde{f}\colon V\longrightarrow V' \qquad \text{by} \qquad \tilde{f}(v) = \big[\alpha,h_\alpha(v)\big] \quad \text{if } \ \pi(v)\in\mathcal{U}_\alpha.$$

[1]Strictly speaking, this is not a chart, since its image is a smooth manifold and not necessarily an open subspace of a Euclidean space. However, such generalized charts are sufficient for verifying that a space is a smooth manifold.

[2]Formally, the overlap map is $(\beta\longrightarrow\alpha)\times h_{\alpha\beta}$.

If $\pi(v)\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta$, then

$$\big[\beta, h_\beta(v)\big] = \big[\alpha, h_{\alpha\beta}(h_\beta(v))\big] = \big[\alpha, h_\alpha(v)\big] \in V',$$

i.e. the map $\tilde{f}$ is well-defined (depends only on v and not on α). It is immediate that $\pi'\circ\tilde{f}=\pi$. Since the map

$$q_\alpha^{-1}\circ\tilde{f}\circ h_\alpha^{-1}\colon \mathcal{U}_\alpha\times\mathbb{R}^k \longrightarrow \mathcal{U}_\alpha\times\mathbb{R}^k$$

is the identity (and thus smooth), $\tilde{f}$ is a smooth map. Since the restrictions of q_α and h_α to every fiber are vector space isomorphisms, it follows that so is $\tilde{f}$. We conclude that $\tilde{f}$ is a vector bundle isomorphism.

In summary, a real rank-k vector bundle over M determines a set of transition data with values in $\mathrm{GL}_k\mathbb{R}$ satisfying (i)-(iii) above (many such sets, of course) and a set of transition data satisfying (i)-(iii) determines a real rank-k vector bundle over M. These two processes are well-defined and are inverses of each other when applied to the set of equivalence classes of vector bundles and the set of equivalence classes of transition data satisfying (i)-(iii). Two vector bundles over M are defined to be equivalent if they are isomorphic as vector bundles over M. Two sets of transition data

$$\big\{g_{\alpha\beta}\big\}_{\alpha,\beta\in\mathcal{A}} \qquad\text{and}\qquad \big\{g'_{\alpha\beta}\big\}_{\alpha,\beta\in\mathcal{A}},$$

with $\mathcal{A}$ consisting of *all* sufficiently small open subsets of M, are said to be equivalent if there exists a collection of smooth functions $\{f_\alpha\colon\mathcal{U}_\alpha\longrightarrow\mathrm{GL}_k\mathbb{R}\}_{\alpha\in\mathcal{A}}$ such that

$$g_{\alpha\beta}^{-1}g'_{\alpha\beta} = f_\alpha f_\beta^{-1}, \qquad \forall\alpha,\beta\in\mathcal{A},$$ [3]

i.e. the two sets of transition data differ by a `Čech boundary` (more in Chapter 5). Along with the cocycle condition on the gluing data, this means that isomorphism classes of real rank-k vector bundles over M can be identified with $\check{H}^1(M;\mathrm{GL}_k\mathbb{R})$, the quotient of the space of Čech cocycles of degree one by the subspace of Čech boundaries.

Remark: As we will see in Chapter 5, Čech cohomology groups, $\check{H}^m$, are normally defined for (sheafs of) abelian groups. However, the first two groups, $\check{H}^0$ and $\check{H}^1$ easily generalize to non-abelian groups as well.

If $\pi\colon V\longrightarrow M$ is a complex rank-k vector bundle over M, we can similarly obtain transition data for V consisting of an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\big\{g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathrm{GL}_k\mathbb{C}\big\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfies (i)-(iii). Conversely, given such transition data, we can construct a complex rank-k vector bundle over M. The set of isomorphism classes of complex rank-k vector bundles over M can be identified with $\check{H}^1(M;\mathrm{GL}_k\mathbb{C})$.

[3] Such a collection $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ corresponds, via trivializations, to an isomorphism between the vector bundles determined by $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ and $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$.

4 Operations on Vector Bundles

Vector bundles can be restricted to smooth submanifolds and pulled back by smooth maps. All natural operations on vector spaces, such as taking quotient vector space, dual vector space, direct sum of vector spaces, tensor product of vector spaces, and exterior powers also carry over to vector bundles via transition functions.

4.1 Restrictions and Pullbacks

If N is a smooth manifold, $M \subset N$ is an embedded submanifold, and $\pi\colon V \longrightarrow N$ is a vector bundle of rank k (real or complex) over N, then its restriction to M,

$$\pi\colon V|_M \equiv \pi^{-1}(M) \longrightarrow M,$$

is a vector bundle of rank k over N. It inherits smooth structure from V by the Slice Lemma or the Implicit Function Theorem. If $\{(\mathcal{U}_\alpha, h_\alpha)\}$ is a collection of trivializations for $V \longrightarrow N$, then $\{(\mathcal{U}_\alpha \cap M, h_\alpha|_{\pi^{-1}(\mathcal{U}\cap M)})\}$ is a collection of trivializations for $V|_M \longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V \longrightarrow N$, then $\{g_{\alpha\beta}|_{\mathcal{U}_\alpha\cap\mathcal{U}_\beta\cap M}\}$ is transition data for $V|_M \longrightarrow M$.

More generally, if $f\colon M \longrightarrow N$ is a smooth map and $\pi\colon V \longrightarrow N$ is a vector bundle of rank k, there is a pullback bundle over M:

$$f^*V \equiv \big\{(m,v) \in M \times V : f(m) = \pi(v)\big\}.$$

Note that V is the maximal subspace of $M \times V$ so that the diagram

$$\begin{tikzcd} f^*V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & V \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. By the Implicit Function Theorem, f^*V is a smooth submanifold of $M \times V$. By construction, the fiber of π_1 over $m \in M$ is $V_{f(m)}$, i.e. the fiber of π over $f(m) \in N$. If $\{(\mathcal{U}_\alpha, h_\alpha)\}$ is a collection of trivializations for $V \longrightarrow N$, then $\{(f^{-1}(\mathcal{U}_\alpha), h_\alpha \circ f)\}$ is a collection of trivializations for $f^*V \longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V \longrightarrow N$, then $\{g_{\alpha\beta} \circ f\}$ is transition data for $f^*V \longrightarrow M$. The case discussed in the previous paragraph corresponds to f being the inclusion map. If $f\colon M \longrightarrow N$ is a smooth map, then df defines a bundle homomorphism from TM to f^*TN:

$$\begin{tikzcd} TM \arrow[rr, "df"] \arrow[dr, "\pi"'] & & f^*TN \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

The smooth map f is an *immersion* if the restriction of df to every fiber of π is injective.

4.2 Quotient Bundles

If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and $V' \subset V$ is a linear subspace, then we can form the quotient vector space, V/V'. If W is another vector space, $W' \subset W$ is a linear subspace, and $g\colon V \longrightarrow W$ is a linear map such that $g(V')=W'$, then g descends to a linear map between the quotient spaces:

$$\bar{g}\colon V/V' \longrightarrow W/W'.$$

If we choose bases for V and W such that the first few vectors in each basis form bases for V' and W', then the matrix for g with respect to these bases is of the form:

$$g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

The matrix for $\bar{g}$ is then D. If g is an isomorphism from V to W that restricts to an isomorphism from V' to W', then $\bar{g}$ is an isomorphism from V/V' to W/W'.

Suppose $\pi\colon V \longrightarrow M$ is a smooth vector bundle of rank k (say, over $\mathbb{R}$). A `subbundle of` V `of rank` k' is a smooth submanifold V' of V such that

$$\pi|_{V'}\colon V' \longrightarrow M$$

is a vector bundle of rank k'. Of course, $k' \le k$. If $V' \subset V$ is a subbundle, we can form a quotient bundle, $V/V' \longrightarrow M$, such that

$$(V/V')_m = V_m/V'_m \qquad \forall\, m \in M.$$

The topology and smooth structure on V/V' are determined from those of V and V' by requiring that if s is a smooth section of V, then the induced section of V/V' is also smooth. More explicitly, we can choose a system of trivializations $\{(\mathcal{U}_\alpha, h_\alpha)\}$ such that

$$h_\alpha(V') = \mathcal{U}_\alpha \times (\mathbb{R}^{k'} \times 0) \subset \mathcal{U}_\alpha \times \mathbb{R}^k \qquad \forall \alpha.$$

Let $q_{k'}\colon \mathbb{R}^k \longrightarrow \mathbb{R}^{k-k'}$ be the projection onto the last $(k-k')$ coordinates. Then, the trivializations for V/V' are given by $\{(\mathcal{U}_\alpha, \{\mathrm{id}\times q_{k'}\}\circ h_\alpha)\}$. Alternatively, if $\{g_{\alpha\beta}\}$ is transition data for V such that the upper-left $k'\times k'$-submatrices of $g_{\alpha\beta}$ correspond to V' (as is the case for the above trivializations h_α) and $\bar{g}_{\alpha\beta}$ is the lower-right $(k-k')\times(k-k')$ matrix of $g_{\alpha\beta}$, then $\{\bar{g}_{\alpha\beta}\}$ is transition data for V/V'.

For example, if N is a k-manifold and $M \subset N$ is a k'-submanifold, then $TN|_M \longrightarrow M$ is a vector bundle of rank k containing the subbundle TN. The quotient bundle in this case,

$$\mathcal{N}_M \equiv TN|_M/TM,$$

is called `the normal bundle of` M `in` N. It describes a neighborhood of M in N. More generally, if $f\colon M \longrightarrow N$ is an immersion, the image of df is a subbundle of f^*TN; see above. In this case, the quotient bundle,

$$\mathcal{N}_f \equiv f^*TN\big/\mathrm{Im}\, df,$$

is called `the normal bundle for the immersion` f. It is a vector bundle over M.

4.3 Dual Bundles

If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$), the dual vector space is the space of the linear homomorphisms to the field ($\mathbb{R}$ or $\mathbb{C}$, respectively):

$$V^* = \mathrm{Hom}\big(V, \mathbb{R}\big) \qquad \text{or} \qquad V^* = \mathrm{Hom}\big(V, \mathbb{C}\big).$$

A linear map $g: V \longrightarrow W$ between two vector spaces, induces a dual map in the "opposite" direction:

$$g^* : W^* \longrightarrow V^*, \qquad \{g^*(L)\}(v) = L\big(g(v)\big) \quad \forall\, L \in W^*,\ v \in V.$$

If $V = \mathbb{R}^k$ and $W = \mathbb{R}^n$, then g is given by an $n \times k$-matrix, and its dual is given by the transposed $k \times n$-matrix.

If $\pi: V \longrightarrow M$ is a smooth vector bundle of rank k (say, over $\mathbb{R}$), the dual bundle of V is a vector bundle $V^* \longrightarrow M$ such that

$$(V^*)_m = V_m^* \qquad \forall\, m \in M.$$

The topology and smooth structure on V^* are determined from those of V by requiring that if s and ψ are smooth sections of V and V^*, then $\psi(s)$ is a smooth function on M. More explicitly, suppose $\{g_{\alpha\beta}\}$ is transition data for V, i.e. the transitions between charts are given by

$$h_\alpha \circ h_\beta^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^k \longrightarrow \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^k, \qquad (m, v) \longrightarrow \big(m, g_{\alpha\beta}(m)v\big).$$

The dual transition maps are then given by

$$\mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^k \longrightarrow \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^k, \qquad (m, v) \longrightarrow \big(m, g_{\alpha\beta}(m)^t v\big).$$

However, these maps reverse the direction, i.e. they go from the α-side to the β-side. To fix this problem, we simply take the inverse of $g_{\alpha\beta}(m)^t$:

$$\mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^k \longrightarrow \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathbb{R}^k, \qquad (m, v) \longrightarrow \big(m, \{g_{\alpha\beta}(m)^t\}^{-1} v\big).$$

So, transition data for V^* is $\{(g_{\alpha\beta}^t)^{-1}\}$. As an example, if V is a line bundle, then $g_{\alpha\beta}$ is a smooth nowhere-zero function on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $(g^*)_{\alpha\beta}$ is the smooth function given by $1/g_{\alpha\beta}$.

4.4 Direct Sums

If V and V' are two vector spaces, we can form a new vector space, $V \oplus V'$, the direct sum of V and V'. If $g: V \longrightarrow W$ and $g': V' \longrightarrow W'$ are linear maps, they induce a linear map

$$g \oplus g' : V \oplus V' \longrightarrow W \oplus W'.$$

If we choose bases for V, V', W, and W' so that g and g' correspond to some matrices A and D, then in the induced bases for $V \oplus V'$ and $W \oplus W'$,

$$g \oplus g' = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

If $\pi: V \longrightarrow M$ and $\pi': V' \longrightarrow M$ are smooth vector bundles, we can form their direct sum, $V \oplus V'$, so that

$$(V \oplus V')_m = V_m \oplus V'_m \qquad \forall\, m \in M.$$

The topology and smooth structure on $V \oplus V'$ are determined from those of V and V' by requiring that if s and s' are smooth sections of V and V', then $s \oplus s'$ is a smooth section of $V \oplus V'$. More explicitly, suppose $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and V'. Then, transition data for

$V \oplus V'$ is given by $\{g_{\alpha\beta} \oplus g'_{\alpha\beta}\}$, i.e. we put the first matrix in the top left corner and the second matrix in the bottom right corner. Alternatively, let

$$d\colon M \longrightarrow M \times M, \qquad d(m) = (m, m),$$

be the diagonal embedding. Then,

$$\pi \times \pi' \colon V \times V' \longrightarrow M \times M'$$

is a smooth vector bundle (with the product structure), and

$$V \oplus V' = d^*(V \times V').$$

Please check that the two definitions of $V \oplus V'$ agree!

If $V, V' \longrightarrow M$ are vector bundles, then V and V' are vector subbundles of $V \oplus V'$. It is immediate from the previous subsection that

$$\big(V \oplus V'\big)/V = V' \qquad \text{and} \qquad \big(V \oplus V'\big)/V' = V.$$

These equalities hold in the holomorphic category as well (i.e. when the bundles and the base manifold carry complex structures and all trivializations and transition maps are holomorphic). Conversely, if W is a subbundle of V, one can show that

$$V \approx (V/W) \oplus W$$

as smooth vector bundles, real or complex (more later). However, if V and W are holomorphic bundles, V may not have the same holomorphic structure as $(V/W) \oplus W$ (i.e. the two bundles are isomorphic as smooth vector bundles, but not as holomorphic ones).

4.5 Tensor Products

If V and V' are two vector spaces, we can form a new vector space, $V \otimes V'$, the tensor product of V and V'. If $g\colon V \longrightarrow W$ and $g'\colon V' \longrightarrow W'$ are linear maps, they induce a linear map

$$g \otimes g' \colon V \otimes V' \longrightarrow W \otimes W'.$$

If we choose bases $\{e_j\}$, $\{e'_n\}$, $\{f_i\}$, and $\{f'_m\}$ for V, V', W, and W', respectively, then $\{e_j \otimes e'_n\}_{(j,n)}$ and $\{f_i \otimes f'_m\}_{(i,m)}$ are bases for $V \otimes V'$ and $W \otimes W'$. If the matrices for g and g' with respect to the chosen bases for V, V', W, and W' are $(g_{ij})_{i,j}$ and $(g'_{mn})_{m,n}$, then the matrix for $g \otimes g'$ with respect to the induced bases for $V \otimes V'$ and $W \otimes W'$ is $(g_{ij} g'_{mn})_{(i,m),(j,n)}$. The rows of this matrix are indexed by the pairs (i, m) and the columns by the pairs (j, n). In order to actually write down the matrix, we need to order all pairs (i, m) and (j, n). If all four vector spaces are one-dimensional, g and g' correspond to single numbers g_{ij} and g'_{mn}, while $g \otimes g'$ corresponds to the single number $g_{ij} g'_{mn}$.

If $\pi : V \longrightarrow M$ and $\pi' : V' \longrightarrow M$ are smooth vector bundles, we can form their tensor product, $V \otimes V'$, so that

$$(V \otimes V')_m = V_m \otimes V'_m \qquad \forall\, m \in M.$$

The topology and smooth structure on $V \otimes V'$ are determined from those of V and V' by requiring that if s and s' are smooth sections of V and V', then $s \otimes s'$ is a smooth section of $V \otimes V'$. More explicitly, suppose $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and V'. Then, transition data for $V\otimes V'$ is given by $\{g_{\alpha\beta}\otimes g'_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $g_{\alpha\beta}\otimes g'_{\alpha\beta}$ from $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ as in the previous paragraph. As an example, if V and V' are line bundles, then $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are smooth nowhere-zero functions on $\mathcal{U}_\alpha\cap\mathcal{U}_\beta$ and $(g\otimes g')_{\alpha\beta}$ is the smooth function given by $g_{\alpha\beta}g'_{\alpha\beta}$.

4.6 Exterior Products

If V is a vector space and k is a nonnegative integer, we can form the kth exterior power, $\Lambda^k V$, of V. A linear map $g\colon V\longrightarrow W$ induces a linear map

$$\Lambda^k g\colon \Lambda^k V \longrightarrow \Lambda^k W.$$

If n is a nonnegative integer, let $S_k(n)$ be the set of increasing k-tuples of integers between 1 and n:

$$S_k(n) = \big\{(i_1,\ldots,i_k)\in\mathbb{Z}^k\colon 1\le i_1<i_2<\ldots< i_k\le n\big\}.$$

If $\{e_j\}_{j=1,\ldots,n}$ and $\{f_i\}_{i=1,\ldots,m}$ are bases for V and W, then $\{e_\eta\}_{\eta\in S_k(n)}$ and $\{f_\mu\}_{\mu\in S_k(m)}$ are bases for $\Lambda^k V$ and $\Lambda^k W$, where

$$e_{(\eta_1,\ldots,\eta_k)} = e_{\eta_1}\wedge\ldots\wedge e_{\eta_k} \qquad\text{and}\qquad f_{(\mu_1,\ldots,\mu_k)} = f_{\mu_1}\wedge\ldots\wedge f_{\mu_k}.$$

If $(g_{ij})_{i=1,\ldots,m,j=1,\ldots,n}$ is the matrix for g with respect to the chosen bases for V and W, then

$$\big(\det\big((g_{\mu_r\mu_s})_{r,s=1,\ldots,k}\big)\big)_{(\mu,\eta)\in I_k(m)\times I_k(n)}$$

is the matrix for $\Lambda^k g$ with respect to the induced bases for $\Lambda^k V$ and $\Lambda^k W$. The rows and columns of this matrix are indexed by the sets $S_k(m)$ and $S_k(n)$, respectively. The (μ,η)-entry of the matrix is the determinant of the $k\times k$-submatrix of $(g_{ij})_{i,j}$ with the rows and columns indexed by the entries of μ and η, respectively. In order to actually write down the matrix, we need to order the sets $S_k(m)$ and $S_k(n)$. If $k=m=n$, then $\Lambda^k V$ and $\Lambda^k W$ are one-dimensional vector spaces, called the `top exterior power of` V and W, with bases

$$\big\{e_1\wedge\ldots\wedge e_k\big\} \qquad\text{and}\qquad \big\{f_1\wedge\ldots\wedge f_k\big\}.$$

With respect to these bases, the homomorphism $\Lambda^k g$ corresponds to the number $\det(g_{ij})_{i,j}$. If $k>n$ (or $k>m$), then $\Lambda^k V$ (or $\Lambda^k W$) is the zero vector space and the corresponding matrix is empty.

If $\pi\colon V\longrightarrow M$ is a smooth vector bundle, we can form its kth exterior power, $\Lambda^k V$, so that

$$(\Lambda^k V)_m = \Lambda^k V_m \qquad \forall\, m\in M.$$

The topology and smooth structure on $\Lambda^k V$ are determined from those of $\Lambda^k V$ by requiring that if $s_1,\ldots,s_k$ are smooth sections of V, then $s_1\wedge\ldots\wedge s_k$ is a smooth section of $\Lambda^k V$. More explicitly, suppose $\{g_{\alpha\beta}\}$ is transition data for V. Then, transition data for $\Lambda^k V$ is given by $\{\Lambda^k g_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $\Lambda^k g_{\alpha\beta}$ from each matrix $g_{\alpha\beta}$ as in the previous paragraph. As an example, if the rank of V is k, then the transition data for the line bundle $\Lambda^k V$, called the `top exterior power of` V, is $\{\det g_{\alpha\beta}\}$.

It follows directly from the definitions that if $V \longrightarrow M$ is a vector bundle of rank k and $L \longrightarrow M$ is a line bundle (vector bundle of rank one), then

$$\Lambda^{\mathrm{top}}(V \oplus L) \equiv \Lambda^{\mathrm{top}}(V \oplus L) = \Lambda^{\mathrm{top}} V \otimes L \equiv \Lambda^k V \otimes L.$$

More generally, if $V, W \longrightarrow M$ are any two vector bundles, then

$$\Lambda^{\mathrm{top}}(V \oplus W) = (\Lambda^{\mathrm{top}} V) \otimes (\Lambda^{\mathrm{top}} W) \qquad \text{and} \qquad \Lambda^k(V \oplus W) = \bigoplus_{i+j=k} (\Lambda^i V) \otimes (\Lambda^j W).$$

Remark: For complex vector bundles, the above constructions (exterior power, tensor product, direct sum, etc.) are always done over $\mathbb{C}$, unless specified otherwise. So if V is a complex vector bundle of rank k over M, the top exterior power of V is the complex line bundle $\Lambda^k V$ over M (could also be denoted as $\Lambda^k_{\mathbb{C}} V$). In contrast, if we forget the complex structure of V (so that it becomes a real vector bundle of rank $2k$), then its top exterior power is the real line bundle $\Lambda^{2k} V$ (could also be denoted as $\Lambda^{2k}_{\mathbb{R}} V$).

5 Metrics on Fibers

If V is a vector space over $\mathbb{R}$, a `positive-definite inner-product` on V is a symmetric bilinear map

$$\langle \cdot, \cdot \rangle \colon V \times V \longrightarrow \mathbb{R}, \quad (v, w) \longrightarrow \langle v, w \rangle, \qquad \text{s.t.} \qquad \langle v, v \rangle > 0 \;\; \forall\, v \in V - 0.$$

If $\langle , \rangle$ and $\langle , \rangle'$ are positive-definite inner-products on V and $a, a' \in \bar{\mathbb{R}}^+$ are not both zero, then

$$a\langle , \rangle + a'\langle , \rangle' \colon V \times V \longrightarrow \mathbb{R}, \qquad \big\{ a\langle , \rangle + a'\langle , \rangle' \big\}(v, w) = a\langle v, w \rangle + a'\langle v, w \rangle',$$

is also a positive-definite inner-product. If W is a subspace of V and $\langle , \rangle$ is a positive-definite inner-product on V, let

$$W^\perp = \big\{ v \in V : \langle v, w \rangle = 0 \;\forall\, w \in W \big\}$$

be the orthogonal complement of W in V. In particular,

$$V = W \oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi \colon V \longrightarrow V/W$$

induces an isomorphism from $W^\perp$ to V/W so that

$$V \approx W \oplus (V/W).$$

If M is a smooth manifold and $V \longrightarrow M$ is a smooth real vector bundle of rank k, a `Riemannian metric` on V is a positive-definite inner-product in each fiber $V_x \approx \mathbb{R}^k$ of V that varies smoothly with $x \in M$. More formally, the smoothness requirement is one of the following equivalent conditions:

(a) the map $\langle,\rangle\colon V\times_M V \longrightarrow \mathbb{R}$ is smooth;
(b) the section $\langle,\rangle$ of the vector bundle $(V\otimes V)^* \longrightarrow M$ is smooth;
(c) if s_1, s_2 are smooth sections of the vector bundle $V\longrightarrow M$, then the map

$$\big\langle s_1, s_2\big\rangle\colon M \longrightarrow \mathbb{R}, \qquad m \longrightarrow \big\langle s_1(m), s_2(m)\big\rangle,$$

is smooth;
(d) if $h\colon V|_{\mathcal{U}} \longrightarrow \mathcal{U}\times\mathbb{R}^k$ is a trivialization of V, then the matrix-valued function,

$$B\colon \mathcal{U} \longrightarrow \mathrm{Mat}_k\mathbb{R}, \quad \text{s.t.} \quad \big\langle h^{-1}(m,v), h^{-1}(m,w)\big\rangle = v^t B(m) w \quad \forall\, m\in\mathcal{U},\ v,w\in\mathbb{R}^k,$$

is smooth.

Every real vector bundle admits a Riemannian metric. Such a metric can be constructed by covering M by a locally finite collection of trivializations for V and patching together positive-definite inner-products on each trivialization using a partition of unity. If W is a subspace of V and $\langle,\rangle$ is a Riemannian metric on V, let

$$W^\perp = \big\{v\in V\colon \langle v,w\rangle = 0\ \forall\, w\in W\big\}$$

be the orthogonal complement of W in V. Then $W^\perp \longrightarrow M$ is a vector subbundle of V and

$$V = W\oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V \longrightarrow V/W$$

induces a vector bundle isomorphism from $W^\perp$ to V/W so that

$$V \approx W\oplus (V/W).$$

If V is a vector space over $\mathbb{C}$, a `nondegenerate Hermitian inner-product` on V is a map

$$\langle\cdot,\cdot\rangle\colon V\times V \longrightarrow \mathbb{C}, \quad (v,w) \longrightarrow \langle v,w\rangle,$$

which is $\mathbb{C}$-antilinear in the first input, $\mathbb{C}$-linear in the second input,

$$\langle w,v\rangle = \overline{\langle v,w\rangle} \qquad \text{and} \qquad \langle v,v\rangle > 0 \ \ \forall\, v\in V-0.$$

If $\langle,\rangle$ and $\langle,\rangle'$ are nondegenerate Hermitian inner-products on V and $a, a'\in\bar{\mathbb{R}}^+$ are not both zero, then $a\langle,\rangle + a'\langle,\rangle'$ is also a nondegenerate Hermitian inner-product on V. If W is a complex subspace of V and $\langle,\rangle$ is a nondegenerate Hermitian inner-product on V, let

$$W^\perp = \big\{v\in V\colon \langle v,w\rangle = 0\ \forall\, w\in W\big\}$$

be the orthogonal complement of W in V. In particular,

$$V = W\oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V \longrightarrow V/W$$

induces an isomorphism from $W^\perp$ to V/W so that

$$V \approx W \oplus (V/W).$$

If M is a smooth manifold and $V \longrightarrow M$ is a smooth complex vector bundle of rank k, a `Hermitian metric` on V is a nondegenerate Hermitian inner-product in each fiber $V_x \approx \mathbb{C}^k$ of V that varies smoothly with $x \in M$. More formally, the smoothness requirement is one of the following equivalent conditions:

(a) the map $\langle,\rangle\colon V \times_M V \longrightarrow \mathbb{C}$ is smooth;

(b) the section $\langle,\rangle$ of the vector bundle $(V \otimes_{\mathbb{R}} V)^* \longrightarrow M$ is smooth;

(c) if s_1, s_2 are smooth sections of the vector bundle $V \longrightarrow M$, then the function $\langle s_1, s_2 \rangle$ on M is smooth;

(d) if $h\colon V|_{\mathcal{U}} \longrightarrow \mathcal{U} \times \mathbb{C}^k$ is a trivialization of V, then the matrix-valued function,

$$B\colon \mathcal{U} \longrightarrow \mathrm{Mat}_k\mathbb{C}, \quad \text{s.t.} \quad \big\langle h^{-1}(m,v), h^{-1}(m,w) \big\rangle = \bar{v}^t B(m) w \quad \forall\, m \in M,\ v, w \in \mathbb{C}^k,$$

is smooth.

Similarly to the real case, every complex vector bundle admits a Hermitian metric. If W is a subspace of V and $\langle,\rangle$ is a Hermitian metric on V, let

$$W^\perp = \big\{ v \in V : \langle v, w \rangle = 0\ \forall\, w \in W \big\}$$

be the orthogonal complement of W in V. Then $W^\perp \longrightarrow M$ is a complex vector subbundle of V and

$$V = W \oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V \longrightarrow V/W$$

induces an isomorphism of complex vector bundles over M so that

$$V \approx W \oplus (V/W).$$

If $V \longrightarrow M$ is a real vector bundle of rank k with a Riemannian metric $\langle,\rangle$ or a complex vector bundle of rank k with a Hermitian metric $\langle,\rangle$, let

$$SV \equiv \big\{ v \in V : \langle v, v \rangle = 1 \big\} \longrightarrow M$$

be `the sphere bundle of` V. In the first case, the fiber of SV over every point of M is S^{k-1}. Furthermore, if $\mathcal{U}$ is a small open subset of M, then $SV|_{\mathcal{U}} \approx \mathcal{U} \times S^{k-1}$ as bundles over $\mathcal{U}$, i.e. SV is an S^{k-1}`-fiber bundle over` M. In the complex case, SV is an S^{2k-1}-fiber bundle over M. If $V \longrightarrow M$ is a real line bundle (vector bundle of rank one) with a Riemannian metric $\langle,\rangle$, then $SV \longrightarrow M$ is an S^0-fiber bundle. In particular, if $\mathcal{U}$ is a small open subset of M, $SV|_{\mathcal{U}}$ is diffeomorphic to $\mathcal{U} \times \{\pm 1\}$. Thus, $SV \longrightarrow M$ is a 2:1-covering map. If M is connected, the covering space SV is connected if and only if V is *not* orientable; see below.

6 Orientations

If V is a real vector space of dimension k, the top exterior power of V, i.e.

$$\Lambda^{\mathrm{top}}V \equiv \Lambda^k V$$

is a one-dimensional vector space. Thus, $\Lambda^{\mathrm{top}}V-0$ has exactly two connected components. An `orientation on` V is a component $\mathcal{C}$ of V. If $\mathcal{C}$ is an orientation on V, then a basis $\{e_i\}$ for V is called `oriented` (with respect to $\mathcal{C}$) if

$$e_1\wedge\ldots\wedge e_k \in \mathcal{C}.$$

If $\{f_j\}$ is another basis for V and A is the change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, i.e.

$$(f_1,\ldots,f_k)=(e_1,\ldots,e_k)A \qquad\Longleftrightarrow\qquad f_j=\sum_{i=1}^{i=k}A_{ij}e_i,$$

then

$$f_1\wedge\ldots\wedge f_k=(\det A)e_1\wedge\ldots\wedge e_k.$$

Thus, two different bases for V belong to the same orientation on V if and only of the determinant of the corresponding change-of-basis matrix is positive.

Suppose $V\longrightarrow M$ is a real vector bundle of rank k. An `orientation for` V is an orientation for each fiber $V_x\approx\mathbb{R}^k$, which varies smoothly (or continuously, or is locally constant) with $x\in M$. This means that if

$$h\colon V|_{\mathcal{U}}\longrightarrow \mathcal{U}\times\mathbb{R}^k$$

is a trivialization of V and $\mathcal{U}$ is connected, then h is either orientation-preserving or orientation-reversing (with respect to the standard orientation of $\mathbb{R}^k$) on every fiber. If V admits an orientation, V is called `orientable`.

Lemma 6 *Suppose $V\longrightarrow M$ is a smooth real vector bundle.*
(1) V is orientable if and only if there exists a collection $\{\mathcal{U}_\alpha,h_\alpha\}$ of trivializations that covers M such that

$$\det g_{\alpha\beta}\colon \mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathbb{R}^+,$$

where $\{g_{\alpha\beta}\}$ is the corresponding transition data.
(2) V is orientable if and only if the line bundle $\Lambda^{\mathrm{top}}V\longrightarrow M$ is orientable.
(3) If V is a line bundle, V is orientable if and only if V is (isomorphic to) the trivial line bundle $M\times\mathbb{R}$.
(4) If V is a line bundle with a Riemannian metric $\langle,\rangle$, V is orientable if and only if SV is not connected.

Proof: (1) If V has an orientation, we can choose a collection $\{\mathcal{U}_\alpha,h_\alpha\}$ of trivializations that covers M such that the restriction of h_α to each fiber is orientation-preserving (if it is orientation-preserving, simply multiply its first component by -1). Then, the corresponding transition data $\{g_{\alpha\beta}\}$ is orientation-preserving, i.e.

$$\det g_{\alpha\beta}\colon \mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathbb{R}^+.$$

Conversely, suppose $\{\mathcal{U}_\alpha, h_\alpha\}$ is a collection of trivializations that covers M such that

$$\det g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathbb{R}^+.$$

Then, if $x \in \mathcal{U}_\alpha$ for some α, define an orientation on V_x by requiring that

$$h_\alpha : V_x \longrightarrow x \times \mathbb{R}^k$$

is orientation-preserving. Since $\det g_{\alpha\beta}$ is $\mathbb{R}^+$-valued, the orientation on V_x is independent of α such that $x \in \mathcal{U}_\alpha$. Each of the trivializations h_α is then orientation-preserving on each fiber.
(2) An orientation for V is the same as an orientation for Λ^{top}, since

$$\Lambda^{\text{top}} V = \Lambda^{\text{top}}\big(\Lambda^{\text{top}} V\big).$$

Furthermore, if $\{(\mathcal{U}_\alpha, h_\alpha\}$ is a collection of trivializations for V such that the corresponding transition functions $g_{\alpha\beta}$ have positive determinant, then $\{(\mathcal{U}_\alpha, \Lambda^{\text{top}} h_\alpha\}$ is a collection of trivialization for $\Lambda^{\text{top}} V$ such that the corresponding transition functions $\Lambda^{\text{top}} g_{\alpha\beta}$ have positive determinant as well[4].
(3) The trivial line bundle $M \times \mathbb{R}$ is orientable, with an orientation determined by the standard orientation on $\mathbb{R}$. Thus, if V is isomorphic to the trivial line bundle, then V is orientable. Conversely, suppose V is an oriented line bundle. For each $x \in M$, let

$$\mathcal{C}_x \subset \Lambda^{\text{top}} V = V$$

be the chosen orientation of the fiber. Choose a Riemannian metric on V and define a section s of V by requiring that for all $x \in M$

$$\big\langle s(x), s(x) \big\rangle = 1 \qquad \text{and} \qquad s(x) \in \mathbb{C}_x.$$

This section is well-defined and smooth (as can be seen by looking on a trivialization). Since it does not vanish, the line bundle V is trivial by Lemma 5.
(4) If V is orientable, then V is isomorphic to $M \times \mathbb{R}$, and thus

$$SV = S(M \times \mathbb{R}) = M \times S^0 = M \sqcup M$$

is not connected. Conversely, if M is connected and SV is not connected, let SV^+ be one of the components of V. Since $SV \longrightarrow M$ is a covering projection, so is $SV^+ \longrightarrow M$. Since the latter is one-to-one, it is a diffeomorphism, and its inverse determines a nowhere-zero section of V. Thus, V is isomorphic to the trivial line bundle by Lemma 5.

If V is a complex vector space of dimension k, V has a canonical orientation as a real vector space of dimension $2k$. If $\{e_i\}$ is a basis for V over $\mathbb{C}$, then

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\}$$

is a basis for V over $\mathbb{R}$. The orientation determined by such a basis is the canonical orientation for the underlying real vector space V. If $\{f_j\}$ is another basis for V over $\mathbb{C}$, B is the complex change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, A is the real change-of-basis matrix from

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\} \qquad \text{to} \qquad \big\{f_1, \mathrm{i}f_1, \ldots, f_k, \mathrm{i}f_k\big\},$$

then

$$\det A = (\det B)\overline{\det B} \in \mathbb{R}^+.$$

Thus, the two bases over $\mathbb{R}$ induced by complex bases for V determine the same orientation for V. This implies that every complex vector bundle $V \longrightarrow M$ is orientable as a real vector bundle.

[4]The transition functions $\Lambda^{\text{top}} g_{\alpha\beta}$ are simply $\det g_{\alpha\beta}$.