MAT531 Geometry/Topology Final Exam Review Sheet

Program of the final exam. The final exam will consist of problems on the following:

- Smooth manifolds, atlases, smooth structures, smooth submanifolds, manifolds with boundaries.
- Smooth functions, smooth maps, immersions, submersions, embeddings. The Inverse and Implicit Function theorems.
- Tangent vectors, tangent spaces, (smooth) vector fields, commutators of vector fields, flows of vector fields.
- Tensors, tensor fields, differential forms.
- Integration of differential forms, the Stokes theorem.
- Distributions, the Frobenius integrability theorem (two versions: via vector fields and via 1-forms). Foliations, fibrations.
- Closed and exact forms. Poincaré lemma. De Rham cohomology. The Mayer-Vietoris exact sequence.

Some key definitions (not all of them!) are gathered below for your reference:

Smooth manifolds. The central notion of the course is that of a *smooth manifold*. A Hausdorff second countable topological space X is called a *smooth manifold*, if it is equipped with a *smooth atlas* consisting of *coordinate charts* that cover X. A *coordinate chart* is an open subset $U \subseteq X$ together with a homeomorphism $\phi : U \to V \subseteq \mathbb{R}^n$ of U to an open subset V of $\mathbb{R}^n$. An atlas is a collection of coordinate charts (U_α, ϕ_α) such that $X = \bigcup_\alpha U_\alpha$. If two coordinate charts U_α and U_β intersect, then we can define the *transition function*

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta).$$

An atlas is *smooth* (C^∞-differentiable) if all transition functions are smooth. A smooth atlas is also called a *smooth structure*. Thus any topological space homeomorphic to a smooth manifold, has a smooth structure (e.g. any polygon, any polyhedron, etc.)

Two atlases are *compatible* if their union is also an atlas. We say that compatible atlases define the same smooth structure. A topological space can have (and usually has) many different smooth structures: to obtain a different smooth structure on X, it is enough to apply any homeomorphism $X \to X$ that is not a diffeomorphism.

A smooth manifold *with boundary* is defined in almost the same way. We only need to replace $\mathbb{R}^n$ with the closed upper half-space. The *boundary* of X is defined as a set of points $x \in X$ that are mapped to the boundary of the upper half-space under coordinate maps.

Smooth geometric objects. *Smoothness* of any geometric object (a map, a vector field, a tensor field, a differential form, etc) on a manifold means the following: the object restricted to each coordinate chart is given by a sequence of coordinates (which are just functions on an open subset of $\mathbb{R}^n$), and these coordinates are supposed to be smooth. Of course, a rigorous definition should be given in each particular case. E.g. a function $f : X \to \mathbb{R}$ is smooth if $f \circ \phi_\alpha^{-1}$ is a smooth function on U_α for any coordinate chart (U_α, ϕ_α).

Tangent spaces. A *tangent vector* to a smooth manifold X at a point $x \in X$ is defined as an equivalence class of curves passing through x, OR as a derivation of smooth functions on X. The two definitions are equivalent. Two smooth curves $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to X$ such that $\gamma_1(0) = \gamma_2(0) = x$ are *equivalent* if

$$\frac{d}{dt}(\phi \circ \gamma_1(t))|_{t=0} = \frac{d}{dt}(\phi \circ \gamma_2(t))|_{t=0}$$

for some coordinate chart (U, ϕ) such that $x \in U$. This equality is independent of the choice of a coordinate chart: if it is true in some coordinate chart, then it is true in any other. A *derivation* at $x \in X$ is a linear functional $D : C^\infty(X) \to \mathbb{R}$ such that

$$D(fg) = f(x)(Dg) + (Df)g(x).$$

All tangent vectors at a given point $x \in X$ form the *tangent space* T_xX to X at x. This is a vector space.

Any smooth map $f : X \to Y$ of a smooth manifold X to a smooth manifold Y gives rise to a linear map $d_xf : T_xX \to T_{f(x)}Y$, which is called the *differential* of f at x. A smooth map $f : X \to Y$ is called an *immersion* if d_xf is injective for all $x \in X$, a *submersion* if d_xf has maximal rank for all x, an *embedding* if it is an immersion and a homomorphism to its image. The image of a smooth manifold under a smooth embedding is called a *smooth submanifold*.

Tensor fields. A tensor of type (k, m) in a vector space V is by definition an element of the space $V^{\otimes k} \otimes V^{*\otimes m}$. A *tensor field* of type (k, m) on a smooth manifold X is a tensor of type (k, m) in T_xX smoothly depending on $x \in X$. In particular, a *vector field* is a tensor field of type $(1, 0)$, a *covector field* is a tensor field of type $(0, 1)$, a *differential m-form* is a skew-symmetric tensor field of type $(0, m)$.

Any vector field v on a manifold X gives rise to a one-parameter family of diffeomorphisms $\phi_v^t : X \to X$ such that

$$\frac{d}{dt}\phi_v^t(x)|_{t=0} = v_x$$

for any point $x \in X$. The diffeomorphism ϕ_v^t is called the *time-t flow* of v.

For any vector field v and a tensor field T, the *Lie derivative* L_vT is defined as

$$(L_vT)_x = \lim_{t \to 0} \frac{T_x - (\phi_v^t)^* T_{\phi_v^t(x)}}{t}.$$

In particular, for a pair of vector fields v, w, we have

$$L_v(w) = -L_w(v) =: [v, w]$$

The vector $[v, w]$ is called the *commutator* of v and w. In coordinates, if $v = v^i\partial_i$ and $w = w^j\partial_j$,

$$[v, w] = (v^i\partial_i w^j - w^i\partial_i v^j)\partial_j.$$

Analysis of differential forms. The *differential* d is the operator mapping differential k-forms to differential $(k+1)$-forms and satisfying the following properties:

- For any smooth function f, the differential of f as a 0-form and the differential of f as a function coincide.
- $d \circ d = 0$.
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta$.

For a 1-form α and vector fields v and w, we have

$$d\alpha(v, w) = \partial_v\alpha(w) - \partial_w\alpha(v) - \alpha([v, w]).$$

A differential form α is called *closed* if $d\alpha = 0$ and *exact* if $\alpha = d\beta$ for some form β. If $Y \subset X$ is a submanifold with boundary then, by the Stokes formula,

$$\int_Y d\alpha = \int_{\partial Y} \alpha$$

for any form α such that $\deg(\alpha) = \dim(Y)$.

Distributions, the Frobenius theorem. A k-distribution on a manifold X is a choice of k-dimensional vector submanifold Δ_x of the tangent space T_xX such that Δ_x depends smoothly on x. A distribution can be given either as a linear span of several vector fields, or as the common zero set of several 1-forms. A distribution Δ is called *integrable* if any point $x \in X$ has a neighborhood U such that there is a diffeomorphism $\phi : U \to V \subseteq \mathbb{R}^n$, where V is an open subset of $\mathbb{R}^n$ and $d\phi_x(\Delta_x)$ is the same for all $x \in U$. By the *Frobenius integrability theorem*, a distribution Δ is integrable if and only if one of the following equivalent statements holds:

(1) For any pair of smooth vector fields $v, w \in \Delta$, we have $[v, w] \in \Delta$.

(2) For any 1-form α such that $\alpha_x(\Delta_x) = 0$ for all $x \in X$, we have $d\alpha_x|_{\Delta_x} = 0$ for all $x \in X$.

An *integral manifold* of a distribution Δ is a submanifold $Y \subset X$ such that $T_xY = \Delta_x$ for any $x \in Y$. Any integrable distribution has many integral submanifolds. If a distribution has at least one integral submanifold Y, then the statements (1) and (2) hold on Y.

De Rham cohomology. The de Rham cohomology space $H^k(X)$ is defined as the quotient of the space of all closed k-forms on X by the space of all exact k-forms on X. For a connected manifold X, we have $H^0(X) = \mathbb{R}$. For a compact orientable manifold X of dimension n, we have $H^n(X) = \mathbb{R}$. For a nonorientable or a noncompact manifold X of dimension n, we have $H^n(X) = 0$.