MAT 531 Geometry/Topology Homework 6

1. Consider a complex polynomial $f : \mathbb{C} \to \mathbb{C}$. Prove that it has only finitely many critical values. Deduce that the mapping degree of f is independent of the choice of a regular value.

2. Prove the Fundamental Theorem of Algebra: any nonconstant complex polynomial has at least one complex root. Find the mapping degree of a complex polynomial in terms of its algebraic degree.

3. Let X be a smooth manifold of dimension m and Y a smooth manifold of dimension $n \leq m$. Consider a smooth map $f : X \to Y$. A point $x \in X$ is called a *critical point* of f if the differential $d_x f$ is not onto (i.e., it does not have the maximal rank). The image of any critical point is called a *critical value*. A *regular value* of f is any point $y \in Y$ that is not a critical value. Prove that for any regular value $y \in Y$, the subset $f^{-1}(y) \subseteq X$ is a smooth submanifold of dimension $m - n$. *Hint:* use the Implicit Function Theorem.

4. Let X and Y be smooth manifolds. A *smooth homotopy* between two smooth maps $f, g : X \to Y$ is defined as a smooth map $F : X \times [0,1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Suppose that both X and Y are oriented, and that smooth maps $f, g : X \to Y$ are smoothly homotopic. Prove that f and g have the same mapping degree. You can use the following fact without proof: there exists a point $y \in Y$ that is a regular value of F, f and g.

5. Let $f, g : X \to Y$ be two diffeomorphisms of a smooth manifold X to a smooth manifold Y. A smooth homotopy F connecting f with g is called a *smooth isotopy* if the map $F(\cdot, t) : x \in X \mapsto F(x, t) \in Y$ is a diffeomorphism for each $t \in [0, 1]$. Prove that the time 1 flow $\phi_v^1 : X \to X$ of any smooth vector field v on X is smoothly isotopic to the identity map.

6*. Let X be a connected smooth manifold. Prove that for any pair of points $y_1, y_2 \in Y$, there exists a smooth self-map $h : Y \to Y$ smoothly isotopic to the identity and such that $h(y_1) = y_2$. Deduce that the mapping degree of a smooth map $f : X \to Y$ does not depend on the choice of a regular value in Y, i.e. $\mathrm{mdeg}_{y_1}(f) = \mathrm{mdeg}_{y_2}(f)$.

Hint: define a smooth vector field on X, whose time 1 flow maps x_1 to x_2. It may be convenient first to define this vector field on a neighborhood of the path connecting x_1 to x_2.