MAT 531 Geometry/Topology Homework 3

Let X be a smooth manifold and $\lambda_p : T_pX \to \mathbb{R}$ a functional (not necessarily linear) depending smoothly on $p \in X$ and satisfying the following homogeneity property:

$$\lambda_p(\xi v) = \xi \lambda_p(v) \quad \forall \xi > 0,\ v \in T_pX.$$

For a smooth path $\gamma : [0,1] \to X$, define the *integral of* λ *over* γ as follows:

$$\int_\gamma \lambda := \int_0^1 \lambda_{\gamma(t)}(\dot{\gamma}(t))dt.$$

The functional λ is called a *1-form* (or a *covector field*) if λ_p is linear for each $p \in X$.

1. Let $\tau : [0,1] \to [0,1]$ be a diffeomorphism such that $\tau(0) = 0$ and $\tau(1) = 1$ (i.e. τ is increasing, or *orientation-preserving*). Prove that

$$\int_{\gamma\circ\tau} \lambda = \int_\gamma \lambda.$$

2. Suppose that λ satisfies the relation $\lambda_p(-v) = -\lambda_p(v)$ for all $p \in X$ and $v \in T_pX$. Let $\tau : [0,1] \to [0,1]$ be a diffeomorphism such that $\tau(0) = 1$ and $\tau(1) = 0$ (i.e. τ is decreasing, or *orientation-reversing*). Prove that

$$\int_{\gamma\circ\tau} \lambda = -\int_\gamma \lambda.$$

3. Let V be a finite dimensional vector space over $\mathbb{R}$ with a basis $(e_1, \dots, e_n)$. Denote by $(e^1, \dots, e^n)$ the dual basis in the dual space V^*. Prove that n^2 vectors $e^i \otimes e^j$, $i, j = 1, \dots, n$ form a basis in the space $V^* \otimes V^*$.

4. In the notation of problem 3, prove that the vectors $e_i \wedge e_j$, $i < j$, form a basis in the vector space $\Lambda^2 V$, and the vectors $e_i \cdot e_j$, $i \le j$, form a basis in the vector space $Sym^2 V$. Find dimensions of the spaces $\Lambda^2 V$ and $Sym^2 V$.

5. Prove that there are canonical isomorphisms

$$V \otimes W \cong W \otimes V, \quad (U \otimes V) \otimes W \cong U \otimes (V \otimes W), \quad V^* \otimes W \cong Hom(V, W).$$

   Here U, V, W are vector spaces, and $Hom(V, W)$ denotes the space of all linear maps (homomorphisms) of V to W.