MAT 531 Geometry/Topology Homework 2

A vector field on a smooth manifold is said to be *smooth*, if it is smooth in any local chart of this manifold.

1. Let v be a smooth vector field on a smooth manifold X. Prove that for any smooth function f on X, the function $\partial_v f$ is also smooth.

2. Let ϕ_v^t be the flow (for time t) of a smooth vector field v on an open subset of $\mathbb{R}^n$. Prove that
$$\phi_v^\epsilon(x_0) = x_0 + \epsilon v(x_0) + \frac{\epsilon^2}{2} d_{x_0} v[v(x_0)] + o(\epsilon^2).$$
Here $d_{x_0} v[v(x_0)]$ means the image of $v(x_0)$ under the action of the linear operator $d_{x_0} v$. In coordinates,
$$\phi_v^\epsilon(x_0)^i = x_0^i + \epsilon v^i(x_0) + \frac{\epsilon^2}{2} \frac{\partial v^i}{\partial x^j}|_{x_0} v^j(x_0) + o(\epsilon^2).$$

3. For any pair of smooth vector fields v and w on an open subset of $\mathbb{R}^n$, prove that
$$\lim_{(\epsilon,\delta)\to 0} \frac{\phi_v^\epsilon \circ \phi_w^\delta(x) - \phi_v^\delta \circ \phi_w^\epsilon(x)}{\epsilon\delta} = d_x v[w] - d_x w[v] = [v, w](x).$$

4. Compute the commutator $[x\partial_x + y\partial_y, x\partial_y - y\partial_x]$. Sketch the corresponding vector fields in the plane.

5. Introduce the spherical coordinates (r, ϕ, ψ) in $\mathbb{R}^3$:
$$x = r\cos(\phi)\cos(\psi), \quad y = r\sin(\phi)\cos(\psi), \quad z = r\sin(\psi).$$
Express the vector fields ∂_x, ∂_y and ∂_z in spherical coordinates.

6. Let $f : X \to Y$ be a diffeomorphism of a smooth manifold X to a smooth manifold Y. Prove that
$$f_*[v, w] = [f_*(v), f_*(w)]$$
for any pair of smooth vector fields v and w on X. Here $f_*(v)$ is a vector field on Y, which is defined as follows. For any point $q \in Y$, the tangent vector $f_*(v)_q$ at q is the image of the tangent vector v_p at $p = f^{-1}(q) \in X$ under the linear map $d_p f : T_pX \to T_qY$.