MAT 531 Geometry/Topology Homework 1

Let U and V be open subsets of $\mathbb{R}^n$. A smooth map $f : U \to V$ is called a *diffeomorphism* if it is invertible, and the inverse map $f^{-1} : V \to U$ is also smooth.

A smooth submanifold in $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ satisfying the following assumption. For any point $p \in M$, there exists a neighborhood U of p in $\mathbb{R}^n$ and a diffeomorphism ϕ of U to an open subset $V \subset \mathbb{R}^n$ such that

$$\phi(U \cap M) = V \cap \{x_1 = \cdots = x_k = 0\}.$$

Here $(x_1, \ldots, x_n)$ is a coordinate system in $\mathbb{R}^n$, and k is a natural number.

(1) Let X be a complete metric space and $f : X \to X$ a continuous map. Suppose that some iterate f^k of f is a contraction. Then f has a fixed point. Is this fixed point unique?

(2) Prove that the boundary of the unit square is not a smooth submanifold of $\mathbb{R}^2$.

(3) Prove that the unit circle is a smooth submanifold of $\mathbb{R}^2$, and that the unit 2-sphere is a smooth submanifold of $\mathbb{R}^3$.

(4) Suppose that a smooth map $f : \mathbb{R} \to \mathbb{R}^2$ has nowhere vanishing derivative, that the image of f is closed in $\mathbb{R}^2$, and that f is a homeomorphism of $\mathbb{R}$ to $f(\mathbb{R})$. Then the image of f is a smooth submanifold in $\mathbb{R}^2$.

(5) Suppose that a subset $M \subset \mathbb{R}^2$ is given by a smooth equation $f(x, y) = 0$ such that the gradient of f vanishes nowhere on M. Then M is a smooth submanifold of $\mathbb{R}^2$.