MAT 531: Topology&Geometry, II
Spring 2006

Problem Set 7
Due on Thursday, 3/30, in class

Note: This problem set has two pages. It covers 1.5 weeks, and so it is longer than usual. The first problem is a leftover from Chapter 4.

1. Let X be a path-connected topological space and let $(\mathcal{S}_*(X), \partial)$ be the singular chain complex of *continuous* simplices into X with *integer* coefficients. Denote by $H_1(X;\mathbb{Z})$ the corresponding first homology group.
(a) Show that there exists a well-defined surjective homomorphism
$$h\colon \pi_1(X, x_0) \longrightarrow H_1(X;\mathbb{Z}).$$
(b) Show that the kernel of this homomorphism is the commutator subgroup of $\pi_1(X, x_0)$ so that h induces an isomorphism
$$\Phi\colon \pi_1(X, x_0)/\big[\pi_1(X, x_0), \pi_1(X, x_0)\big] \longrightarrow H_1(X;\mathbb{Z}).$$
This is the first part of the Hurewicz Theorem.
Hint: For each $x \in X$, choose a path from x_0 to x. Use these paths to turn each 1-simplex into a loop based at x_0 and construct a homomorphism
$$\mathcal{S}_1(X) \longrightarrow \pi_1(X, x_0)/\big[\pi_1(X, x_0), \pi_1(X, x_0)\big].$$
Show that it vanishes on $\partial\mathcal{S}_2(X)$, well-defined on $\ker\partial$ (may not be necessary), and its composition with Φ is the identity on $\pi_1(X, x_0)/\big[\pi_1(X, x_0), \pi_1(X, x_0)\big]$. Sketch something.

2. (a) Prove `Mayer-Vietoris for Cohomology`: If M is a smooth manifold, $\mathcal{U}, V \subset M$ open subsets, and $M = \mathcal{U} \cup V$, then there exists an exact sequence
$$\begin{aligned}
0 \longrightarrow H^0_{\mathrm{de\,R}}(M) &\xrightarrow{f_0} H^0_{\mathrm{de\,R}}(\mathcal{U}) \oplus H^0_{\mathrm{de\,R}}(V) \xrightarrow{g_0} H^0_{\mathrm{de\,R}}(\mathcal{U}\cap V) \xrightarrow{\delta_0} \\
\xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(M) &\xrightarrow{f_1} H^1_{\mathrm{de\,R}}(\mathcal{U}) \oplus H^1_{\mathrm{de\,R}}(V) \xrightarrow{g_1} H^1_{\mathrm{de\,R}}(\mathcal{U}\cap V) \xrightarrow{\delta_1} \\
\xrightarrow{\delta_1} \dots & \\
\vdots &
\end{aligned}$$
where
$$f_i(\alpha) = \big(\alpha|_{\mathcal{U}}, \alpha|_V\big) \qquad \text{and} \qquad g_i(\beta, \gamma) = \beta|_{\mathcal{U}\cap V} - \gamma|_{\mathcal{U}\cap V}.$$
(b) Suppose M is a compact connected orientable n-dimensional submanifold of $\mathbb{R}^{n+1}$. Show that $\mathbb{R}^{n+1} - M$ has exactly two connected components. How is the compactness of M used?

3. (a) Show that the inclusion map $S^n \longrightarrow \mathbb{R}^{n+1}-0$ induces an isomorphism in cohomology.
(b) Show that for all $n \geq 0$ and $p \in \mathbb{Z}$,

$$H^p_{\mathrm{de\,R}}(S^n) \approx \begin{cases} \mathbb{R}^2, & \text{if } p=n=0; \\ \mathbb{R}, & \text{if } p=0,n,\ n\neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Discuss the $p \leq 0$, $p > n$, $n=0,1$ cases separately, before starting an induction on n. The case $n=1$ was the subject of Problem 2 on PS6.
(c) Show that S^n is not a product of two positive-dimensional manifolds.
Note: Do *not* use the Kunneth formula, unless you are intending to prove it. However, the cup/wedge product can be used and might be useful here.

4. (a) Use Mayer-Vietoris (*not* Kunneth formula) to compute $H^*_{\mathrm{de\,R}}(T^2)$, where T^2 is the two-torus, $S^1 \times S^1$. Find a basis for $H^*_{\mathrm{de\,R}}(T^2)$; justify your answer.
(b) Let Σ_g be a compact connected orientable surface of genus g (donut with g holes). Let $B \subset \Sigma_g$ be a small closed ball or a single point. Relate $H^*_{\mathrm{de\,R}}(\Sigma_g - B)$ to $H^*_{\mathrm{de\,R}}(\Sigma_g)$ (do not compute $H^1_{\mathrm{de\,R}}$ explicitly).
(c) Show that

$$H^p_{\mathrm{de\,R}}(\Sigma_g) = \begin{cases} \mathbb{R}, & \text{if } p=0,2; \\ \mathbb{R}^{2g}, & \text{if } p=1; \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Discuss the cases $g=0,1$ before starting an induction on g. Note that $\Sigma_{g_1+g_2} \approx \Sigma_{g_1} \# \Sigma_{g_2}$.

5. (a) Suppose $q\colon \tilde{M} \longrightarrow M$ is a regular covering projection with a finite group of deck transformations G (so that $M = \tilde{M}/G$). Show that

$$q^*\colon H^*_{\mathrm{de\,R}}(M) \longrightarrow H^*_{\mathrm{de\,R}}(\tilde{M})^G \equiv \{\alpha \in H^*_{\mathrm{de\,R}}(\tilde{M})\colon g^*\alpha = \alpha\ \forall\, g \in G\}$$

is an isomorphism. Does the statement continue to hold if G is not assumed to be finite?
(b) Determine $H^*_{\mathrm{de\,R}}(K)$, where K is the Klein bottle. Find a basis for $H^*_{\mathrm{de\,R}}(K)$; justify your answer.
Hint: see Exercise 3 on p454 of Munkres.

6. Chapter 5, #4 (p216)

7. Let $K = \mathbb{Z}$ and let $\pi\colon \mathcal{S}_0 \longrightarrow \mathbb{R}$ be the corresponding skyscraper sheaf, with the only non-trivial stack over $0 \in \mathbb{R}$; see Subsection 5.11. What is $\mathcal{S}_0$ as a topological space?
Hint: it is something familiar.

Exercises (*figure these out, but do not hand them in*): Chapter 5, #11, 13, 16, 17 (pp 216,217); verify Lemma 5.14 (p172). The kernel of the first map in (2) of Lemma 5.14 is denoted by $A'' * B$ or $\mathrm{Tor}(A'', B)$ and known as the torsion product of A'' and B; $A'' * B = B * A''$.