MAT 531: Topology&Geometry, II
Spring 2006

Problem Set 6
Due on Friday, 3/10, by 11am, in Math 3-117

Note: This problem set has two pages. It covers 1.5 weeks, and so it is longer than usual. Please review the newest version of *Notes on Vector Bundles* before starting on Problem 4. It may also help with Problem 3.

1. Chapter 4, #19 (p160)

2. (a) Show that a one-form α on S^1 is exact if and only if

$$\int_{[0,1]} f^*\alpha = 0$$

for every smooth function $f\colon [0,1] \longrightarrow S^1$ such that $f(0)=f(1)$.
(b) Let

$$q\colon \mathbb{R} \longrightarrow S^1, \qquad q(t) = e^{it},$$

be the standard covering map. Show the differential of the "angular coordinate function",

$$\theta\colon S^1 \longrightarrow \mathbb{R}, \qquad \theta(q(t))=t,$$

is well-defined (even though θ is not) and generates $H^1_{\mathrm{de\,R}}(S^1)$. Determine all de Rham cohomology groups of S^1.

3. (a) Suppose $\varphi\colon M \longrightarrow \mathbb{R}^N$ is an immersion. Show that M is orientable if and only if the normal bundle to the immersion φ is orientable.
(b) Chapter 4, #1 (p157)

4. Let M be a smooth manifold.
(a) Show that every real vector bundle $V \longrightarrow M$ admits a Riemannian metric and every complex vector bundle admits a hermitian metric.
(b) Show that if M is connected and there exists a non-orientable vector bundle $V \longrightarrow M$, then M admits a connected double-cover (2:1 covering map).
(c) Show that if the order of $\pi_1(M)$ is finite and odd, then M is orientable.

5. (a) Show that the antipodal map on $S^n \subset \mathbb{R}^{n+1}$ (i.e. $x \longrightarrow -x$) is orientation-preserving if n is odd and orientation-reversing if n is even.
(b) Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.
(c) Describe the orientable double cover of $\mathbb{R}P^n \times \mathbb{R}P^n$ with n even.

6. (a) Show that every diffeomorphism $f\colon S^n \longrightarrow S^n$ that has no fixed points is smoothly homotopic to the antipodal map (x is a *fixed point* of f if $f(x)=x$).
(b) Show that if $\pi\colon S^n \longrightarrow M$ is a covering projection onto a smooth manifold M and $|\pi_1(M)| \neq 2$, then M is orientable.

7. (a) Show that if X is a smooth nowhere-vanishing[1] vector field on a compact manifold M, then the flow $X_t\colon M \longrightarrow M$ of X has no fixed points for some $t \in \mathbb{R}$.
(b) Show that S^n admits a nowhere vanishing vector field if and only if n is odd.
(c) Show that the tangent bundle of S^n is not trivial if $n \geq 1$ is even.
Note: In fact, TS^n is trivial if and only if $n=1,3,7$.

8. Suppose M is a 3-manifold with boundary and $\partial M = T^2 = S^1 \times S^1$. Let

$$\pi_1, \pi_2\colon T^2 \longrightarrow S^1$$

be the two projection maps. Show that it is impossible to extend both (as opposed to at least one of) $\alpha_1 \equiv \pi_1^* d\theta$ and $\alpha_2 \equiv \pi_2^* d\theta$ to closed forms on M ($d\theta$ is as in Problem 2).

[1] $X(m) \neq 0$ for all $m \in M$