MAT 531: Topology&Geometry, II
Spring 2006

Final Exam Solutions

Part I (*choose 2 problems from 1,2, and 3*)

1. *Suppose M is a compact manifold and α is a nowhere-zero closed one-form. Show that*

$$[\alpha] \neq 0 \in H^1_{\mathrm{deR}}(M).$$

We need to show that α is not exact, i.e. $\alpha \neq df$ for any $f \in C^\infty(M)$. Suppose $f\colon M \longrightarrow \mathbb{R}$ is a smooth function. Since M is compact, f must achieve its maximum at some point $m \in M$ (not necessarily unique). Then, $df|_m = 0$, since in local coordinates near m all partial derivatives of f must vanish at m. Since $\alpha|_m \neq 0$, $df \neq \alpha$.

2. *Let Y and Z be the vector fields on $\mathbb{R}^3$ given by*

$$Y(x_1,x_2,x_3) = \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} \qquad \text{and} \qquad Z(x_1,x_2,x_3) = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_3},$$

where a, b, and c are constants.
(a) *Compute the Lie bracket $[Y,Z]$.*
(b) *Describe the flows φ_s of Y and ψ_t of Z.*
(c) *For what constants a, b, and c do these two flows commute?*

(a) The Lie bracket of two smooth vector fields Y and Z on a manifold M is defined by

$$[Y,Z]\colon C^\infty(M) \longrightarrow C^\infty(M), \qquad [Y,Z]f = Y(Zf) - Z(Yf).$$

Since the Lie bracket of two coordinate vector fields vanishes, in this case we obtain

$$\begin{aligned}[Y,Z] &= \Big(Y(a)\frac{\partial}{\partial x_1} + Y(b)\frac{\partial}{\partial x_2} + Y(c)\frac{\partial}{\partial x_3}\Big) - \Big(Z(1)\frac{\partial}{\partial x_1} + Z(x_3)\frac{\partial}{\partial x_2} - Z(x_2)\frac{\partial}{\partial x_3}\Big) \\ &= 0 - \Big(0 + c\frac{\partial}{\partial x_2} - b\frac{\partial}{\partial x_3}\Big) = -c\frac{\partial}{\partial x_2} + b\frac{\partial}{\partial x_3}.\end{aligned}$$

(b) Since Z is a constant vector field, its flow ψ_t are translations with the velocity (a,b,c), i.e.

$$\psi_t(x_1,x_2,x_3) = (x_1{+}at, x_2{+}bt, x_3{+}ct).$$

If $(a,b,c)=0$, ψ_t is just the identity map, i.e. the flow is stationary. The effect of the flow φ_s of Y on the x_2 and x_3-coordinates is the clockwise rotation by angle s. To see that this rotation is clockwise, notice that

$$Y(0,1,0) = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3},$$

i.e. points down in x_3 at $(x_2,x_3)=(1,0)$. The first component of Y shifts the planes $x_1 = const$ at the unit speed in the direction of increasing x_1. Thus, $\varphi_s(x_1,x_2,x_3)$ is the clockwise spiral around the x_1-axis moving to the right. Explicitly,

$$\varphi_s\big(x_1, r\cos\theta, r\sin\theta\big) = \varphi_s\big(x_1{+}s, r\cos(\theta{-}s), r\sin(\theta{-}s)\big).$$

(c) By Problem 4 on PS5, the flows φ_s and ψ_t for Y and Z commute if and only if $[Y,Z]=0$. By part (a), this is the case if and only if $b=c=0$, so that ψ_t is a sideways translation (i.e. in x_1 only).

Alternatively, φ_s is a composition of a nontrivial (if $s\neq 0$) sideways translation (i.e. in x_1) and a nontrivial rotation in x_2 and x_3. On the other hand, ψ_t is a composition of a sideways translation and a translation in x_2 and x_3, either of which might be trivial. Since all translations commute and (x_2,x_3)-rotations commute with x_1-translations, the flows φ_s and ψ_t commute if and only if the nontrivial (x_2,x_3)-rotation for φ_s commutes with the (x_2,x_3)-translation for ψ_t. Since rotations and translations in $\mathbb{R}^2$ do not commute, the latter is the case if and only if the (x_2,x_3)-translation for ψ_t is trivial, i.e. $b=c=0$.

3. *Describe explicitly trivializations and transition data for the vector bundle $TS^2 \longrightarrow S^2$.*

Suppose M is a smooth manifold and $\pi\colon V\longrightarrow M$ is a vector bundle of rank k. A trivialization of V over an open subset $\mathcal{U}$ of M can be constructed by finding

$$s_1,\ldots,s_k \in \Gamma(\mathcal{U};V)$$

such that $s_1(x),\ldots,s_k(x)\in V_x$ are linearly independent vectors (and thus a basis for V_x) for all $x\in\mathcal{U}$. Such sections $s_1,\ldots,s_k$ must exist if $\mathcal{U}$ is sufficiently small. Then, for every $x\in\mathcal{U}$ and $v\in V_x$, there exist unique

$$c_1(v),\ldots,c_k(v)\in\mathbb{R} \qquad \text{s.t.} \qquad v=c_1(v)s_1(x)+\ldots+c_k(v)s_k(x).$$

A trivialization of V over $\mathcal{U}$ can then be defined by

$$h\colon V|_{\mathcal{U}}\longrightarrow \mathcal{U}\times\mathbb{R}^k, \qquad h(v)=\big(\pi(x),c_1(v),\ldots,c_k(v)\big).$$

If $h_\alpha\colon V|_{\mathcal{U}_\alpha}\longrightarrow\mathcal{U}_\alpha\times\mathbb{R}^k$ and $h_\beta\colon V|_{\mathcal{U}_\beta}\longrightarrow\mathcal{U}_\beta\times\mathbb{R}^k$ are two trivializations, then

$$h_\alpha\circ h_\beta^{-1}\colon (\mathcal{U}_\alpha\cap\mathcal{U}_\beta)\times\mathbb{R}^k \longrightarrow (\mathcal{U}_\alpha\cap\mathcal{U}_\beta)\times\mathbb{R}^k$$

is a diffeomorphism which commutes with the projection map onto the first component and is linear on the fibers of this projection. Thus, for every $x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta$ there exists a unique $g_{\alpha\beta}(x)\in\mathrm{GL}_k\mathbb{R}$ such that

$$\big\{h_\alpha\circ h_\beta^{-1}\big\}(x,v)=\big(x,g_{\alpha\beta}(x)\cdot v\big) \qquad \forall\, v\in\mathbb{R}^k.$$

The smooth map

$$g_{\alpha\beta}\colon \mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow \mathrm{GL}_k\mathbb{R}$$

is then the transition map from $(\mathcal{U}_\beta,h_\beta)$ to $(\mathcal{U}_\alpha,h_\alpha)$.

Suppose next that $V=TM$ is the tangent bundle. If

$$\varphi_\alpha=(x_1,\ldots,x_n)\colon\mathcal{U}_\alpha\longrightarrow\mathbb{R}^n$$

is a coordinate chart, then the coordinate tangent vectors

$$\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\in\Gamma(\mathcal{U}_\alpha;TM)$$

are linearly independent at each point of $\mathcal{U}$. Furthermore, for all $x\in\mathcal{U}$ and $v\in T_xM$,

$$v = v(x_1)\frac{\partial}{\partial x_1}\Big|_x + \ldots + v(x_n)\frac{\partial}{\partial x_n}\Big|_x.$$

It is sufficient to check this identity on the functions $x_1,\ldots,x_n$ on $\mathcal{U}$ as they form a basis for F_x/F_x^2 for all $x\in\mathcal{U}$; see Theorem 1.17 of Warner. Thus, we obtain a trivialization of TM over $\mathcal{U}_\alpha$:

$$h_\alpha\colon TM|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{U}_\alpha\times\mathbb{R}^n, \qquad h_\alpha(v) = \big(\pi(v), v(x_1),\ldots,v(x_n)\big).$$

Suppose

$$\varphi_\beta=(y_1,\ldots,y_n)\colon \mathcal{U}_\alpha \longrightarrow \mathbb{R}^n$$

is another coordinate chart. By the above,

$$\frac{\partial}{\partial y_j} = \sum_{i=1}^{i=n}\Big(\frac{\partial}{\partial y_j}x_i\Big)\frac{\partial}{\partial x_i} = \sum_{i=1}^{i=n}\mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)_{ij}\frac{\partial}{\partial x_i} \qquad \text{and}$$

$$\begin{aligned}\sum_{i=1}^{i=n}v(x_i)\frac{\partial}{\partial x_i} = v = \sum_{j=1}^{j=n}v(y_j)\frac{\partial}{\partial y_j} &= \sum_{j=1}^{j=n}v(y_j)\sum_{i=1}^{i=n}\mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)_{ij}\frac{\partial}{\partial x_i}\\ &= \sum_{i=1}^{i=n}\Big(\sum_{j=1}^{j=n}\mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)_{ij}v(y_j)\Big)\frac{\partial}{\partial x_i},\end{aligned}$$

where $\mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)$ is the Jacobian of the map $\varphi_\alpha\circ\varphi_\beta^{-1}$ between open subsets of $\mathbb{R}^n$. Thus,

$$\big(v(x_1),\ldots,v(x_n)\big)^t = \mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)\big(v(y_1),\ldots,v(y_n)\big)^t \qquad\Longrightarrow\qquad g_{\alpha\beta}(x) = \mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)\big|_{\varphi_\beta(x)}.$$

In other words, the transition map from $(\mathcal{U}_\beta,h_\beta)$ to $(\mathcal{U}_\alpha,h_\alpha)$ is $g_{\alpha\beta}=\mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)\circ\varphi_\beta$.

In the present case, $M=S^2\subset\mathbb{R}^3$. We take

$$\mathcal{U}_\alpha = S^2-(0,0,1), \qquad \mathcal{U}_\beta = S^2-(0,0,-1),$$

and $\varphi_\alpha\colon\mathcal{U}_\alpha\longrightarrow\mathbb{R}^2$ and $\varphi_\beta\colon\mathcal{U}_\beta\longrightarrow\mathbb{R}^2$ to be the projections from the north and south poles. In other words, for each $(x,y,z)\in\mathcal{U}_\alpha$, $\varphi_\alpha(x,y,z)\in\mathbb{R}^2$ is the intersection of the line passing through $(0,0,1)$ and (x,y,z) with the xy-plane. Explicitly,

$$\varphi_\alpha(x,y,z) = \Big(\frac{x}{1-z},\frac{y}{1-z}\Big) \qquad \text{and} \qquad \varphi_\beta(x,y,z) = \Big(\frac{x}{1+z},\frac{y}{1+z}\Big).$$

Thus,

$$\big\{\varphi_\alpha\circ\varphi_\beta^{-1}\big\}^{-1}(s,t) = \Big(\frac{s}{s^2+t^2},\frac{t}{s^2+t^2}\Big) \qquad\Longrightarrow$$

$$g_{\alpha\beta}=\mathcal{J}\big(\varphi_\alpha\circ\varphi_\beta^{-1}\big)\circ\varphi_\beta = \frac{1}{(s^2+t^2)^2}\begin{pmatrix}-s^2+t^2 & -2st\\ -2st & s^2-t^2\end{pmatrix} = \frac{(1+z)^2}{(x^2+y^2)^2}\begin{pmatrix}-x^2+y^2 & -2xy\\ -2xy & x^2-y^2\end{pmatrix}.$$

Thus, the transition data for the trivializations $\{(\mathcal{U}_\alpha, h_\alpha), (\mathcal{U}_\beta, h_\beta)\}$ of $TS^2 \longrightarrow S^2$ is

$$\begin{gathered} g_{\alpha\beta}, g_{\beta\alpha} = g_{\alpha\beta}^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{GL}_2\mathbb{R}, \\ g_{\alpha\beta}(x,y,z) = \frac{(1+z)^2}{(x^2+y^2)^2} \begin{pmatrix} -x^2+y^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix}, \\ g_{\beta\alpha}(x,y,z) = \frac{1}{(1+z)^2} \begin{pmatrix} -x^2+y^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix} = \frac{(1-z)^2}{(x^2+y^2)^2} \begin{pmatrix} -x^2+y^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix}, \end{gathered}$$

since $x^2+y^2+z^2=1$.

Alternatively, we can view S^2 as the complex one-manifold

$$\mathbb{C}P^1 = \big\{[X_0, X_1] : (X_0, X_1) \in \mathbb{C}^2 - \{0\}\big\},$$

or the Riemann sphere. We then take

$$\begin{gathered} \mathcal{U}_0 = \big\{[X_0, X_1] \in \mathbb{C}P^1 : X_0 \neq 0\big\}, \qquad \mathcal{U}_1 = \big\{[X_0, X_1] \in \mathbb{C}P^1 : X_1 \neq 0\big\}, \\ \varphi_0 : \mathcal{U}_0 \longrightarrow \mathbb{C}, \quad \varphi_0\big([X_0, X_1]\big) = X_1/X_0, \qquad \varphi_1 : \mathcal{U}_1 \longrightarrow \mathbb{C}, \quad \varphi_1\big([X_0, X_1]\big) = X_0/X_1 \\ \Longrightarrow \quad \varphi_0 \circ \varphi_1^{-1} : \mathbb{C}^* \longrightarrow \mathbb{C}^*, \quad \big\{\varphi_0 \circ \varphi_1^{-1}\big\}(z) = z^{-1}, \quad g_{01} = \mathcal{J}\big(\varphi_0 \circ \varphi_1^{-1}\big) \circ \varphi_1 = -1/z^2 = -X_1^2/X_0^2. \end{gathered}$$

Thus, the transition data for the trivializations $\{(\mathcal{U}_0, h_0), (\mathcal{U}_0, h_0)\}$ of the complex line bundle $T\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is given by

$$g_{01}, g_{10} = g_{01}^{-1} : \mathcal{U}_0 \cap \mathcal{U}_1 \longrightarrow \mathrm{GL}_1\mathbb{C} = \mathbb{C}^*, \quad g_{01}\big([X_0, X_1]\big) = -X_1^2/X_0^2, \quad g_{10}\big([X_0, X_1]\big) = -X_0^2/X_1^2.$$

Forgetting the complex structure in the complex line bundle $T\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$, we obtain transition data for the real bundle $TS^2 \longrightarrow S^2$ of rank 2 by viewing $\mathbb{C}^*$ as subspace of $\mathrm{GL}_2\mathbb{R}$. An explicit (and standard in complex analysis) identification of S^2 with $\mathbb{C}P^1$ is given by

$$S^2 \longrightarrow \mathbb{C}P^1, \qquad (x,y,z) \longrightarrow \big[x+\mathrm{i}y, 1+z\big] = \big[1-z, x-\mathrm{i}y\big],$$

i.e. by extending the chart φ_β. Since $\mathcal{U}_\alpha = \mathcal{U}_0$ and $\mathcal{U}_\beta = \mathcal{U}_1$ with this identification, plugging this into the above transition data we obtain

$$\begin{gathered} g_{\alpha\beta} = g_{01}, g_{\beta\alpha} = g_{10} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathbb{C}^* \subset \mathrm{GL}_2\mathbb{R}, \\ g_{\alpha\beta}(x,y,z) = -\Big(\frac{1+z}{x+\mathrm{i}y}\Big)^2 = -\frac{(1+z)^2}{(x^2+y^2)^2}\big((x^2-y^2) - 2\mathrm{i}xy\big) = \frac{(1+z)^2}{(x^2+y^2)^2} \begin{pmatrix} -x^2+y^2 & 2xy \\ -2xy & -x^2+y^2 \end{pmatrix}, \\ g_{\beta\alpha}(x,y,z) = -\Big(\frac{1-z}{x-\mathrm{i}y}\Big)^2 = -\frac{(1-z)^2}{(x^2+y^2)^2}\big((x^2-y^2) + 2\mathrm{i}xy\big) = \frac{(1-z)^2}{(x^2+y^2)^2} \begin{pmatrix} -x^2+y^2 & -2xy \\ 2xy & -x^2+y^2 \end{pmatrix}. \end{gathered}$$

This transition data differs from $g_{\alpha\beta}$ and $g_{\beta\alpha}$ obtained previously, because under the above identification of S^2 with $\mathbb{C}P^1$ the chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ defined previously corresponds to the chart $(\mathcal{U}_0, \bar{\varphi}_0)$, while $(\mathcal{U}_\beta, \varphi_\beta)$ corresponds to $(\mathcal{U}_1, \varphi_1)$.

Part II (*choose 2 problems from 4,5, and 6*)

4. *Compute the singular homology of a point directly from the definition.*

If R is any ring (e.g. $\mathbb{R}$), we will show that

$$H_p(pt; R) \approx \begin{cases} R, & \text{if } p=0; \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$S_p(pt) \equiv S_p(X; R)$$

be the free R-module with a basis consisting of smooth maps $f: \Delta^p \longrightarrow pt$, where $\Delta^p \subset \mathbb{R}^p$ is the standard p-simplex. In this case, there is only one such map, which we denote by f_p. Thus,

$$S_p(pt) = R\{f_p\} \approx R \qquad \forall\, p \geq 0.$$

By definition, $H_p(pt; R)$ is the pth homology of the chain complex $(S_p(pt), \partial)$, where

$$\partial_p \colon S_p(pt) \longrightarrow S_{p-1}(pt)$$

is the boundary operator.

If $p \geq 1$,

$$\partial_p f_p = \sum_{i=0}^{i=p} (-1)^i f_p \circ k_i^p,$$

where $k_i^p \colon \Delta^{p-1} \longrightarrow \Delta^p$ are the maps described in Section 4.6 of Warner. However, in this case it does not matter what these maps are, since the map

$$f_p \circ k_i^p \colon \Delta^{p-1} \longrightarrow pt$$

must be f_{p-1}. Thus,

$$\partial_p f_p = \sum_{i=0}^{i=p} (-1)^i f_p \circ k_i^p = \sum_{i=0}^{i=p} (-1)^i f_{p-1} = \begin{cases} 0, & \text{if } p \text{ is odd;} \\ f_{p-1}, & \text{if } p>0 \text{ is odd.} \end{cases}$$

By definition, $\partial_0 \equiv 0$. Therefore, the homomorphism

$$\partial_p \colon S_p(pt) \longrightarrow S_{p-1}(pt)$$

is an isomorphism if $p>0$ is even and trivial otherwise. We conclude that

$$\ker \partial_p \approx \begin{cases} R, & \text{if } p>0 \text{ is odd or } p=0; \\ 0, & \text{otherwise;} \end{cases} \qquad \operatorname{Im} \partial_{p+1} \approx \begin{cases} R, & \text{if } p>0 \text{ is odd;} \\ 0, & \text{otherwise;} \end{cases}$$

$$\Longrightarrow \qquad H_p(pt; R) \equiv \ker \partial_p / \operatorname{Im} \partial_{p+1} \approx \begin{cases} R, & \text{if } p=0; \\ 0, & \text{otherwise.} \end{cases}$$

5. (a) *Show that* $\mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4$ *is not orientable.*
(b) *Describe the orientable double cover of* $\mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4$.

(a) By Problem 5 on the midterm,

$$\mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4 = \mathbb{R}P^2\times\big(\mathbb{R}P^3\times\mathbb{R}P^4\big)$$

is orientable if and only if $\mathbb{R}P^2$ and $\mathbb{R}P^3\times\mathbb{R}P^4$ are orientable. On the other hand, by Problem 5b on PS6, $\mathbb{R}P^n$ is orientable if and only if n is odd. Thus, $\mathbb{R}P^2$ is not orientable, and neither is $\mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4$.

Alternatively, let

$$\pi\colon S^2\times S^3\times S^4 \longrightarrow \mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4$$

be the natural covering map. It is a regular covering map and the group of deck transformations is the subgroup G of diffeomorphisms of $S^2\times S^3\times S^4$ generated by the diffeomorphisms

$$a_2\times\mathrm{id}_{S^3}\times\mathrm{id}_{S^4},\qquad \mathrm{id}_{S^2}\times a_3\times\mathrm{id}_{S^4},\qquad \mathrm{id}_{S^2}\times\mathrm{id}_{S^3}\times a_4,$$

where

$$a_n\colon S^n\longrightarrow S^n,\qquad a_n(x)=-x,$$

is the antipodal map. In particular, G acts without fixed points and

$$\mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4 = \big(S^2\times S^3\times S^4\big)/G.$$

Since a_n is orientation-preserving if and only if n is odd (see Problem 5a on PS6), the diffeomorphism $a_2\times\mathrm{id}_{S^3}\times\mathrm{id}_{S^4}$ is orientation-preserving. Therefore, $\mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4$ is not orientable.

(b) Let a_n and G be as in the previous paragraph. Denote by $G_0\subset G$ the subgroup consisting of orientable diffeomorphisms. The oriented double cover of the non-orientable space

$$\mathbb{R}P^2\times\mathbb{R}P^3\times\mathbb{R}P^4 = \big(S^2\times S^3\times S^4\big)/G$$

is given by

$$\tilde{M} = \big(S^2\times S^3\times S^4\big)/G_0.$$

Since $\mathrm{id}_{S^2}\times a_3\times\mathrm{id}_{S^4}$ is orientation-preserving, while $a_2\times\mathrm{id}_{S^3}\times\mathrm{id}_{S^4}$ and $\mathrm{id}_{S^2}\times\mathrm{id}_{S^3}\times a_4$ are orientation-reversing,

$$\begin{aligned} G_0 &= \big\{\mathrm{id}_{S^2}\times\mathrm{id}_{S^3}\times\mathrm{id}_{S^4}, \mathrm{id}_{S^2}\times a_3\times\mathrm{id}_{S^4}, a_2\times\mathrm{id}_{S^3}\times a_4, a_2\times a_3\times a_4\big\}\\ &= \big\{\mathrm{id}_{S^2}\times\mathrm{id}_{S^3}\times\mathrm{id}_{S^4}, \mathrm{id}_{S^2}\times a_3\times\mathrm{id}_{S^4}\big\}\times\big\{\mathrm{id}_{S^2}\times\mathrm{id}_{S^3}\times\mathrm{id}_{S^4}, a_2\times\mathrm{id}_{S^3}\times a_4\big\}. \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{M} &= \big(S^2\times S^3\times S^4\big)/G_0 = \big(S^2\times\mathbb{R}P^3\times S^4\big)/G_0',\\ \text{where}\qquad G_0' &= \big\{\mathrm{id}_{S^2}\times\mathrm{id}_{\mathbb{R}P^3}\times\mathrm{id}_{S^4}, a_2\times\mathrm{id}_{\mathbb{R}P^3}\times a_4\big\}. \end{aligned}$$

6. (a) *Determine the de Rham cohomology of* $\mathbb{R}P^2$.
(b) *Determine the de Rham cohomology of* $\mathbb{R}P^2\#\mathbb{R}P^2$.

(a) Since $\mathbb{R}P^2$ is connected,

$$H^0_{\mathrm{deR}}(\mathbb{R}P^2) \approx \mathbb{R}.$$

Since $\mathbb{R}P^2$ is a connected and non-orientable 2-manifold,

$$H^2_{\mathrm{deR}}(\mathbb{R}P^2) = 0.$$

Since $\mathbb{R}P^2$ is connected and $\pi_1(\mathbb{R}P^2)\approx\mathbb{Z}_2$, by Hurewicz and de Rham Theorems

$$\begin{aligned} H_1(\mathbb{R}P^2;\mathbb{Z}) \approx \mathrm{Abel}\big(\pi_1(M)\big) \approx \mathbb{Z}_2 \qquad &\Longrightarrow \qquad H_1(\mathbb{R}P^2;\mathbb{R}) \approx H_1(\mathbb{R}P^2;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R} = 0 \\ &\Longrightarrow \qquad H^1_{\mathrm{deR}}(\mathbb{R}P^2) \approx H^1(\mathbb{R}P^2;\mathbb{R}) \approx \big(H_1(\mathbb{R}P^2;\mathbb{R})\big)^* = 0. \end{aligned}$$

Since $\mathbb{R}P^2$ is a two-dimensional manifold, all other de Rham groups are zero.

Alternatively, let

$$\pi\colon S^2 \longrightarrow \mathbb{R}P^2$$

be the standard covering projection. It is regular and its group of covering transformations is

$$G = \big\{\mathrm{id}, a_2\big\} \approx \mathbb{Z}_2,$$

where a_2 is as in Problem 5, so that G acts without fixed points and $\mathbb{R}P^2 = S^2/G$. Since G is finite,

$$\begin{aligned} \pi^*\colon H^*_{\mathrm{deR}}(\mathbb{R}P^2) \longrightarrow H^*_{\mathrm{deR}}(S^2)^G &\equiv \big\{[\tilde\alpha]\in H^*_{\mathrm{deR}}(S^2)\colon g^*[\tilde\alpha]=[\tilde\alpha]\ \forall\, g\in G\big\} \\ &= \big\{[\tilde\alpha]\in H^*_{\mathrm{deR}}(S^2)\colon a_2^*[\tilde\alpha]=[\tilde\alpha]\big\} \end{aligned}$$

is an isomorphism. Thus,

$$\begin{aligned} H^0_{\mathrm{deR}}(\mathbb{R}P^2) \approx H^0_{\mathrm{deR}}(S^2)^G &= \big\{f\in C^\infty(S^2)\colon f \text{ is const}, f\circ a_2 = f\big\} \\ &= \big\{f\in C^\infty(S^2)\colon f \text{ is const}\big\} \approx \mathbb{R}; \\ H^1_{\mathrm{deR}}(\mathbb{R}P^2) \approx H^1_{\mathrm{deR}}(S^2)^G &\approx 0^G = 0. \end{aligned}$$

Finally, since $H^2_{\mathrm{deR}}(S^2)\approx\mathbb{R}$ and $a_2\colon S^2\longrightarrow S^2$ is orientation-reversing,

$$a_2^*\colon H^2_{\mathrm{deR}}(S^2) \longrightarrow H^2_{\mathrm{deR}}(S^2)$$

is multiplication by a negative number (actually -1). Thus, a_2^* does not fixed any nonzero element of $H^2_{\mathrm{deR}}(S^2)$ and

$$H^2_{\mathrm{deR}}(\mathbb{R}P^2) \approx H^2_{\mathrm{deR}}(S^2)^G = 0.$$

(b) We begin by computing the de Rham cohomology of the complement $\mathcal{U}$ in $\mathbb{R}P^2$ of a point or a small closed ball. Let V be a slightly larger open ball. Since $\mathcal{U}$ is a connected non-compact two-manifold,

$$H^0_{\mathrm{deR}}(\mathcal{U}) \approx \mathbb{R}, \qquad H^2_{\mathrm{deR}}(\mathcal{U}) = 0.$$

Furthermore,

$$H^p_{\mathrm{deR}}(V) \approx \begin{cases} \mathbb{R}, & \text{if } p=0; \\ 0, & \text{otherwise;} \end{cases} \qquad H^p_{\mathrm{deR}}(\mathbb{R}P^2) \approx \begin{cases} \mathbb{R}, & \text{if } p=0; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{U}\cap V$ is an annulus and thus homotopy equivalent to S^1,

$$H^p_{\mathrm{deR}}(\mathcal{U}\cap V) \approx \begin{cases} \mathbb{R}, & \text{if } p=0,1; \\ 0, & \text{otherwise.} \end{cases}$$

By Mayer-Vietoris, we then have an exact sequence

$$H^1_{\mathrm{deR}}(\mathbb{R}P^2) \longrightarrow H^1_{\mathrm{deR}}(\mathcal{U}) \oplus H^1_{\mathrm{deR}}(V) \xrightarrow{g_1} H^1_{\mathrm{deR}}(\mathcal{U}\cap V) \longrightarrow H^2_{\mathrm{deR}}(\mathbb{R}P^2).$$

Plugging in for the known groups, we obtain an exact sequence

$$0 \longrightarrow H^1_{\mathrm{deR}}(\mathcal{U}) \oplus 0 \xrightarrow{g_1} \mathbb{R} \longrightarrow 0; \qquad \Longrightarrow \qquad H^p_{\mathrm{deR}}(\mathcal{U}) \approx \begin{cases} \mathbb{R}, & \text{if } p=0,1; \\ 0, & \text{otherwise.} \end{cases}$$

The 2-manifold $\mathbb{R}P^2\#\mathbb{R}P^2$ is obtained by joining two copies of $\mathcal{U}$, which we will call $\mathcal{U}_1$ and $\mathcal{U}_2$, along boundary annuli, so that

$$H^p_{\mathrm{deR}}(\mathcal{U}_1\cap\mathcal{U}_2) \approx H^p_{\mathrm{deR}}(S^1) \approx \begin{cases} \mathbb{R}, & \text{if } p=0,1; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, since $\mathbb{R}P^2\#\mathbb{R}P^2$ is connected,

$$H^0_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \approx \mathbb{R}.$$

By Mayer-Vietoris, we obtain

$$\begin{aligned} 0 \longrightarrow\ & H^0_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow H^0_{\mathrm{deR}}(\mathcal{U}_1) \oplus H^0_{\mathrm{deR}}(\mathcal{U}_2) \longrightarrow H^0_{\mathrm{deR}}(\mathcal{U}_1\cap\mathcal{U}_2) \\ \xrightarrow{\delta_0}\ & H^1_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow H^1_{\mathrm{deR}}(\mathcal{U}_1) \oplus H^1_{\mathrm{deR}}(\mathcal{U}_2) \xrightarrow{g_1} H^1_{\mathrm{deR}}(\mathcal{U}_1\cap\mathcal{U}_2) \\ \xrightarrow{\delta_1}\ & H^2_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow H^2_{\mathrm{deR}}(\mathcal{U}_1) \oplus H^2_{\mathrm{deR}}(\mathcal{U}_2). \end{aligned}$$

Plugging in for known groups, we obtain

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}\oplus\mathbb{R} \longrightarrow \mathbb{R} \xrightarrow{\delta_0} H^1_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow \mathbb{R}\oplus\mathbb{R} \xrightarrow{g_1} \mathbb{R} \xrightarrow{\delta_1} H^2_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow 0.$$

By exactness of the sequence, the homomorphism δ_0 is trivial, and the above sequence reduces to

$$0 \longrightarrow H^1_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow \mathbb{R}\oplus\mathbb{R} \xrightarrow{g_1} \mathbb{R} \xrightarrow{\delta_1} H^2_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow 0.$$

The restriction of the homomorphism g_1 to each component of the domain is the restriction homomorphism

$$g_1\colon H^1_{\mathrm{deR}}(\mathcal{U}) \longrightarrow H^1_{\mathrm{deR}}(\mathcal{U}\cap V)$$

of the previous paragraph, which is nontrivial. Thus, in this case g_1 must be nontrivial and δ_1 trivial. Therefore, we obtain exact sequences

$$0 \longrightarrow H^1_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow \mathbb{R}\oplus\mathbb{R} \xrightarrow{g_1} \mathbb{R} \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow H^2_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2) \longrightarrow 0.$$

It follows that

$$H^1_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2)\approx\mathbb{R} \qquad\text{and}\qquad H^0_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2)=0.$$

Alternatively, we can observe that $\mathbb{R}P^2\#\mathbb{R}P^2$ is not orientable and therefore $H^0_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2)=0$.

Here is another solution. Since $\mathbb{R}P^2\#\mathbb{R}P^2$ is a connected and non-orientable 2-manifold (being the connect-sum of such manifolds),

$$H^0_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2)\approx\mathbb{R}, \qquad H^2_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2)=0.$$

On the other hand, by MAT 530, Hurewicz Theorem, and de Rham Theorem,

$$\begin{aligned}\pi_1(\mathbb{R}P^2\#\mathbb{R}P^2)=\langle a,b|a^2b^2\rangle \qquad &\Longrightarrow \qquad H_1(\mathbb{R}P^2\#\mathbb{R}P^2;\mathbb{Z})=\mathrm{Abel}\big(\pi_1(\mathbb{R}P^2\#\mathbb{R}P^2)\big)=\mathbb{Z}\oplus\mathbb{Z}_2\\ &\Longrightarrow \qquad H_1(\mathbb{R}P^2\#\mathbb{R}P^2;\mathbb{R})\approx H_1(\mathbb{R}P^2\#\mathbb{R}P^2;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\approx\mathbb{R}\\ &\Longrightarrow \qquad H^1_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2)\approx H^1(\mathbb{R}P^2\#\mathbb{R}P^2;\mathbb{R})\approx\big(H_1(\mathbb{R}P^2\#\mathbb{R}P^2;\mathbb{R})\big)^*\approx\mathbb{R}.\end{aligned}$$

Yet another approach is to observe that $\mathbb{R}P^2\#\mathbb{R}P^2$ is homeomorphic to the Klein bottle and therefore by the de Rham Theorem

$$H^*_{\mathrm{deR}}(\mathbb{R}P^2\#\mathbb{R}P^2)\approx H^*_{\mathrm{deR}}(K).$$

On the other hand, the orientable double cover of K is the 2-torus and we can compute its cohomology as in Problem 5 on PS7.

Part III (*choose 1 problem from 7 and 8*)

7. (a) *Show that the normal bundle of S^n in $\mathbb{R}^{n+1}$ is trivial.*
(b) *Show that $S^3\times S^4$ can be embedded into $\mathbb{R}^8$.*
(c) *Let $n_1,\ldots,n_k$ be nonnegative integers and N their sum. Show that*

$$S^{n_1}\times S^{n_2}\times\ldots\times S^{n_k}$$

can be embedded into $\mathbb{R}^{N+1}$.

(a) Since $\mathbb{R}^{n+1}$ and S^n are orientable, the normal bundle of S^n in $\mathbb{R}^{n+1}$ is orientable; see Problem 3a on PS6. Since an orientable line bundle is necessarily trivial, we conclude that the normal bundle of S^n in $\mathbb{R}^{n+1}$ is trivial.

Alternatively, the normal bundle of S^n in $\mathbb{R}^{n+1}$ is isomorphic to the bundle of vectors normal to S^n:

$$\mathcal{N}\equiv\big\{(x,v)\colon x\in S^n,\ v\in T_x\mathbb{R}^{n+1},\ \langle v,w\rangle=0\ \forall\, w\in T_xS^n\big\}\longrightarrow S^n.$$

It consists of radial directions along S^n. In particular, $\mathcal{N}$ admits a nowhere-vanishing section, $\partial/\partial r$. Since $\mathcal{N}$ is a line bundle, it follows that $\mathcal{N}$ is trivial.

(b) The normal bundle of

$$S^3=S^3\times 0\subset\mathbb{R}^4\times 0\subset\mathbb{R}^4\times\mathbb{R}^4=\mathbb{R}^8$$

in $\mathbb{R}^8$ is the direct sum of the normal bundle of S^3 in $\mathbb{R}^4$ with the normal bundle of $\mathbb{R}^4$ in $\mathbb{R}^8$:

$$\mathcal{N}_{\mathbb{R}^8}S^3 = \mathcal{N}_{\mathbb{R}^4}S^3 \oplus \mathcal{N}_{\mathbb{R}^8}\mathbb{R}^4\big|_{S^3} \approx (S^3\times\mathbb{R}) \oplus (S^3\times\mathbb{R}^4) \approx S^3\times\mathbb{R}^5.$$

On the other hand, a (tubular) neighborhood of S^3 in $\mathbb{R}^8$ is diffeomorphic to $\mathcal{N}_{\mathbb{R}^8}S^3$. Thus, this neighborhood contains a submanifold diffeomorphic to the sphere bundle of $\mathcal{N}_{\mathbb{R}^8}S^3$:

$$S(\mathcal{N}_{\mathbb{R}^8}S^3) \equiv \{v\in\mathcal{N}_{\mathbb{R}^8}S^3 \colon |v|=1\} \approx S(S^3\times\mathbb{R}^5) \approx S^3\times S^4.$$

(c) For each $i=1,\ldots,k$, let

$$N_i = n_1+\ldots+n_i \qquad \text{and} \qquad X_i = S^{n_1}\times\ldots\times S^{n_i}.$$

The sphere X_1 is an embedded submanifold of $\mathbb{R}^{N_1+1}$ by definition. Suppose $2\le i\le k$ and X_{i-1} has been embedded into $\mathbb{R}^{N_{i-1}+1}$. Since X_{i-1} is an orientable N_{i-1}-manifold (being a product of orientable manifolds), the normal bundle of X_{i-1} in the orientable manifold $\mathbb{R}^{N_{i-1}+1}$ is an orientable line bundle and therefore a trivial line. The normal bundle of

$$X_{i-1}=X_{i-1}\times 0 \subset \mathbb{R}^{N_{i-1}+1}\times 0 \subset \mathbb{R}^{N_{i-1}+1}\times\mathbb{R}^{n_i}=\mathbb{R}^{N_i+1}$$

in $\mathbb{R}^{N_i+1}$ is the direct sum of the normal bundle of X_{i-1} in $\mathbb{R}^{N_{i-1}+1}$ with the normal bundle of $\mathbb{R}^{N_{i-1}+1}$ in $\mathbb{R}^{N_i+1}$:

$$\begin{aligned}\mathcal{N}_{\mathbb{R}^{N_i+1}}X_{i-1} &= \mathcal{N}_{\mathbb{R}^{N_{i-1}+1}}X_{i-1} \oplus \mathcal{N}_{\mathbb{R}^{N_i+1}}\mathbb{R}^{N_{i-1}+1}\big|_{X_{i-1}} \\ &\approx (X_{i-1}\times\mathbb{R}) \oplus (X_{i-1}\times\mathbb{R}^{n_i}) \approx X_{i-1}\times\mathbb{R}^{n_i+1}.\end{aligned}$$

On the other hand, a (tubular) neighborhood of X_{i-1} in $\mathbb{R}^{N_i+1}$ is diffeomorphic to $\mathcal{N}_{\mathbb{R}^{N_i+1}}X_{i-1}$. Thus, this neighborhood contains a submanifold diffeomorphic to the sphere bundle of $\mathcal{N}_{\mathbb{R}^{N_i+1}}X_{i-1}$:

$$\begin{aligned}S(\mathcal{N}_{\mathbb{R}^{N_i+1}}X_{i-1}) &\equiv \{v\in\mathcal{N}_{\mathbb{R}^{N_i+1}}X_{i-1} \colon |v|=1\} \\ &\approx S(X_{i-1}\times\mathbb{R}^{n_i+1}) \approx X_{i-1}\times S^{n_i} = X_i.\end{aligned}$$

Thus, X_i embeds into $\mathbb{R}^{N_i+1}$. By induction this implies the claim.

8. *Let M be a smooth Riemannian manifold.*
(a) *What is the symbol of the differential,*

$$d_p \colon E^p(M) \longrightarrow E^{p+1}(M)?$$

Under what conditions is this operator elliptic?
(b) *What is the symbol of the (formal) adjoint of the differential,*

$$\delta_p \colon E^p(M) \longrightarrow E^{p-1}(M)?$$

Under what conditions is this operator elliptic?
(c) *What is the symbol of the operator*

$$d+\delta \colon E^*(M) \longrightarrow E^*(M)?$$

Under what conditions is this operator elliptic?

(a) Suppose $x\in M$, $\alpha\in T_x^*M$, $\beta\in E^p(M)$, and $f\in C^\infty(M)$ is such that $f(x)=0$ and $df|_x=\alpha$. Since d_p is a first-order differential operator, by definition

$$\{\sigma_{d_p}(\alpha)\}\big(\beta(x)\big) \equiv d\big(f\cdot\beta\big)\big|_x = df|_x\wedge\beta|_x + (-1)^0 f|_m\cdot d\beta|_x = \alpha\wedge\beta(x)+0.$$

Thus, the symbol σ_{d_p} of d_p is the bundle homomorphism over M given by

$$\sigma_{d_p}\colon T^*M \longrightarrow \mathrm{Hom}\big(\Lambda^pT^*M,\Lambda^{p+1}T^*M\big), \quad \{\sigma_{d_p}(\alpha)\}(\beta)=\alpha\wedge\beta \quad \forall\ \alpha\in T_x^*M,\ \beta\in\Lambda^pT_x^*M,\ x\in M.$$

The operator d_p is elliptic if the homomorphism

$$\sigma_{d_p}(\alpha)\colon \Lambda^pT_x^*M \longrightarrow \Lambda^{p+1}T_x^*M$$

is an isomorphism for all $\alpha\in T_xM$, $\alpha\neq 0$, and $x\in M$. If this is the case, then

$$\binom{n}{p} = \dim\Lambda^pT_x^*M = \dim\Lambda^{p+1}T_x^*M = \binom{n}{p+1} = \frac{n-p}{p+1}\cdot\binom{n}{p},$$

where n is the dimension of M. This means that $p=(n-1)/2$, so that n is odd. On the other hand, if $p\geq 1$, then the homomorphism

$$\sigma_{d_p}(\alpha)\colon \Lambda^pT_x^*M \longrightarrow \Lambda^{p+1}T_x^*M, \qquad \beta\longrightarrow\alpha\wedge\beta,$$

is not injective; its kernel contains any element of the form $\alpha\wedge\gamma$. Thus, $\sigma_{d_p}(\alpha)$ is not an isomorphism in this case. The remaining case is $p=0$ and $n=1$. The homomorphism

$$\sigma_{d_p}(\alpha)\colon \Lambda^0T_x^*M=\mathbb{R} \longrightarrow \Lambda^1T_x^*M\approx\mathbb{R}, \qquad \beta\longrightarrow\alpha\wedge\beta,$$

is then an isomorphism if $\alpha\neq 0$. Thus, d_p is elliptic if and only if M is a one-dimensional manifold and $p=0$.

(b) Since δ_p is the formal adjoint of d_{p-1} with respect to the inner-product $\langle\langle\cdot,\cdot\rangle\rangle$ on $E^*(M)$ induced via integration by the point-wise inner-product $\langle\cdot,\cdot\rangle$ on Λ^*T^*M, $\sigma_{\delta_p}(\alpha)$ is the adjoint of $\sigma_{d_{p-1}}(\alpha)$ with respect to $\langle\cdot,\cdot\rangle$:

$$\sigma_{\delta_p}(\alpha) = \big\{\sigma_{d_{p-1}}(\alpha)\big\}^*\colon \Lambda^pT_x^*M \longrightarrow \Lambda^{p-1}T_x^*M \qquad \forall\ \alpha\in T_x^*M,\ x\in M.$$

Since the inner-product $\langle\cdot,\cdot\rangle$ is nondegenerate, the homomorphism

$$T_xM \longrightarrow T_x^*M, \qquad v\longrightarrow\langle v,\cdot\rangle,$$

is an isomorphism. For each $\alpha\in T_x^*M$, denote by $v_\alpha\in T_xM$ its preimage under this isomorphism. Since $\sigma_{d_{p-1}}(\alpha)$ is the left-wedging with α and $\sigma_{\delta_p}(\alpha)$ is its adjoint with respect to $\langle v,\cdot\rangle$, it follows that

$$\sigma_{\delta_p}(\alpha) = \iota_{v_\alpha}\colon \Lambda^pT_x^*M \longrightarrow \Lambda^{p-1}T_x^*M \qquad \forall\ \alpha\in T_x^*M,\ x\in M,$$

where ι_{v_α} is the contraction with respect to v_α; see Section 2.11 in Warner.

Since δ_p is the formal adjoint of d_{p-1}, δ_p is elliptic if and only if d_{p-1} is elliptic. By part (a), this is the case if and only if M is a one-dimensional manifold and $p=1$.

(c) Since d and δ are differential operators of the same order (first), the symbol of their sum (as long as it is defined) is the sum of their symbols. By parts (a) and (b), this means that

$$\sigma_{d+\delta} = \sigma_d + \sigma_\delta \colon T^*M \longrightarrow \mathrm{Hom}\big(\Lambda^*T^*M, \Lambda^*T^*M\big),$$
$$\beta \longrightarrow \alpha \wedge \beta + \iota_{v_\alpha}\beta, \qquad \forall\, \alpha \in T_x^*M,\ \beta \in \Lambda^*T_x^*M,\ x \in M.$$

Since $d^2=0$ and δ is the adjoint of d, $\delta^2=0$. Therefore,

$$(d+\delta)^2 = d\delta + \delta d = \Delta \colon E^*(M) \longrightarrow E^*(M).$$

Since Δ is an elliptic operator (see Section 63.5 in Warner), $d+\delta$ must be an elliptic operator as well. The reason for this is that

$$\sigma_\Delta(\alpha) = \sigma_{(d+\delta)\circ(d+\delta)}(\alpha) = \big\{\sigma_{d+\delta}(\alpha)\big\} \circ \big\{\sigma_{d+\delta}(\alpha)\big\} \colon \Lambda^*T^*M \longrightarrow \Lambda^*T^*M \quad \forall\, \alpha \in T_x^*M,\ x \in M.$$

Therefore, if $\sigma_\Delta(\alpha)$ is an isomorphism, then so is $\sigma_{d+\delta}(\alpha)$.

Bonus Problem

Determine the cohomology ring of $\mathbb{C}P^2$.

Since $\mathbb{C}P^2$ is connected (being a quotient of S^5),

$$H^0_{\mathrm{deR}}(\mathbb{C}P^2) \approx \mathbb{R}.$$

Since $\mathbb{C}P^2$ is a complex 2-manifold, it is also an orientable real 4-manifold. Since $\mathbb{C}P^2$ is compact, by Poincare Duality

$$H^4_{\mathrm{deR}}(\mathbb{C}P^2) \approx \mathbb{R}.$$

It remains to compute $H^1_{\mathrm{deR}}(\mathbb{C}P^2)$, $H^2_{\mathrm{deR}}(\mathbb{C}P^2)$, and $H^3_{\mathrm{deR}}(\mathbb{C}P^2)$.

In order to compute these cohomology groups, we'll break $\mathbb{C}P^2$ into pieces. One natural choice is to break it into three standard coordinate patches:

$$\mathcal{U}_i \equiv \big\{[X_0, X_1, X_2] \in \mathbb{C}P^2 \colon X_i \neq 0\big\} \approx \mathbb{C}^2, \qquad i = 0, 1, 2.$$

Another possibility is to split $\mathbb{C}P^2$ into a tubular neighborhood V of

$$\mathbb{C}P^1 = \big\{[X_0, X_1, X_2] \in \mathbb{C}P^2 : X_2 = 0\big\}$$

and the complement of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. The latter is precisely $\mathcal{U}_2$, while the former is diffeomorphic to the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. By Problem 4 on PS2, this normal bundle is isomorphic to γ_1^*, where $\gamma_1 \longrightarrow \mathbb{C}P^1$ is the tautological line bundle. In fact, we can take a rather large tubular neighborhood of $\mathbb{C}P^1$:

$$V \equiv \big\{[X_0, X_1, X_2] \in \mathbb{C}P^2 : (X_0, X_1) \neq 0\big\} \approx \gamma_1^*, \qquad [X_0, X_1, \alpha(X_0, X_1)] \longleftrightarrow \big([X_0, X_1], \alpha\big).$$

Note that $V = \mathcal{U}_0 \cup \mathcal{U}_1$. While we could determine the cohomology of V via Mayer-Vietoris, this turns out to be unnecessary.

We will determine the cohomology of $\mathbb{C}P^2$ using the splitting $\mathbb{C}P^2 = \mathcal{U}_2 \cup V$. First,

$$\mathcal{U}_2 \cap V = \big\{[X_0, X_1, X_2] \in \mathbb{C}P^2 : (X_0, X_1) \neq 0, X_2 \neq 0\big\} \approx \mathbb{C}^2 - 0, \quad [X_0, X_1, X_2] \longrightarrow \big(X_0/X_2, X_1/X_2\big).$$

Thus, $\mathcal{U}_2 \cap V$ is homotopy equivalent to $S^3 \subset \mathbb{C}^2$. Therefore, $\mathcal{U}_2 \cap V$ is connected and

$$H^p_{\mathrm{deR}}(\mathcal{U}_2 \cap V) \approx \begin{cases} \mathbb{R}, & \text{if } p = 0, 3; \\ 0, & \text{otherwise.} \end{cases}$$

Since V is diffeomorphic to a vector bundle over $\mathbb{C}P^1 \approx S^2$, V is homotopy equivalent to S^2. In particular, V is connected, simply-connected, and

$$H^p_{\mathrm{deR}}(V) \approx \begin{cases} \mathbb{R}, & \text{if } p = 0, 2; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, $\mathcal{U}_2$ is diffeomorphic to $\mathbb{C}^2$. Therefore, $\mathcal{U}_2$ is connected, simply-connected, and

$$H^p_{\mathrm{deR}}(\mathcal{U}_2) \approx \begin{cases} \mathbb{R}, & \text{if } p = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, by van Kampen Theorem, $\mathbb{C}P^2 = \mathcal{U}_2 \cup V$ is also simply connected. By Hurewicz Theorem, de Rham Theorem, and Poincare Duality,

$$H_1(\mathbb{C}P^2; \mathbb{Z}) = \mathrm{Abel}\big(\pi_1(\mathbb{C}P^2)\big) = 0, \qquad H_1(\mathbb{C}P^2; \mathbb{R}) \approx H_1(\mathbb{C}P^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = 0,$$
$$H^1_{\mathrm{deR}}(\mathbb{C}P^2) \approx H^1(\mathbb{C}P^2; \mathbb{R}) = \big(H_1(\mathbb{C}P^2; \mathbb{R})\big)^* = 0, \qquad H^3_{\mathrm{deR}}(\mathbb{C}P^2) \approx \big(H^1_{\mathrm{deR}}(\mathbb{C}P^2)\big)^* = 0.$$

By Mayer-Vietoris, we have an exact sequence

$$H^1_{\mathrm{deR}}(\mathcal{U}_2 \cap V) \longrightarrow H^2_{\mathrm{deR}}(\mathbb{C}P^2) \longrightarrow H^2_{\mathrm{deR}}(\mathcal{U}_2) \oplus H^2_{\mathrm{deR}}(V) \longrightarrow H^2_{\mathrm{deR}}(\mathcal{U}_2 \cap V).$$

Plugging in for the known groups, we obtain an exact sequence

$$0 \longrightarrow H^2_{\mathrm{deR}}(\mathbb{C}P^2) \longrightarrow \mathbb{R} \longrightarrow 0; \qquad \Longrightarrow \qquad H^2_{\mathrm{deR}}(\mathbb{C}P^2) \approx \mathbb{R}.$$

Finally, by Poincare Duality, the pairing

$$H^2_{\mathrm{deR}}(\mathbb{C}P^2) \otimes H^2_{\mathrm{deR}}(\mathbb{C}P^2) \longrightarrow \mathbb{R}, \qquad \big([\alpha], [\beta]\big) \longrightarrow \int_{\mathbb{C}P^2} \alpha \wedge \beta,$$

is nondegenerate. Therefore, we have an isomorphism of *graded rings*

$$H^*_{\mathrm{deR}}(\mathbb{C}P^2) \approx \mathbb{R}[u]/u^3,$$

where the degree of u is defined to be 2.

Note: We can compute $H^1_{\mathrm{deR}}(\mathbb{C}P^2)$ and $H^3_{\mathrm{deR}}(\mathbb{C}P^2)$ from MV as well. In MAT 566, we will see that the natural generator u for $H^*_{\mathrm{deR}}(\mathbb{C}P^2)$ is the first chern class of the line bundle $\gamma_2^* \longrightarrow \mathbb{C}P^2$. You should now be able to guess what the de Rham cohomology of $\mathbb{C}P^n$ is and to prove your guess inductively.