MAT 531: Topology&Geometry, II
Spring 2010

Midterm Solutions

Problem 1 (15pts)

Let $f: M \longrightarrow N$ and $g: N \longrightarrow Z$ be smooth maps between smooth manifolds. State the chain rule for the differential of the map $g\circ f: M \longrightarrow Z$ and obtain it directly from the relevant definitions (state the relevant definition(s); you do not need to show that they define the required objects).

If $h\colon X \longrightarrow Y$ is a smooth map and $x\in X$, the differential $\mathrm{d}_x h\colon T_xX \longrightarrow T_{h(x)}Y$ is defined by

$$\big\{\mathrm{d}_x h(v)\big\}(\phi) = v(\phi\circ h) \qquad \forall\, v\in T_xX,\, \phi\in C^\infty(Y).$$

Thus, for all $p\in M$, $v\in T_pM$, and $\phi\in C^\infty(Z)$,

$$\begin{aligned}\big\{\mathrm{d}_p(g\circ f)(v)\big\}(\phi) &\equiv v\big(\phi\circ g\circ f\big) \equiv \big\{\mathrm{d}_p f(v)\big\}(\phi\circ g) \equiv \big\{\mathrm{d}_{f(p)} g\big(\mathrm{d}_p f(v)\big)\big\}(\phi)\\ &= \big\{\{\mathrm{d}_{f(p)} g\circ \mathrm{d}_p f\}(v)\big\}(\phi).\end{aligned}$$

Since this equality holds for all $v\in T_pM$ and $\phi\in C^\infty(Z)$,

$$\mathrm{d}_p(g\circ f) = \mathrm{d}_{f(p)} g\circ \mathrm{d}_p f\colon T_pM \longrightarrow T_{g(f(p))}Z \qquad \forall\, p\in M;$$

this statement is the chain rule for $g\circ f$.

Problem 2 (20pts)

Let M be a smooth manifold and $p\in M$ a fixed point of a smooth map $f\colon M\longrightarrow M$, i.e. $f(p)=p$. Show that if all eigenvalues of the linear transformation

$$\mathrm{d}_p f\colon T_pM \longrightarrow T_pM$$

are different from 1 (so $\mathrm{d}_p f(v)\neq v$ for all $v\in T_pM-0$), then p is an isolated fixed point (has a neighborhood that contains no other fixed point).

We will show that the map

$$h\colon M \longrightarrow M\times M, \qquad h(x) = \big(x, f(x)\big),$$

is transverse to the diagonal $\Delta_M=\{(x,x)\colon x\in M\}$ at p, i.e.

$$T_{h(p)}(M\times M) = \operatorname{Im} \mathrm{d}_p h + T_{h(p)}\Delta_M\,.$$

This condition is equivalent to the surjectivity of the composite homomorphism

$$\mathrm{d}_p h\colon T_pM \longrightarrow T_{h(p)}(M\times M) \longrightarrow T_{h(p)}(M\times M)\big/T_{h(p)}\Delta_M\,, \tag{1}$$

where the second map is the natural projection to the quotient. Since the dimensions of the domain and target vector spaces in (1) are the same, the map (1) is not surjective if and only if

$$\mathrm{d}_p h(v) \equiv \big(v, \mathrm{d}_p f(v)\big) \in T_{h(p)}\Delta_M \qquad \Longleftrightarrow \qquad \mathrm{d}_p f(v) = v$$

for some $v \in T_pM - 0$. Thus, by our assumption on $\mathrm{d}_p f$ the homomorphism (1) is surjective. Since the map $q \longrightarrow \mathrm{d}_q h$ is continuous on M, there exists a neighborhood U of p such that

$$\mathrm{d}_q h \colon T_qM \longrightarrow T_{h(q)}(M\times M) \longrightarrow T_{h(q)}(M\times M)\big/T_{h(q)}\Delta_M$$

is onto for all $q \in h^{-1}(\Delta_M)$. Thus, $h : U \longrightarrow M \times M$ is transverse to Δ_M and by the Implicit Function Theorem

$$U \cap h^{-1}(\Delta_M) = \big\{q \in U : \, f(q) = q\big\}$$

is a smooth submanifold of U of codimension $\dim(M\times M) - \dim M = \dim M$; so $U \cap h^{-1}(\Delta_M)$ consists of isolated fixed points (and p is one of them).

Here is a direct approach. Let $\varphi \colon (U,p) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ be a smooth chart, $V = U \cap f^{-1}(U)$, and

$$g = \varphi \circ f \circ \varphi^{-1} \colon \varphi(V) \longrightarrow V \longrightarrow U \longrightarrow \mathbb{R}^n .$$

This is a smooth map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$g(\mathbf{0}) = \mathbf{0}, \qquad \mathrm{d}_{\mathbf{0}} g = \mathrm{d}_p\varphi \circ \mathrm{d}_p f \circ \mathrm{d}_{\mathbf{0}}\varphi^{-1} = \mathrm{d}_p\varphi \circ \mathrm{d}_p f \circ (\mathrm{d}_p\varphi)^{-1} \colon T_{\mathbf{0}}\mathbb{R}^n = \mathbb{R}^n \longrightarrow \mathbb{R}^n;$$

thus, $\mathrm{d}_{\mathbf{0}} g(v) \neq v$ for all $v \in \mathbb{R}^n - 0$ by our assumption on $\mathrm{d}_p f$. Furthermore, $q \in V$ is a fixed point of f if and only if $\varphi(q)$ is a fixed point of g. Thus, it is sufficient to show that $\mathbf{0}$ is an isolated fixed point of g. Suppose instead that there is a sequence $x_k \in \varphi(V) - \mathbf{0}$ converging to $\mathbf{0}$ such that $g(x_k) = x_k$ for all $k \in \mathbb{Z}^+$. Since S^{n-1} is compact, a subsequence of the sequence $v_k = x_k/|x_k|$, which we'll continue to call v_k, converges to some $v \in S^{n-1}$. Since $g(\mathbf{0}) = \mathbf{0}$, $g(x_k) = x_k$, $|x_k| \longrightarrow 0$, and g is smooth,

$$\begin{aligned} d_{\mathbf{0}} g(v) &\equiv \lim_{t\longrightarrow 0} \frac{g(tv) - g(\mathbf{0})}{t} = \lim_{k\longrightarrow\infty} \frac{g(|x_k|v)}{|x_k|} = \lim_{k\longrightarrow\infty} \frac{g(x_k) + \big(g(|x_k|v) - g(|x_k|v_k)\big)}{|x_k|} \\ &= \lim_{k\longrightarrow\infty} v_k + \lim_{k\longrightarrow\infty} \frac{g(|x_k|v) - g(|x_k|v_k)}{|x_k|} = v + 0. \end{aligned} \tag{2}$$

The reason the last limit vanishes is the following. If $g_k(t) = g(|x_k|(v_k + t(v - v_k)))$, then

$$g(|x_k|v) - g(|x_k|v_k) = g(1) - g(0) = g'(t_k) = \vec{\nabla} g|_{|x_k|(v_k + t_k(v - v_k))} \cdot |x_k|(v - v_k)$$

for some $t_k \in [0,1]$ by the Mean Value Theorem and the Chain Rule. Equation (2) contradicts our assumptions.

Problem 3 (20pts)

Let $\alpha = \mathrm{d}x_1 + f\mathrm{d}x_2$ be a smooth 1-form on $\mathbb{R}^3$ (so $f \in C^\infty(\mathbb{R}^3)$). Show that for every $p \in \mathbb{R}^3$ there exists a diffeomorphism

$$\varphi = (y_1, y_2, y_3) : U \longrightarrow V$$

from a neighborhood U of p to an open subset V of $\mathbb{R}^3$ such that $\alpha|_U = g\mathrm{d}y_1$ for some $g \in C^\infty(U)$ if and only if f does not depend on x_3.

The form $g\mathrm{d}y_1$ vanishes on the two-dimensional slices $y_1 = \mathit{const}$ and any one-form vanishing on these slices has this form. Thus, the existence of the desired charts is equivalent to the existence of two-dimensional integral submanifolds for α through every point; this is the setting of Frobenius Theorem. Since $\mathbb{R}^3$ is 3-dimensional and α is a nowhere zero one-form on $\mathbb{R}^3$, the line subbundle $\mathbb{R}\alpha \subset T^*\mathbb{R}^3$ satisfies the condition of Frobenius Theorem if and only if

$$0 = \alpha \wedge d\alpha = (\mathrm{d}x_1 + f\mathrm{d}x_2) \wedge \big(f_{x_1}\mathrm{d}x_1 \wedge \mathrm{d}x_2 + f_{x_3}\mathrm{d}x_3 \wedge \mathrm{d}x_2\big) = -f_{x_3}\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3;$$

see Problem 5 on PS5. So, the desired charts exist around every point if and only if $f_{x_3} = 0$, i.e. f does not depend on x_3.

Problem 4 (20pts)

Let $D \subset \mathbb{R}^2$ be the closed unit disk centered at the origin.
(a) *State Stokes' Theorem (for integration of top forms on manifold; no singular chains) for D.*
(b) *Show that it reduces to Green's theorem of calculus (if you do not remember what the latter says, make sure your final statement is in calculus notation).*

(a) If $\alpha \in E^1(D)$,

$$\int_D \mathrm{d}\alpha = \int_{S^1} \alpha \,,$$

where D has the standard orientation and the unit circle $S^1 = \partial D$ has the induced orientation. If $p \in S^1$ and $v \in T_pD = T_p\mathbb{R}^2$ is an outer normal vector to T_pS^1, then a vector $v_1 \in T_pS^1$ forms an oriented basis for T_pS^1 if $\{v, v_1\}$ is an oriented basis for T_pD. Thus, S^1 is oriented counter-clockwise.

(b) A one-form α on $E^1(D)$ must be of the form $\alpha = f\mathrm{d}x + g\mathrm{d}y$; then,

$$\mathrm{d}\alpha = f_y\mathrm{d}y \wedge \mathrm{d}x + g_x\mathrm{d}x \wedge \mathrm{d}y = (g_x - f_y)\mathrm{d}x \wedge \mathrm{d}y.$$

Thus, Stokes' theorem reduces to

$$\int_{S^1} f\mathrm{d}x + g\mathrm{d}y = \int_D (g_x - f_y)\mathrm{d}x \wedge \mathrm{d}y = \int_D (g_x - f_y)\mathrm{d}x\mathrm{d}y = \int_D (g_x - f_y)\mathrm{d}A.$$

In geometry, LHS is computed by taking a smooth $\sigma=(x,y)\colon[0,1]\longrightarrow S^1$ with $\sigma(0)=\sigma(1)$ which is a diffeomorphism from $(0,1)$ to $S^1-f(0)$ and traverses S^1 counter-clockwise; then

$$\int_{S^1} f\mathrm{d}x+g\mathrm{d}y = \int_{[0,1]}\sigma^*(f\mathrm{d}x+g\mathrm{d}y) = \int_{[0,1]}(f\circ\sigma\mathrm{d}x + g\circ\sigma\mathrm{d}y) = \int_0^1 \big((f\circ\sigma)x'(t)+(g\circ\sigma)y'(t)\big)\mathrm{d}t.$$

The last expression is the calculus definition of the line integral

$$\oint_{S^1} f\mathrm{d}x+g\mathrm{d}y = \oint_{S^1}(f,g)\cdot\mathrm{d}\vec{r}$$

with S^1 oriented counter-clockwise. So, Stokes' theorem reduces to

$$\oint_{S^1}(f,g)\cdot\mathrm{d}\vec{r} = \int_D (g_x-f_y)\mathrm{d}A.$$

This is Green's theorem for the disk. Note that

$$g_x - f_y = \det\begin{pmatrix}\partial_x & \partial_y\\ f & g\end{pmatrix}.$$

Problem 5 (25pts)

(a) *State the usual definition of the tautological line bundle γ_n over the real projective space $\mathbb{R}P^n$, making clear the topology on the total space and the projection map.*
(b) *Show that $\gamma_1\longrightarrow\mathbb{R}P^1$ is isomorphic to the line bundle formed by projecting the infinite Mobius Band to the circle S^1.*
(c) *Show that the line bundle $\gamma_n\longrightarrow\mathbb{R}P^n$ is not orientable (for $n\geq 1$).*

(a) $\gamma_n=\big\{(\ell,v)\in\mathbb{R}P^n\times\mathbb{R}^{n+1}\colon\ v\in\ell\big\}$ The topology on $\gamma_n\subset\mathbb{R}P^n\times\mathbb{R}^{n+1}$ is the subspace topology, and the projection map $\gamma_n\longrightarrow\mathbb{R}P^n$ is the restriction of the first-component projection.

(b) The Mobius band bundle is described by

$$\begin{gathered} MB=(\mathbb{R}\times\mathbb{R})/\mathbb{Z}\longrightarrow S^1=\mathbb{R}/\mathbb{Z}, \qquad [x,y]\longrightarrow[x], \\ k\cdot(x,y)=(x+k,(-1)^k y),\quad k\cdot x=x+k \qquad \forall\ x,y\in\mathbb{R},\ k\in\mathbb{Z}^+. \end{gathered}$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, define

$$h\colon MB=(\mathbb{R}\times\mathbb{R})/\mathbb{Z}\longrightarrow\gamma_1\subset\mathbb{R}P^1\times\mathbb{R}^2=S^1/\mathbb{Z}_2\times\mathbb{R}^2 \qquad \text{by}\quad [x,y]\longrightarrow\big([\mathrm{e}^{\pi\mathrm{i}x}],\mathrm{e}^{\pi\mathrm{i}x}y\big).$$

This map is well-defined because

$$\begin{aligned}[x,y]=\big[x+k,(-1)^k y\big]\longrightarrow\big([\mathrm{e}^{\pi\mathrm{i}(x+k)}],\mathrm{e}^{\pi\mathrm{i}(x+k)}(-1)^k y\big) &= \big([(-1)^k\mathrm{e}^{\pi\mathrm{i}x}],(-1)^k\mathrm{e}^{\pi\mathrm{i}x}(-1)^k y\big)\\ &= \big([\mathrm{e}^{\pi\mathrm{i}x}],\mathrm{e}^{\pi\mathrm{i}x}y\big).\end{aligned}$$

It is smooth because it is induced by a smooth map $\tilde{h}\colon \mathbb{R}\times\mathbb{R} \longrightarrow S^1\times\mathbb{R}^2$ followed by the projection to $\mathbb{R}P^1\times\mathbb{R}^2$ and $\gamma_1\subset\mathbb{R}P^1\times\mathbb{R}^2$ is an embedded submanifold. Since $\tilde{h}$ is an immersion, so is h. Since h is a bijection, it is a diffeomorphism. Furthermore, it maps each fiber of $MB\longrightarrow S^1$ to a fiber of $\gamma_1\longrightarrow\mathbb{R}P^1$, i.e. the diagram

$$\begin{tikzcd} MB \arrow[r, "h"] \arrow[d] & \gamma_1 \arrow[d] \\ S^1 \arrow[r, "{[x]\longrightarrow[\mathrm{e}^{\pi \mathrm{i} x}]}"] & \mathbb{R}P^1 \end{tikzcd}$$

commutes. Thus, h is an isomorphism of vector bundles.

Alternatively, one can start with

$$MB = \big([0,1]\times\mathbb{R}\big)/\sim\longrightarrow S^1, \quad (0,t)\sim(1,-t), \qquad [s,t]\longrightarrow \mathrm{e}^{2\pi \mathrm{i} s},$$

with the smooth structure specified by the two standard charts on MB. Define

$$h\colon \big([0,1]\times\mathbb{R}\big)/\sim\longrightarrow \gamma_1\subset\mathbb{R}P^1\times\mathbb{R}^2 = S^1/\mathbb{Z}_2\times\mathbb{R}^2 \qquad \text{by} \quad [s,t]\longrightarrow \big([\mathrm{e}^{\pi \mathrm{i} s}], \mathrm{e}^{\pi \mathrm{i} s}t\big).$$

This map is well-defined because

$$h\big([1,-t]\big) = \big([-1], -(-t)\big) = \big([1],t\big) = h\big([0,t]\big).$$

It is smooth because the composition of h with the inverse of each of the two charts on MB is the composition of a smooth map

$$\tilde{h}_i\colon (0,1)\times\mathbb{R} \longrightarrow S^1\times\mathbb{R}^2$$

followed by the projection to $\mathbb{R}P^1\times\mathbb{R}^2$ and $\gamma_1\subset\mathbb{R}P^1\times\mathbb{R}^2$ is an embedded submanifold. Since $\tilde{h}_i$ is an immersion, so is h. Since h is a bijection, it is a diffeomorphism. Furthermore, it maps each fiber of $MB\longrightarrow S^1$ to a fiber of $\gamma_1\longrightarrow\mathbb{R}P_1$ and thus provides an isomorphism of vector bundles.

(c) Since the Mobius Band line is not orientable, neither is $\gamma_1\longrightarrow\mathbb{R}P^1$ by part (b). Since $\mathbb{R}P^1\subset\mathbb{R}P^n$ and

$$\gamma_1 = \gamma_n\big|_{\mathbb{R}P^1}\longrightarrow\mathbb{R}P^1$$

is not orientable, neither is γ_n.

Alternatively, since γ_n is a line bundle, it is enough to show that it is not trivial. For the latter, it is enough to show that the complement of the zero section $s_0(\mathbb{R}P^n)$ in γ_n is connected (since the complement of $\mathbb{R}P^n\times 0$ in $\mathbb{R}P^n\times\mathbb{R}$ is not connected). By definition of γ_n in part (a),

$$\gamma_n - s_0(\mathbb{R}P^n) = \big\{(\ell,v)\in\mathbb{R}P^n\times(\mathbb{R}^{n+1}-0)\colon\ v\in\ell\big\}.$$

Since each $v\in\mathbb{R}^{n+1}-0$ determines a unique element $\pi(v)\in\mathbb{R}P^n$, the projection $\gamma_n\longrightarrow\mathbb{R}^{n+1}-0$ is a continuous map with inverse

$$\mathbb{R}^{n-1}-0\longrightarrow\gamma_n, \qquad v\longrightarrow\big(\pi(v),v\big).$$

Since π is continuous and $\gamma_n\subset\mathbb{R}P^n\times(\mathbb{R}^{n+1}-0)$ has the subspace topology, the inverse map is also continuous. Thus, $\gamma_n - s_0(\mathbb{R}P^n)$ is homeomorphic to $\mathbb{R}^{n+1}-0$, which is connected for $n\geq 1$, and so $\gamma_n\longrightarrow\mathbb{R}P^n$ is not orientable for $n\geq 1$.