MAT 531: Topology&Geometry, II
Spring 2011

Midterm

Give concise proofs, quoting established facts as appropriate; no treatises. You can do the problems and parts of problems in any order. You do not need to copy the statements of the problems. Please write legibly.

Problem 1 (15pts)

Suppose M is a smooth manifold, $X,Y \in \Gamma(M;TM)$ are smooth vector fields on M, and $g \in C^\infty(M)$ is a smooth function on M. Show *directly from the definition* that

$$[gX,Y] = g[X,Y] - Y(g)X\,.$$

(You can assume that $[\cdot,\cdot]$ is whatever object it is supposed to be, but do state what you are taking it to be.)

Problem 2 (20pts)

Let $f\colon M \longrightarrow N$ be a smooth surjective map.

(a) Suppose f is a submersion ($\mathrm{d}_p f$ is onto for all $p \in M$). Show that a map $h\colon N \longrightarrow \mathbb{R}$ is smooth if and only if the map $h \circ f : M \longrightarrow \mathbb{R}$ is smooth.

(b) Which of the two implications can fail if f is not assumed to be a submersion? Give an example.

Problem 3 (20pts)

Let $\alpha = \mathrm{d}x_1 + f\mathrm{d}x_2$ be a smooth 1-form on $\mathbb{R}^3$ (so $f \in C^\infty(\mathbb{R}^3)$). Show that for every $p \in \mathbb{R}^3$ there exists a diffeomorphism

$$\varphi = (y_1, y_2, y_3)\colon U \longrightarrow V$$

from a neighborhood U of p to an open subset V of $\mathbb{R}^3$ such that $\alpha|_U = \mathrm{d}y_1$ if and only if f does not depend on x_1 or x_3 (depends on x_2 only).

Problem 4 (20pts)

Let M and N be smooth nonempty manifolds and $\pi_1 : M \times N \longrightarrow M$ and $\pi_2 : M \times N \longrightarrow N$ the projection maps. Show directly from the definitions that the homomorphism

$$\Phi\colon H^1_{deR}(M) \oplus H^1_{deR}(N) \longrightarrow H^1_{deR}(M \times N), \qquad \big([\alpha],[\beta]\big) \longrightarrow \big[\pi_1^*\alpha + \pi_2^*\beta\big],$$

is well-defined and injective.

Problem 5 (25pts)

Let $V, W \longrightarrow M$ be smooth vector bundles over a smooth manifold M.

(a) Suppose V is orientable. Show that W is orientable if and only if $V \oplus W$ is.

(b) Give an example of $V, W \longrightarrow M$ non-orientable so that $V \oplus W$ is orientable.

(c) Give an example of $V, W \longrightarrow M$ non-orientable so that $V \oplus W$ is non-orientable.

For (b) and (c), specify M, V, and W and justify your answer; M need not be the same.