Notes on Smooth Manifolds and Vector Bundles

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Chapter 0

Notation and Terminology

If M is a topological space and $p \in M$, a neighborhood of p in M is an open subset U of M that contains p.

The identity element in the groups $\mathrm{GL}_k\mathbb{R}$ and $\mathrm{GL}_k\mathbb{C}$ of invertible $k \times k$ real and complex matrices will be denoted $\mathbb{I}_k$. For any set M, id_M will denote the identity map on M.

If $h\colon M \longrightarrow N$ and $f\colon V \longrightarrow X$ are maps and $V \subset N$, we will denote by $f \circ h$ the map

$$h^{-1}(V) \xrightarrow{h} V \xrightarrow{f} X\,.$$

Chapter 1

Smooth Manifolds and Maps

1 Smooth Manifolds: Definition and Examples

Definition 1.1. *A topological space M is a* topological m-manifold *if*

(TM1) *M is Hausdorff and second-countable, and*

(TM2) *every point $p \in M$ has a neighborhood U homeomorphic to $\mathbb{R}^m$.*

A chart *around p on M is a pair (U, φ), where U is a neighborhood of p in M and $\varphi: U \longrightarrow U'$ is a homeomorphism onto an open subset of $\mathbb{R}^m$.*

Thus, the set of rational numbers, $\mathbb{Q}$, in the discrete topology is a 0-dimensional topological manifold. However, the set of real numbers, $\mathbb{R}$, in the discrete topology is not a 0-dimensional manifold because it does not have a countable basis. On the other hand, $\mathbb{R}$ with its standard topology is a 1-dimensional topological manifold, since

(TM1:$\mathbb{R}$) $\mathbb{R}$ is Hausdorff (being a metric space) and second-countable;

(TM2:$\mathbb{R}$) the map $\varphi = \mathrm{id}: U = \mathbb{R} \longrightarrow \mathbb{R}$ is a homeomorphism; thus, $(\mathbb{R}, \mathrm{id})$ is a chart around every point $p \in \mathbb{R}$.

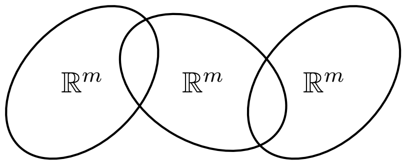
A topological space satisfying (TM2) in Definition 1.1 is called locally Euclidean; such a space is made up of copies of $\mathbb{R}^m$ glued together; see Figure 1.1. While every point in a locally Euclidean space has a neighborhood which is homeomorphic to $\mathbb{R}^m$, the space itself need not be Hausdorff; see Example 1.2 below. A Hausdorff locally Euclidean space is easily seen to be regular, while a regular second-countable space is normal [7, Theorem 32.1], metrizable (Urysohn Metrization Theorem [7, Theorem 34.1]), paracompact [7, Theorem 41.4], and thus admits partitions of unity (see Definition 11.1 below).

Example 1.2. Let $M = (0 \times \mathbb{R} \sqcup 0' \times \mathbb{R})/\sim$, where $(0, s) \sim (0', s)$ for all $s \in \mathbb{R} - 0$. As sets, $M = \mathbb{R} \sqcup \{0'\}$. Let $\mathcal{B}$ be the collection of all subsets of $\mathbb{R} \sqcup \{0'\}$ of the form

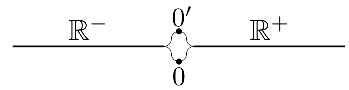
$$(a, b) \subset \mathbb{R}, \quad a, b \in \mathbb{R}, \qquad (a, b)' \equiv \big((a, b) - 0\big) \sqcup \{0'\} \quad \text{if } a < 0 < b.$$

This collection $\mathcal{B}$ forms a basis for the quotient topology on M. Note that

(TO1) any neighborhoods U of 0 and U' of $0'$ in M intersect, and thus M is not Hausdorff;



locally Euclidean space



line with two origins

Figure 1.1: A locally Euclidean space M, such as an m-manifold, consists of copies of $\mathbb{R}^m$ glued together. The line with two origins is a non-Hausdorff locally Euclidean space.

(TO2) the subsets $M-0'$ and $M-0$ of M are open in M and homeomorphic to $\mathbb{R}$; thus, M is locally Euclidean.

This example is illustrated in the right diagram in Figure 1.1. The two thin lines have length zero: $\mathbb{R}^-$ continues through 0 and $0'$ to $\mathbb{R}^+$. Since M is not Hausdorff, it cannot be topologically embedded into $\mathbb{R}^m$ (and thus cannot be accurately depicted in a diagram). Note that the quotient map

$$q\colon 0\times\mathbb{R}\sqcup 0'\times\mathbb{R} \longrightarrow M$$

is open (takes open sets to open sets); so open quotient maps do not preserve separation properties. In contrast, the image of a *closed* quotient map from a normal topological space is still normal [7, Lemma 73.3].

Definition 1.3. *A smooth m-manifold is a pair $(M,\mathcal{F})$, where M is a topological m-manifold and $\mathcal{F}=\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of charts on M such that*

(SM1) $M = \bigcup\limits_{\alpha\in\mathcal{A}} U_\alpha$,

(SM2) $\varphi_\alpha\circ\varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha\cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha\cap U_\beta)$ *is a smooth map (between open subsets of $\mathbb{R}^m$) for all $\alpha,\beta\in\mathcal{A}$;*

(SM3) $\mathcal{F}$ *is maximal with respect to* (SM2).

The collection $\mathcal{F}$ is called a smooth structure on M.

Since the maps φ_α and φ_β in Definition 1.3 are homeomorphisms, $\varphi_\beta(U_\alpha\cap U_\beta)$ and $\varphi_\alpha(U_\alpha\cap U_\beta)$ are open subsets of $\mathbb{R}^m$, and so the notion of a *smooth* map between them is well-defined; see Figure 1.2. Since $\{\varphi_\alpha\circ\varphi_\beta^{-1}\}^{-1}=\varphi_\beta\circ\varphi_\alpha^{-1}$, *smooth map* in (SM2) can be replaced by *diffeomorphism*. If $\alpha=\beta$,

$$\varphi_\alpha\circ\varphi_\beta^{-1}=\mathrm{id} : \varphi_\beta(U_\alpha\cap U_\beta)=\varphi_\alpha(U_\alpha) \longrightarrow \varphi_\alpha(U_\alpha\cap U_\beta)=\varphi_\alpha(U_\alpha)$$

is of course a smooth map, and so it is sufficient to verify the smoothness requirement of (SM2) only for $\alpha\neq\beta$.

An element of such a collection $\mathcal{F}$ will be called a smooth chart on the smooth manifold on $(M,\mathcal{F})$ or simply M.

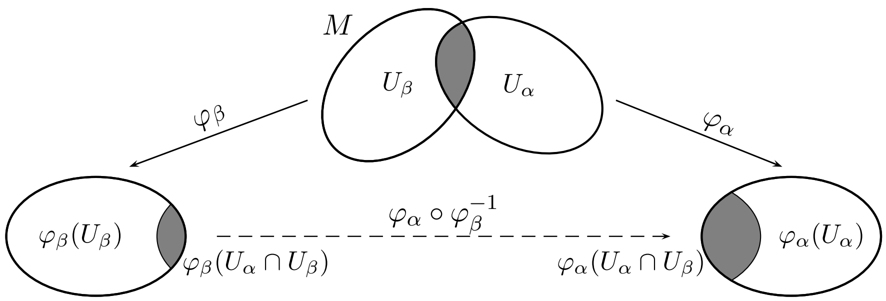


Figure 1.2: The overlap map between two charts is a map between open subsets of $\mathbb{R}^m$.

It is hardly ever practical to specify a smooth structure $\mathcal{F}$ on a manifold M by listing all elements of $\mathcal{F}$. Instead $\mathcal{F}$ can be specified by describing a collection of charts $\mathcal{F}_0 = \{(U,\varphi)\}$ satisfying (SM1) and (SM2) in Definition 1.3 and setting

$$\mathcal{F} = \big\{\text{chart } (V,\psi) \text{ on } M \big|\ \varphi\circ\psi^{-1}\colon \psi(U\cap V) \longrightarrow \varphi(U\cap V) \text{ is diffeomorphism } \forall\, (U,\varphi)\in\mathcal{F}_0\big\}. \quad (1.1)$$

Example 1.4. The map $\varphi = \mathrm{id}\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a homeomorphism, and thus the pair $(\mathbb{R}^m, \mathrm{id})$ is a chart around every point in the topological m-manifold $M = \mathbb{R}^m$. So, the single-element collection $\mathcal{F}_0 = \{(\mathbb{R}^m, \mathrm{id})\}$ satisfies (SM1) and (SM2) in Definition 1.3. It thus induces a smooth structure $\mathcal{F}$ on $\mathbb{R}^m$; this smooth structure is called the standard smooth structure on $\mathbb{R}^m$.

Example 1.5. Every finite-dimensional vector space V has a canonical topology specified by the requirement that any vector-space isomorphism $\varphi\colon V \longrightarrow \mathbb{R}^m$, where $m = \dim V$, is a homeomorphism (with respect to the standard topology on $\mathbb{R}^m$). If $\psi : V \longrightarrow \mathbb{R}^m$ is another vector-space isomorphism, then the map

$$\varphi\circ\psi^{-1}\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m \quad (1.2)$$

is an invertible linear transformation; thus, it is a diffeomorphism and in particular a homeomorphism. So, two different isomorphisms $\varphi, \psi\colon V \longrightarrow \mathbb{R}^m$ determine the same topology on V. Each pair (V,φ) is then a chart on V, and the one-element collection $\mathcal{F}_0 = \{(V,\varphi)\}$ determines a smooth structure $\mathcal{F}$ on V. Since the map (1.2) is a diffeomorphism, $\mathcal{F}$ is independent of the choice of vector-space isomorphism $\varphi : V \longrightarrow \mathbb{R}^m$. Thus, every finite-dimensional vector space carries a canonical smooth structure.

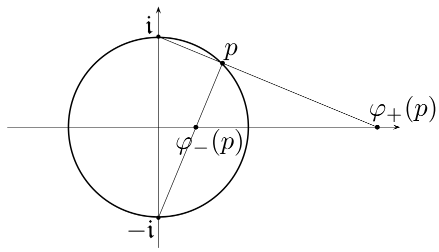
Example 1.6. The map $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(t) = t^3$, is a homeomorphism, and thus the pair $(\mathbb{R},\varphi)$ is a chart around every point in the topological 1-manifold $M = \mathbb{R}$. So, the single-element collection $\mathcal{F}_0' = \{(\mathbb{R},\varphi)\}$ satisfies (SM1) and (SM2) in Definition 1.3. It thus induces a smooth structure $\mathcal{F}'$ on $\mathbb{R}$. While $\mathcal{F}' \neq \mathcal{F}$, where $\mathcal{F}$ is the standard smooth structure on $\mathbb{R}^1$ described in Example 1.4, the smooth manifolds $(\mathbb{R}^1, \mathcal{F})$ and $(\mathbb{R}^1, \mathcal{F}')$ are diffeomorphic in the sense of (2) in Definition 2.1 below.

Example 1.7. Let $M = S^1$ be the unit circle in the complex (s,t)-plane,

$$U_+ = S^1 - \{\mathfrak{i}\}, \qquad U_- = S^1 - \{-\mathfrak{i}\}.$$

For each $p \in U_\pm$, let $\varphi_\pm(p) \in \mathbb{R}$ be the s-intercept of the line through the points $\pm\mathfrak{i}$ and $p \neq \pm\mathfrak{i}$; see Figure 1.3. The maps $\varphi_\pm\colon U_\pm \longrightarrow \mathbb{R}$ are homeomorphisms and $S^1 = U_+ \cup U_-$. Since

$$U_+ \cap U_- = S^1 - \{\mathfrak{i}, -\mathfrak{i}\} = U_+ - \{-\mathfrak{i}\} = U_- - \{\mathfrak{i}\}$$



$$\varphi_+(s,t) = \frac{s}{1-t}$$

$$\varphi_-(s,t) = \frac{s}{1+t}$$

$$\varphi_+\circ\varphi_-^{-1}\colon \mathbb{R}^* \longrightarrow \mathbb{R}^*, \quad \varphi_+\circ\varphi_-^{-1}(s) = 1/s$$

Figure 1.3: A pair of charts on S^1 determining a smooth structure.

and $\varphi_\pm(U_+\cap U_-)=\mathbb{R}-0\equiv\mathbb{R}^*$, the overlap map is

$$\varphi_+\circ\varphi_-^{-1}\colon \varphi_-(U_+\cap U_-)=\mathbb{R}^* \longrightarrow \varphi_+(U_+\cap U_-)=\mathbb{R}^*;$$

by a direct computation, this map is $s\longrightarrow s^{-1}$. Since this map is a diffeomorphism between open subsets of $\mathbb{R}^1$, the collection

$$\mathcal{F}_0 = \big\{(U_+,\varphi_+),(U_-,\varphi_-)\big\}$$

determines a smooth structure $\mathcal{F}$ on S^1.

A smooth structure on the unit sphere $M=S^m\subset\mathbb{R}^{m+1}$ can be defined similarly: take $U_\pm\subset S^m$ to be the complement of the point $q_\pm\in S^m$ with the last coordinate ±1 and $\varphi_\pm(p)\in\mathbb{R}^m$ the intersection of the line through $q_\pm$ and $p\neq q_\pm$ with $\mathbb{R}^m=\mathbb{R}^m\times 0$. This smooth structure is the unique one with which S^m is a submanifold of $\mathbb{R}^{m+1}$; see Definition 5.1 and Corollary 5.8.

Example 1.8. Let $\mathrm{MB}=([0,1]\times\mathbb{R})/\!\sim$, $(0,t)\sim(1,-t)$, be the infinite Mobius Band,

$$U_0=(0,1)\times\mathbb{R}\subset\mathrm{MB}, \qquad \varphi_0=\mathrm{id}\colon U_0\longrightarrow(0,1)\times\mathbb{R},$$

$$\varphi_{1/2}\colon U_{1/2}=\mathrm{MB}-\{1/2\}\times\mathbb{R}\longrightarrow(0,1)\times\mathbb{R}, \quad \varphi_{1/2}([s,t])=\begin{cases}(s-1/2,t), & \text{if } s\in(1/2,1],\\ (s+1/2,-t), & \text{if } s\in[0,1/2),\end{cases}$$

where $[s,t]$ denotes the equivalence class of $(s,t)\in[0,1]\times\mathbb{R}$ in MB. The pairs (U_0,φ_0) and $(U_{1/2},\varphi_{1/2})$ are then charts on the topological 1-manifold MB. The overlap map between them is

$$\varphi_{1/2}\circ\varphi_0^{-1}\colon \varphi_0(U_0\cap U_{1/2})=\big((0,1/2)\cup(1/2,1)\big)\times\mathbb{R}\longrightarrow\varphi_{1/2}(U_0\cap U_{1/2})=\big((0,1/2)\cup(1/2,1)\big)\times\mathbb{R},$$

$$\varphi_{1/2}\circ\varphi_0^{-1}(s,t)=\begin{cases}(s+1/2,-t), & \text{if } s\in(0,1/2);\\ (s-1/2,t), & \text{if } s\in(1/2,1);\end{cases}$$

see Figure 1.4. Since this map is a diffeomorphism between open subsets of $\mathbb{R}^2$, the collection

$$\mathcal{F}_0=\big\{(U_0,\varphi_0),(U_{1/2},\varphi_{1/2})\big\}$$

determines a smooth structure $\mathcal{F}$ on MB.

Example 1.9. The real projective space of dimension n, denoted $\mathbb{R}P^n$, is the space of real one-dimensional subspaces ℓ of $\mathbb{R}^{n+1}$ (or lines through the origin in $\mathbb{R}^{n+1}$) in the natural quotient topology. In other words, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a nonzero vector in

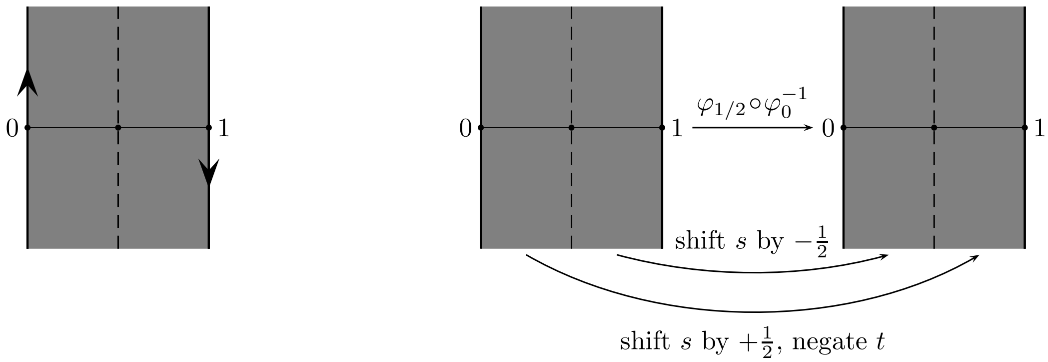


Figure 1.4: The infinite Mobius band MB is obtained from an infinite strip by identifying the two infinite edges in opposite directions, as indicated by the arrows in the first diagram. The two charts on MB of Example 1.8 overlap smoothly.

$\mathbb{R}^{n+1}$, i.e. an element of $\mathbb{R}^{n+1}-0$. Two such vectors determine the same one-dimensional subspace in $\mathbb{R}^{n+1}$ and the same element of $\mathbb{R}P^n$ if and only if they differ by a non-zero scalar. Thus, as sets

$$\begin{gathered}\mathbb{R}P^n = \big(\mathbb{R}^{n+1}-0\big)/\mathbb{R}^* \equiv \big(\mathbb{R}^{n+1}-0\big)/\sim, \qquad \text{where}\\ c\cdot v = cv \in \mathbb{R}^{n+1}-0 \quad \forall\, c\in\mathbb{R}^*,\, v\in\mathbb{R}^{n+1}-0, \qquad v\sim cv \quad \forall\, c\in\mathbb{R}^*,\, v\in\mathbb{R}^{n+1}-0.\end{gathered}$$

Alternatively, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a unit vector in $\mathbb{R}^{n+1}$, i.e. an element of S^n. Two such vectors determine the same element of $\mathbb{R}P^n$ if and only if they differ by a non-zero scalar, which in this case must necessarily be ±1. Thus, as sets

$$\begin{gathered}\mathbb{R}P^n = S^n/\mathbb{Z}_2 \equiv S^n/\sim, \qquad \text{where}\\ \mathbb{Z}_2=\{\pm1\}, \quad c\cdot v = cv \in S^n \quad \forall\, c\in\mathbb{Z}_2,\, v\in S^n, \quad v\sim cv \quad \forall\, c\in\mathbb{Z}_2,\, v\in S^2.\end{gathered} \tag{1.3}$$

Thus, as sets,

$$\mathbb{R}P^n = \big(\mathbb{R}^{n+1}-0\big)/\mathbb{R}^* = S^n/\mathbb{Z}_2.$$

It follows that $\mathbb{R}P^n$ has two natural quotient topologies; these two topologies are the same, however. The space $\mathbb{R}P^n$ has a natural smooth structure, induced from that of $\mathbb{R}^{n+1}-0$ and S^n. It is generated by the $n+1$ charts

$$\begin{gathered}\varphi_i\colon U_i \equiv \big\{[X_0,X_1,\ldots,X_n]\colon\ X_i\neq0\big\} \longrightarrow \mathbb{R}^n,\\ [X_0,X_1,\ldots,X_n] \longrightarrow \bigg(\frac{X_0}{X_i},\ldots,\frac{X_{i-1}}{X_i},\frac{X_{i+1}}{X_i},\ldots,\frac{X_n}{X_i}\bigg).\end{gathered}$$

Note that $\mathbb{R}P^1=S^1$.

Example 1.10. The complex projective space of dimension n, denoted $\mathbb{C}P^n$, is the space of complex one-dimensional subspaces of $\mathbb{C}^{n+1}$ in the natural quotient topology. Similarly to the real case of Example 1.9,

$$\begin{gathered}\mathbb{C}P^n = \big(\mathbb{C}^{n+1}-0\big)/\mathbb{C}^* = S^{2n+1}/S^1, \qquad \text{where}\\ S^1=\big\{c\in\mathbb{C}^*\colon |c|=1\big\}, \qquad S^{2n+1}=\big\{v\in\mathbb{C}^{n+1}-0\colon |v|=1\big\},\\ c\cdot v = cv \in \mathbb{C}^{n+1}-0 \quad \forall\, c\in\mathbb{C}^*,\, v\in\mathbb{C}^{n+1}-0.\end{gathered}$$

The two quotient topologies on $\mathbb{C}P^n$ arising from these quotients are again the same. The space $\mathbb{C}P^n$ has a natural complex structure, induced from that of $\mathbb{C}^{n+1}-0$.

There are a number of canonical ways of constructing new smooth manifolds.

Proposition 1.11. *(1) If $(M,\mathcal{F})$ is a smooth m-manifold, $U\subset M$ is open, and*

$$\mathcal{F}|_U \equiv \big\{(U_\alpha\cap U, \varphi_\alpha|_{U_\alpha\cap U})\colon (U_\alpha,\varphi_\alpha)\in\mathcal{F}\big\} = \big\{(U_\alpha,\varphi_\alpha)\in\mathcal{F}\colon U_\alpha\subset U\big\}, \tag{1.4}$$

then $(U,\mathcal{F}|_U)$ is also a smooth m-manifold.
(2) If $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ are smooth manifolds, then the collection

$$\mathcal{F}_0 = \big\{(U_\alpha\times V_\beta, \varphi_\alpha\times\psi_\beta)\colon (U_\alpha,\varphi_\alpha)\in\mathcal{F}_M,\ (V_\beta,\psi_\beta)\in\mathcal{F}_N\big\} \tag{1.5}$$

satisfies (SM1) *and* (SM2) *of Definition 1.3 and thus induces a smooth structure on $M\times N$.*

It is immediate that the second collection in (1.4) is contained in the first. The first collection is contained in the second because $\mathcal{F}$ is maximal with respect to (SM2) in Definition 1.3 and the restriction of a smooth map from an open subset of $\mathbb{R}^m$ to a smaller open subset is still smooth. Since every element $(U_\alpha,\varphi_\alpha)$ of $\mathcal{F}$ is a chart on M, every such element with $U_\alpha\subset U$ is also a chart on U. Since $\{U_\alpha\colon (U_\alpha,\varphi_\alpha)\in\mathcal{F}\}$ is an open cover of M, $\{U_\alpha\cap U\colon (U_\alpha,\varphi_\alpha)\in\mathcal{F}\}$ is an open cover of U. Since $\mathcal{F}$ satisfies (SM2) in Definition 1.3, so does its subcollection $\mathcal{F}|_U$. Since $\mathcal{F}$ is maximal with respect to (SM2) in Definition 1.3, so is its subcollection $\mathcal{F}|_U$. Thus, $\mathcal{F}|_U$ is indeed a smooth structure on U.

Let $m=\dim M$ and $n=\dim N$. Since each $(U_\alpha,\varphi_\alpha)\in\mathcal{F}_M$ is a chart on M and each $(V_\beta,\psi_\beta)\in\mathcal{F}_N$ is a chart on N,

$$\varphi_\alpha\times\psi_\beta\colon U_\alpha\times V_\beta \longrightarrow \varphi_\alpha(U_\alpha)\times\psi_\beta(V_\beta) \subset \mathbb{R}^m\times\mathbb{R}^n = \mathbb{R}^{m+n}$$

is a homeomorphism between an open subset of $M\times N$ (in the product topology) and an open subset of $\mathbb{R}^{m+n}$. Since the collections $\{U_\alpha\colon (U_\alpha,\varphi_\alpha)\in\mathcal{F}_M\}$ and $\{V_\beta\colon (V_\beta,\psi_\beta)\in\mathcal{F}_N\}$ cover M and N, respectively, the collection

$$\big\{U_\alpha\times V_\beta\colon (U_\alpha,\varphi_\alpha)\in\mathcal{F}_M,\ (V_\beta,\psi_\beta)\in\mathcal{F}_N\big\}$$

covers $M\times N$. If $(U_\alpha\times V_\beta,\varphi_\alpha\times\psi_\beta)$ and $(U_{\alpha'}\times V_{\beta'},\varphi_{\alpha'}\times\psi_{\beta'})$ are elements of the collection (1.5),

$$\begin{gathered}
U_\alpha\times V_\beta\cap U_{\alpha'}\times V_{\beta'} = \big(U_\alpha\cap U_{\alpha'}\big)\times\big(V_\beta\cap V_{\beta'}\big),\\
\big\{\varphi_\alpha\times\psi_\beta\big\}\big(U_\alpha\times V_\beta\cap U_{\alpha'}\times V_{\beta'}\big) = \varphi_\alpha\big(U_\alpha\cap U_{\alpha'}\big)\times\psi_\beta\big(V_\beta\cap V_{\beta'}\big)\subset\mathbb{R}^{m+n},\\
\big\{\varphi_{\alpha'}\times\psi_{\beta'}\big\}\big(U_\alpha\times V_\beta\cap U_{\alpha'}\times V_{\beta'}\big) = \varphi_{\alpha'}\big(U_\alpha\cap U_{\alpha'}\big)\times\psi_{\beta'}\big(V_\beta\cap V_{\beta'}\big)\subset\mathbb{R}^{m+n},
\end{gathered}$$

and the overlap map,

$$\big\{\varphi_\alpha\times\psi_\beta\big\}\circ\big\{\varphi_{\alpha'}\times\psi_{\beta'}\big\}^{-1} = \big\{\varphi_\alpha\circ\varphi_{\alpha'}^{-1}\big\}\times\big\{\varphi_\beta\circ\varphi_{\beta'}^{-1}\big\},$$

is the product of the overlap maps for M and N; thus, it is smooth. So the collection (1.5) satisfies the requirements (SM1) and (SM2) of Definition 1.3 and thus induces a smooth structure on $M\times N$, called the product smooth structure.

Corollary 1.12. *The general linear group,*

$$\mathrm{GL}_n\mathbb{R} = \big\{A\in\mathrm{Mat}_{n\times n}\mathbb{R}\colon \det A\neq 0\big\},$$

is a smooth manifold of dimension n^2.

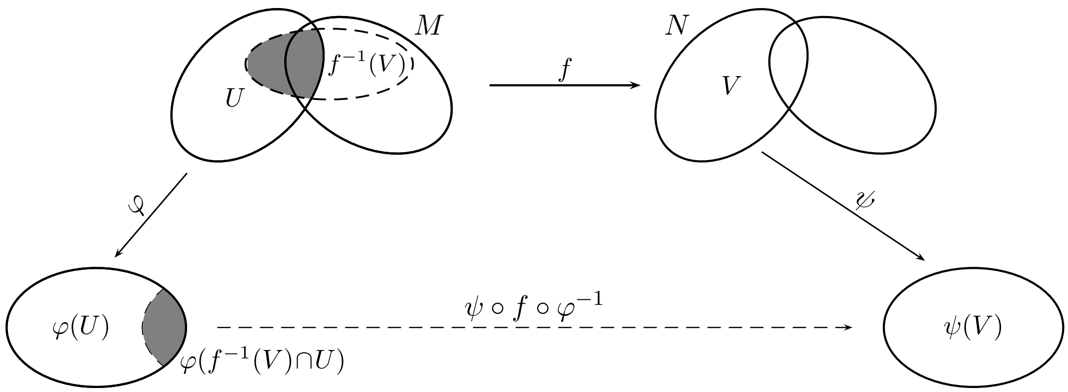


Figure 1.5: A continuous map f between manifolds is smooth if it induces smooth maps between open subsets of Euclidean spaces via the charts.

The map

$$\det\colon \mathrm{Mat}_{n\times n}\mathbb{R} \approx \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$

is continuous. Since $\mathbb{R}-0$ is an open subset of $\mathbb{R}$, its pre-image under det, $\mathrm{GL}_n\mathbb{R}$, is an open subset of $\mathbb{R}^{n^2}$ and thus is a smooth manifold of dimension n^2 by (1) of Proposition 1.11.

2 Smooth Maps: Definition and Examples

Definition 2.1. *Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth manifolds.*

(1) A continuous map $f : M \longrightarrow N$ is a smooth map between $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ if for all $(U,\varphi)\in\mathcal{F}_M$ and $(V,\psi)\in\mathcal{F}_N$ the map

$$\psi\circ f\circ\varphi^{-1}\colon \varphi\big(f^{-1}(V)\cap U\big) \longrightarrow \psi(V) \tag{2.1}$$

is a smooth map (between open subsets of Euclidean spaces).

(2) A smooth bijective map $f : (M,\mathcal{F}_M) \longrightarrow (N,\mathcal{F}_N)$ is a diffeomorphism if the inverse map, $f^{-1}\colon (N,\mathcal{F}_N)\longrightarrow(M,\mathcal{F}_M)$, is also smooth.

(3) A smooth map $f : (M,\mathcal{F}_M) \longrightarrow (N,\mathcal{F}_N)$ is a local diffeomorphism if for every $p\in M$ there are open neighborhoods U_p of p in M and V_p of $f(p)$ in N such that $f|_{U_p} : U_p \longrightarrow V_p$ is a diffeomorphism between the smooth manifolds $(U_p,\mathcal{F}_M|_{U_p})$ and $(V_p,\mathcal{F}_N|_{V_p})$.

If $f : M \longrightarrow N$ is a continuous map and $(V,\psi)\in\mathcal{F}_N$, $f^{-1}(V)\subset M$ is open and $\psi(V)\subset\mathbb{R}^n$ is open, where $n=\dim N$. If in addition $(U,\varphi)\in\mathcal{F}_M$, then $\varphi(f^{-1}(V)\cap U)$ is an open subset of $\mathbb{R}^m$, where $m=\dim M$. Thus, (2.1) is a map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and so the notion of a *smooth* map between them is well-defined; see Figure 1.5.

A bijective local diffeomorphism is a diffeomorphism, and vice versa. In particular, the identity map $\mathrm{id}\colon (M,\mathcal{F}) \longrightarrow (M,\mathcal{F})$ on any manifold is a diffeomorphism, since for all $(U,\varphi),(V,\psi)\in\mathcal{F}_M$ the map (2.1) is simply

$$\psi\circ\varphi^{-1}\colon \varphi\big(U\cap V\big) \longrightarrow \psi\big(U\cap V\big) \subset \psi(V);$$

it is smooth by (SM2) in Definition 1.3. For the same reason, the map

$$\varphi\colon \big(U, \mathcal{F}_M|_U\big) \longrightarrow \varphi(U) \subset \mathbb{R}^m$$

is a diffeomorphism for every $(U,\varphi)\in\mathcal{F}_M$. A composition of two smooth maps (local diffeomorphisms, diffeomorphisms) is again smooth (a local diffeomorphism, a diffeomorphism).

It is generally impractical to verify that the map (2.1) is smooth for all $(U,\varphi)\in\mathcal{F}_M$ and $(V,\psi)\in\mathcal{F}_N$. The following lemma provides a simpler way of checking whether a map between two smooth manifolds is smooth.

Lemma 2.2. *Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth manifolds and $f\colon M\longrightarrow N$ a map.*

(1) If $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of M, then $f: M\longrightarrow N$ is a smooth map (local diffeomorphism) if and only if for every $\alpha\in\mathcal{A}$ the restriction $f|_{U_\alpha}: U_\alpha\longrightarrow N$ is a smooth map (local diffeomorphism) with respect to the induced smooth structure on U_α of Proposition 1.11.

(2) If $\mathcal{F}_{M;0}$ and $\mathcal{F}_{N;0}$ are collections of charts on M and N, respectively, that generate $\mathcal{F}_M$ and $\mathcal{F}_N$ in the sense of (1.1), then $f\colon M\longrightarrow N$ is a smooth map (local diffeomorphism) if and only if (2.1) is a smooth map (local diffeomorphism) for every $(U,\varphi)\in\mathcal{F}_{M;0}$ and $(V,\psi)\in\mathcal{F}_{N;0}$.

Thus, $f: M\longrightarrow N$ is a smooth map (local diffeomorphism) if and only if (2.1) is a smooth map (local diffeomorphism) for every $(U,\varphi)\in\mathcal{F}_{M;0}$ and all $(V,\psi)\in\mathcal{F}_{N;0}$ in a subcollection of $\mathcal{F}_{N;0}$ covering $f(U)$. If follows that for every chart $(U,\varphi)\in\mathcal{F}_M$ the map

$$\varphi\colon U \longrightarrow \varphi(U) \subset \mathbb{R}^m$$

is a diffeomorphism.

By Lemma 2.2, if $f\colon(M,\mathcal{F}_M)\longrightarrow(N,\mathcal{F}_N)$ is smooth, then $\psi\circ f\colon f^{-1}(V)\longrightarrow\mathbb{R}^n$ is also a smooth map from an open subset of M (with the smooth structure induced from $\mathcal{F}_M$ as in Proposition 1.11) for every $(V,\psi)\in\mathcal{F}_N$. If in addition f is a diffeomorphism (and thus $m=n$),

$$\psi\circ f\circ\varphi^{-1}\colon \varphi\big(U\cap f^{-1}(V)\big) \longrightarrow \psi\big(f(U)\cap V\big) \subset \mathbb{R}^m$$

is a diffeomorphism for every $(U,\varphi)\in\mathcal{F}_M$, and thus $(f^{-1}(V),\psi\circ f)\in\mathcal{F}_M$ by the maximality of $\mathcal{F}_M$. It follows that every diffeomorphism $f\colon(M,\mathcal{F}_M)\longrightarrow(N,\mathcal{F}_N)$, which is a map $f\colon M\longrightarrow N$ with certain properties, induces a map

$$f^*\colon \mathcal{F}_N \longrightarrow \mathcal{F}_M, \qquad (V,\psi) \longrightarrow \big(f^{-1}(V),\psi\circ f\big),$$

which is easily seen to be bijective. However, there are lots of bijections $\mathcal{F}_N\longrightarrow\mathcal{F}_M$, and most of them do not arise from a diffeomorphism $f\colon M\longrightarrow N$ (which may not exist at all) or even some map between the underlying spaces.

Example 2.3. Let V and W be finite-dimensional vector spaces with the canonical smooth structures of Example 1.5 and $f: V\longrightarrow W$ a vector-space homomorphism. If $\varphi: V\longrightarrow\mathbb{R}^m$ and $\psi\colon W\longrightarrow\mathbb{R}^n$ are vector-space isomorphisms,

$$\psi\circ f\circ\varphi^{-1}\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

is a linear map and thus smooth. Since $f(V)$ is contained in the domain of ψ, it follows that $f\colon V\longrightarrow W$ is a smooth map. So every homomorphism between finite-dimensional vector spaces is a smooth map with respect to the canonical smooth structures on the vector spaces.

Example 2.4. Let $\mathrm{Mat}_{n\times n}\mathbb{R}$ be the vector space of $n\times n$ real matrices with the canonical smooth structure of Example 1.5. Define

$$f\colon \mathrm{Mat}_{n\times n}\mathbb{R} \longrightarrow \mathrm{Mat}_{n\times n}\mathbb{R} \qquad \text{by} \qquad A \longrightarrow A^{\mathrm{tr}}A, \tag{2.2}$$

where A^{tr} is the transpose of A. If $\varphi\colon \mathrm{Mat}_{n\times n}\mathbb{R} \longrightarrow \mathbb{R}^{n^2}$ is an isomorphism of vector spaces (for example, with each component of f sending a matrix to one of its entries), then each component of the map

$$\varphi\circ f\circ\varphi^{-1}\colon \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$$

is a homogeneous quadratic polynomial on $\mathbb{R}^{n^2}$; so $\varphi\circ f\circ\varphi^{-1}$ is a smooth map. Since the image of f is contained in the domain of φ, it follows that the map (2.2) is smooth. The image of f is actually contained in the linear subspace $\mathrm{SMat}_n\mathbb{R}$ of symmetric $n\times n$ matrices. Thus, f induces a map

$$f_0\colon \mathrm{Mat}_{n\times n}\mathbb{R} \longrightarrow \mathrm{SMat}_n\mathbb{R}, \qquad f_0(A) = f(A),$$

obtained by restricting the range of f; so the diagram

$$\begin{tikzcd} & \mathrm{SMat}_n\mathbb{R} \arrow[d, "\iota"] \\ \mathrm{Mat}_{n\times n}\mathbb{R} \arrow[ur, dashed, "f_0"] \arrow[r, "f"] & \mathrm{Mat}_{n\times n}\mathbb{R} \end{tikzcd}$$

where ι is the inclusion map, commutes. The induced map f_0 is also smooth with respect to the canonical smooth structures on $\mathrm{Mat}_{n\times n}\mathbb{R}$ and $\mathrm{SMat}_n\mathbb{R}$. In fact, if $\psi\colon \mathrm{SMat}_n\mathbb{R} \longrightarrow \mathbb{R}^{n(n+1)/2}$ is an isomorphism of vector spaces (for example, with each component of f sending a matrix to one of its upper-triangular entries), then each component of the map

$$\psi\circ f\circ\varphi^{-1}\colon \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n(n+1)/2}$$

is again a homogeneous quadratic polynomial on $\mathbb{R}^{n^2}$; so $\psi\circ f\circ\varphi^{-1}$ is a smooth map and thus f_0 is smooth. The smoothness of f_0 also follows directly from the smoothness of f because $\mathrm{SMat}_n\mathbb{R}$ is an embedded submanifold of $\mathrm{Mat}_{n\times n}\mathbb{R}$; see Proposition 5.5.

Example 2.5. Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth manifolds and $\mathcal{F}_{M\times N}$ the product smooth structure on $M\times N$ of Proposition 1.11. Let $\mathcal{F}_0$ be as in (1.5).

(1) For each $q\in N$, the inclusion as a "horizontal" slice,

$$\iota_q\colon M \longrightarrow M\times N, \qquad p \longrightarrow (p,q),$$

is smooth, since for every $(U,\varphi)\in\mathcal{F}_M$ and $(U\times V,\varphi\times\psi)\in\mathcal{F}_0$ with $q\in V$ the map

$$\{\varphi\times\psi\}\circ\iota_q\circ\varphi^{-1} = \mathrm{id}\times\psi(q)\colon \varphi\big(\iota_q^{-1}(U\times V)\cap U\big) = \varphi(U) \longrightarrow \{\varphi\times\psi\}\big(U\times V\big) = \varphi(U)\times\psi(V)$$

is smooth and $\iota_q(U)\subset U\times V$. Similarly, for each $p\in M$, the inclusion as a "vertical" slice,

$$\iota_p\colon N \longrightarrow M\times N, \qquad q \longrightarrow (p,q),$$

is also smooth.

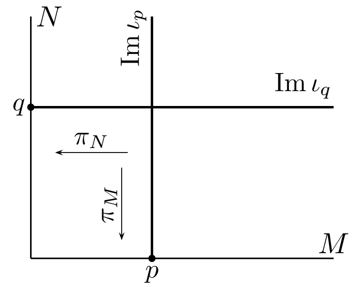


Figure 1.6: A horizontal slice $M\times q=\operatorname{Im}\iota_q$, a vertical slice $p\times N=\operatorname{Im}\iota_p$, and the two component projection maps $M\times N\longrightarrow M,N$

(2) The projection map onto the first component,

$$\pi_1=\pi_M\colon M\times N\longrightarrow M,\qquad (p,q)\longrightarrow p,$$

is smooth, since for every $(U\times V,\varphi\times\psi)\in\mathcal{F}_0$ and $(U,\varphi)\in\mathcal{F}_M$ the map

$$\varphi\circ\pi_M\circ\{\varphi\times\psi\}^{-1}=\pi_1\colon\{\varphi\times\psi\}\big(\pi_M^{-1}(U)\cap U\times V\big)=\varphi(U)\times\psi(V)\longrightarrow\varphi(U)$$

is smooth (being the restriction of the projection $\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}^m$ to an open subset) and $\pi_M(U\times V)\subset U$. Similarly, the projection map onto the second component,

$$\pi_2=\pi_N\colon M\times N\longrightarrow N,\qquad (p,q)\longrightarrow q,$$

is also smooth.

The following lemma, corollary, and proposition provide additional ways of constructing smooth structures. Corollary 2.7 follows immediately from Lemmas 2.6 and B.1.1. It gives rise to manifold structures on the tangent and cotangent bundles of a smooth manifold, as indicated in Example 7.5. Lemma 2.6 can be used in the proof of Proposition 2.8.

Lemma 2.6. *Let M be a Hausdorff second-countable topological space and $\{\varphi_\alpha\colon U_\alpha\longrightarrow M_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of homeomorphisms from open subsets U_α of M to smooth m-manifolds M_α such that*

$$\varphi_\alpha\circ\varphi_\beta^{-1}\colon\varphi_\beta\big(U_\alpha\cap U_\beta\big)\longrightarrow\varphi_\alpha\big(U_\alpha\cap U_\beta\big)\tag{2.3}$$

is a smooth map for all $\alpha,\beta\in\mathcal{A}$. If the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M, then M admits a unique smooth structure such that each map φ_α is a diffeomorphism.

Corollary 2.7. *Let M be a set and $\{\varphi_\alpha\colon U_\alpha\longrightarrow M_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of bijections from subsets U_α of M to smooth m-manifolds M_α such that*

$$\varphi_\alpha\circ\varphi_\beta^{-1}\colon\varphi_\beta\big(U_\alpha\cap U_\beta\big)\longrightarrow\varphi_\alpha\big(U_\alpha\cap U_\beta\big)$$

is a smooth map between open subsets of M_β and M_α, respectively, for all $\alpha,\beta\in\mathcal{A}$. If the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ separates points in M and a countable subcollection $\{U_\alpha\}_{\alpha\in\mathcal{A}_0}$ of $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M, then M admits a unique topology $\mathcal{T}_M$ and smooth structure $\mathcal{F}_M$ such that each map φ_α is a diffeomorphism.

Proposition 2.8. *If a group G acts properly discontinuously on a smooth m-manifold $(\tilde{M}, \mathcal{F}_{\tilde{M}})$ by diffeomorphisms and $\pi\colon \tilde{M} \longrightarrow M = \tilde{M}/G$ is the quotient projection map, then*

$$\mathcal{F}_0 = \big\{(\pi(U), \varphi\circ\{\pi|_U\}^{-1})\colon\ (U,\varphi)\in\mathcal{F}_{\tilde{M}},\ \pi|_U \text{ is injective}\big\}$$

is a collection of charts on the quotient topological space M that satisfies (SM1) *and* (SM2) *in Definition 1.3 and thus induces a smooth structure $\mathcal{F}_M$ on M. This smooth structure on M is the unique one satisfying either of the following two properties:*

(QSM1) *the projection map $\tilde{M} \longrightarrow M$ is a local diffeomorphism;*

(QSM2) *if N is a smooth manifold, a continuous map $f : M \longrightarrow N$ is smooth if and only if the map $f\circ\pi : \tilde{M} \longrightarrow N$ is smooth.*

In the case of Lemma 2.6, $\varphi_\alpha(U_\alpha\cap U_\beta)$ is an open subset of M_α because U_α and U_β are open subsets of M and φ_α is a homeomorphism; thus, smoothness for the map (2.3) is a well-defined requirement in light of (1) of Proposition 1.11 and (1) of Definition 2.1. In the case of Corollary 2.7, $\varphi_\alpha(U_\alpha\cap U_\beta)$ need not be a priori open in M_α, and so this must be one of the assumptions. In both cases, the requirement that $\varphi_\alpha\circ\varphi_\beta^{-1}$ be smooth can be replaced by the requirement that it be a diffeomorphism. We leave proofs of Lemma 2.6, Corollary 2.7, and Proposition 2.8 as exercises.

The smooth structure $\mathcal{F}_M$ on M of Proposition 2.8 is called the quotient smooth structure on M. For example, the group $\mathbb{Z}$ acts on $\mathbb{R}$ and on $\mathbb{R}\times\mathbb{R}$ by

$$\mathbb{Z}\times\mathbb{R} \longrightarrow \mathbb{R}, \qquad (m,s) \longrightarrow s+m, \tag{2.4}$$

$$\mathbb{Z}\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}, \qquad (m,s,t) \longrightarrow \big(s+m, (-1)^m t\big). \tag{2.5}$$

Both of these actions satisfy the assumptions of Proposition 2.8 and thus give rise to quotient smooth structures on $S^1 = \mathbb{R}/\mathbb{Z}$ and $\mathrm{MB} = (\mathbb{R}\times\mathbb{R})/\mathbb{Z}$. These smooth structures are the same as those of Examples 1.7 and 1.8, respectively.

Example 1.6 is a special case of the following phenomenon. If $(M,\mathcal{F})$ is a smooth manifold and $h\colon M \longrightarrow M$ is a homeomorphism, then

$$h^*\mathcal{F} \equiv \big\{\big(h^{-1}(U), \varphi\circ h\big)\colon\ (U,\varphi)\in\mathcal{F}\big\}$$

is also a smooth structure on M, since the overlap maps are the same as for the collection $\mathcal{F}$. The smooth structures $\mathcal{F}$ and $h^*\mathcal{F}$ are the same if and only if $h\colon (M,\mathcal{F}) \longrightarrow (M,\mathcal{F})$ is a diffeomorphism. However, in all cases, the map $h^{-1} : (M,\mathcal{F}) \longrightarrow (M, h^*\mathcal{F})$ is a diffeomorphism; so if a topological manifold admits a smooth structure, it admits many smooth equivalent (diffeomorphic) smooth structures.

This raises the question of which topological manifolds admit smooth structures and if so how many inequivalent ones. Since every connected component of a topological manifold is again a topological manifold, it is sufficient to study this question for connected topological manifolds.

dim=0: every connected topological 0-manifold M consists of a single point, $M = \{pt\}$; the only smooth structure on such a topological manifold is the single-element collection $\{(M,\varphi)\}$, where φ is the unique map $M \longrightarrow \mathbb{R}^0$.

dim=1: every connected topological (smooth) 1-manifold is homeomorphic (diffeomorphic) to either $\mathbb{R}$ or S^1 in the standard topology (and with standard smooth structure); a short proof of the smooth statement is given in [4, Appendix].

dim=2: every topological 2-manifold admits a unique smooth structure; every compact topological 2-manifold is homeomorphic (and thus diffeomorphic) to either a "torus" with $g \geq 0$ handles or to a connected sum of such a "torus" with $\mathbb{R}P^2$ [7, Chapter 8]; every such manifold admits a smooth structure as it is the quotient of either S^2 or $\mathbb{R}^2$ by a group acting properly discontinuously by diffeomorphisms.

dim=3: every topological 3-manifold admits a unique smooth structure [5].

dim=4: there are lots of topological 4-manifolds that admit no smooth structure and lots of other topological 4-manifolds (including $\mathbb{R}^4$) that admit many (even uncountably many) smooth structures.

The first known example of a topological manifold admitting non-equivalent smooth structures is the 7-sphere [3]. Since then the situation in dimensions 5 or greater has been sorted out by topological arguments [8].

Remark 2.9. While topology studies the topological category $\mathcal{TC}$, differential geometry studies the smooth category $\mathcal{SC}$. The objects in the latter are smooth manifolds, while the morphisms are smooth maps. The composition of two morphisms is the usual composition of maps (which is still a smooth map). For each object $(M, \mathcal{F}_M)$, the identity morphism is just the identity map id_M on M (which is a smooth map). The "forgetful map"

$$\mathcal{SC} \longrightarrow \mathcal{TC}, \qquad (M, \mathcal{F}_M) \longrightarrow M, \qquad \big(f\colon (M, \mathcal{F}_M) \longrightarrow (N, \mathcal{F}_N)\big) \longrightarrow \big(f : M \longrightarrow N\big),$$

is a functor from the smooth category to the topological category.

In the remainder of these notes, we will typically denote a smooth manifold in the same way as its underlying set and topological space; so a smooth manifold M will be understood to come with a smooth structure $\mathcal{F}_M$.

3 Tangent Vectors

If M is an m-manifold embedded in $\mathbb{R}^n$, with $m \leq n$, and $\gamma\colon (a,b) \longrightarrow M$ is a smooth map (curve on M), then

$$\dot{\gamma}(t) \equiv \lim_{\tau \longrightarrow 0} \frac{\gamma(t+\tau) - \gamma(t)}{\tau} \in \mathbb{R}^n \tag{3.1}$$

should be a tangent vector of M at $\gamma(t)$. The set of such vectors is an m-dimensional linear subspace of $\mathbb{R}^n$; it is often thought of as having the 0-vector at p; see Figure 1.7. However, this presentation of the tangent space T_pM of M at p depends on the embedding of M in $\mathbb{R}^n$, and not just on M and p.

On the other hand, the tangent space at a point $p \in \mathbb{R}^m$ should be $\mathbb{R}^m$ itself, but based (with the origin) at p. Each vector $v \in \mathbb{R}^m$ acts on smooth functions f defined near p by

$$\partial_v|_p f = \lim_{t \longrightarrow 0} \frac{f(p+tv) - f(p)}{t}\,. \tag{3.2}$$

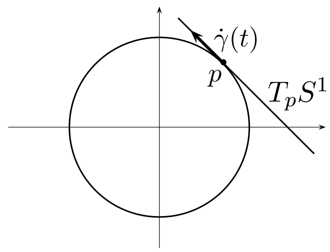


Figure 1.7: The tangent space of S^1 at p viewed as a subspace of $\mathbb{R}^2$.

If $v=e_i$ is the i-th coordinate vector on $\mathbb{R}^m$, then $\partial_v|_p f$ is just the i-th partial derivative $\partial_i f|_p$ of f at p. The map $\partial_v|_p$ defined by (3.2) takes each smooth function defined on a neighborhood of p in $\mathbb{R}^m$ to $\mathbb{R}$ and satisfies:

(TV1) if $f: U \longrightarrow \mathbb{R}$ and $g: V \longrightarrow \mathbb{R}$ are smooth functions on neighborhoods of p such that $f|_W = g|_W$ for some neighborhood W of p in $U\cap V$, then $\partial_v|_p f = \partial_v|_p g$;

(TV2) if $f: U \longrightarrow \mathbb{R}$ and $g: V \longrightarrow \mathbb{R}$ are smooth functions on neighborhoods of p and $a, b\in\mathbb{R}$, then

$$\partial_v|_p\big(af+bg\big) = a\,\partial_v|_p f + b\,\partial_v|_p g\,,$$

where $af+bg$ is the smooth function on the neighborhood $U\cap V$ given by

$$\{af+bg\}(q) = af(q) + bg(q)\,;$$

(TV3) if $f: U \longrightarrow \mathbb{R}$ and $g: V \longrightarrow \mathbb{R}$ are smooth functions on neighborhoods of p, then

$$\partial_v|_p(fg) = f(p)\partial_v|_p g + g(p)\partial_v|_p f\,,$$

where fg is the smooth function on the neighborhood $U\cap V$ given by $\{fg\}(q) = f(q)g(q)$.

It turns out every that $\mathbb{R}$-valued map on the space of smooth functions defined on neighborhood of p satisfying (TV1)-(TV3) is $\partial_v|_p$ for some $v\in\mathbb{R}^m$; see Proposition 3.4 below. At the same time, these three conditions make sense for any smooth manifold, and this approach indeed leads to an intrinsic definition of tangent vectors for smooth manifolds.

The space of functions defined on various neighborhoods of a point does not have a very nice structure. In order to study the space of operators satisfying (TV1)-(TV3) it is convenient to put an equivalence relation on this space.

Definition 3.1. *Let M be a smooth manifold and $p\in M$.*

(1) Functions $f: U\longrightarrow\mathbb{R}$ and $g: V\longrightarrow\mathbb{R}$ defined on neighborhoods of p in M are p-equivalent, or $f\sim_p g$, if there exists a neighborhood W of p in $U\cap V$ such that $f|_W = g|_W$.

(2) The set of p-equivalence classes of smooth functions is denoted $\tilde{F}_p$; the p-equivalence class of a smooth function $f: U\longrightarrow\mathbb{R}$ on a neighborhood of p is called the germ of f at p and is denoted $\underline{f}_p$.

The set $\tilde{F}_p$ has a natural $\mathbb{R}$-algebra structure:

$$a\underline{f}_p + b\underline{g}_p = \underline{af+bg}_p\,, \quad \underline{f}_p \cdot \underline{g}_p = \underline{fg}_p \qquad \forall\, \underline{f}_p, \underline{g}_p \in \tilde{F}_p,\ a,b\in\mathbb{R},$$

where $af+bg$ and fg are functions defined on $U\cap V$ if f and g are defined on U and V, respectively. There is a well-defined valuation homomorphism,

$$\mathrm{ev}_p\colon \tilde{F}_p \longrightarrow \mathbb{R}, \qquad \underline{f}_p \longrightarrow f(p).$$

Let $F_p = \ker \mathrm{ev}_p$; this subset of $\tilde{F}_p$ consists of the germs at p of the smooth functions defined on neighborhoods of p in M that vanish at p. Since ev_p is an $\mathbb{R}$-algebra homomorphisms, F_p is an ideal in $\tilde{F}_p$; this can also be seen directly: if $f(p)=0$, then $\{fg\}(p)=0$. Let $F_p^2 \subset F_p$ be the ideal in $\tilde{F}_p$ consisting of all finite linear combinations of elements of the form $\underline{f}_p\underline{g}_p$ with $\underline{f}_p, \underline{g}_p \in F_p$. If $c\in\mathbb{R}$, let $\underline{c}_p \in \tilde{F}_p$ denote the germ at p of the constant function with value c on M.

Lemma 3.2. *Let M be a smooth manifold and $p\in M$. If v is a derivation on $\tilde{F}_p$ relative to the valuation* ev_p,[1] *then*

$$v|_{F_p^2} \equiv 0, \qquad v(\underline{c}_p) = 0 \ \ \forall\, c\in\mathbb{R}. \tag{3.3}$$

If $\underline{f}_p, \underline{g}_p \in F_p$, then $f(p), g(p)=0$ and thus

$$v\big(\underline{f}_p\underline{g}_p\big) = f(p)v(\underline{g}_p) + g(p)v(\underline{f}_p) = 0;$$

so v vanishes identically on F_p^2. If $c\in\mathbb{R}$,

$$\begin{aligned} v(\underline{c}_p) = v(\underline{1}_p\underline{c}_p) = 1(p)\cdot v(\underline{c}_p) + c(p)\cdot v(\underline{1}_p) &= 1\cdot v(\underline{c}_p) + c\cdot v(\underline{1}_p) \\ &= v(\underline{c}_p) + v(c\cdot \underline{1}_p) = v(\underline{c}_p) + v(\underline{c}_p); \end{aligned}$$

so $v(\underline{c}_p) = 0$.

Corollary 3.3. *If M is a smooth manifold and $p\in M$, the map $v \longrightarrow v|_{F_p}$ induces an isomorphism from the vector space $\mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p)$ of derivations on $\tilde{F}_p$ relative to the valuation ev_p to*

$$\big\{L\in \mathrm{Hom}(F_p,\mathbb{R})\colon\ L|_{F_p^2}\equiv 0\big\} \approx \big(F_p/F_p^2\big)^*.$$

The set $\mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p)$ of derivations on $\tilde{F}_p$ relative to the valuation ev_p indeed forms a vector space:

$$\big\{av + bw\big\}\big(\underline{f}_p\big) = av\big(\underline{f}_p\big) + bw\big(\underline{f}_p\big) \qquad \forall\ v,w\in \mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p),\ \ a,b\in\mathbb{R},\ \ \underline{f}_p \in \tilde{F}_p.$$

If $v\in \mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p)$, the restriction of v to $F_p \subset \tilde{F}_p$ is a homomorphism to $\mathbb{R}$ that vanishes on F_p^2 by Lemma 3.2. Conversely, if $L\colon F_p \longrightarrow \mathbb{R}$ is a linear homomorphism vanishing on F_p^2, define

$$v_L\colon \tilde{F}_p \longrightarrow \mathbb{R} \qquad \text{by} \qquad v_L\big(\underline{f}_p\big) = L\big(\underline{f - f(p)}_p\big);$$

[1] i.e. $v\colon \tilde{F}_p \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-linear map such that

$$v(\underline{f}_p\underline{g}_p) = \mathrm{ev}_p(\underline{f}_p)v(\underline{g}_p) + \mathrm{ev}_p(\underline{g}_p)v(\underline{f}_p) \qquad \forall\ \underline{f}_p, \underline{g}_p \in \tilde{F}_p\,.$$

since the function $f-f(p)$ vanishes at p, $\underline{f-f(p)}_p\in F_p$ and so v_L is well-defined. It is immediate that v_L is a homomorphism of vector spaces. Furthermore, for all $\underline{f}_p,\underline{g}_p$,

$$\begin{aligned} v_L\big(\underline{f}_p\underline{g}_p\big) &= L\big(\underline{fg-f(p)g(p)}_p\big) = L\big(f(p)\underline{g-g(p)}_p + g(p)\underline{f-f(p)}_p + \underline{f-f(p)}_p\underline{g-g(p)}_p\big) \\ &= f(p)L\big(\underline{g-g(p)}_p\big) + g(p)L\big(\underline{f-f(p)}_p\big) + L\big(\underline{f-f(p)}_p\underline{g-g(p)}_p\big) \\ &= f(p)v_L\big(\underline{g}_p\big) + g(p)v_L\big(\underline{f}_p\big) + 0, \end{aligned}$$

since L vanishes on F_p^2; so v_L is a derivation with respect to the valuation ev_p. It is also immediate that the maps

$$\begin{aligned} \mathrm{Der}(\tilde{F}_p,\mathrm{ev}_p) &\longrightarrow \big\{L\in\mathrm{Hom}(F_p,\mathbb{R}):\ L|_{F_p^2}\equiv 0\big\}, & v &\longrightarrow L_v\equiv v|_{F_p}, \\ \big\{L\in\mathrm{Hom}(F_p,\mathbb{R}):\ L|_{F_p^2}\equiv 0\big\} &\longrightarrow \mathrm{Der}(\tilde{F}_p,\mathrm{ev}_p), & L &\longrightarrow v_L, \end{aligned} \tag{3.4}$$

are homomorphisms of vector spaces. If $L\in\mathrm{Hom}(F_p,\mathbb{R})$ and $L|_{F_p^2}\equiv 0$, the restriction of v_L to F_p is L, and so $L_{v_L}=L$. If $v\in\mathrm{Der}(\tilde{F}_p,\mathrm{ev}_p)$ and $\underline{f}_p\in\tilde{F}_p$, by the second statement in (3.3)

$$v\big(\underline{f}_p\big) = v\big(\underline{f}_p\big) - v\big(\underline{f(p)}_p\big) = v\big(\underline{f-f(p)}_p\big) = L_v\big(\underline{f-f(p)}_p\big) = v_{L_v}\big(\underline{f}_p\big);$$

so $v_{L_v}=v$ and the two homomorphisms in (3.4) are inverses of each other. This completes the proof of Corollary 3.3.

Proposition 3.4. *If $p\in\mathbb{R}^m$, the vector space F_p/F_p^2 is m-dimensional and the homomorphism*

$$\mathbb{R}^m \longrightarrow \mathrm{Der}\big(\tilde{F}_p,\mathrm{ev}_p\big) \approx \big(F_p/F_p^2\big)^*, \qquad v\longrightarrow \partial_v|_p\,, \tag{3.5}$$

induced by (3.2), is an isomorphism.

By (TV1), $\partial_v|_p$ induces a well-defined map $\tilde{F}_p\longrightarrow\mathbb{R}$. By (TV2), $\partial_v|_p$ is a vector-space homomorphism. By (TV3), $\partial_v|_p$ is a derivation with respect to the valuation ev_p. Thus, the map (3.5) is well-defined and is clearly a vector-space homomorphism. If $\pi_j:\mathbb{R}^m\longrightarrow\mathbb{R}$ is the projection to the j-th component,

$$\partial_{e_i}|_p\big(\pi_j-\pi_j(p)\big) = \big(\partial_i(\pi_j-\pi_j(p))\big)_p = \delta_{ij}\equiv\begin{cases}1, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases} \tag{3.6}$$

Thus, the homomorphism (3.5) is injective, and the set $\{\underline{\pi_j-\pi_j(p)}_p\}$ is linearly independent in F_p/F_p^2. On the other hand, Lemma 3.5 below implies that

$$f(p+x) = f(p) + \sum_{i=1}^{i=m}(\partial_i f)_p x_i + \sum_{i,j=1}^{i,j=m} x_i x_i\int_0^1(1-t)(\partial_i\partial_j f)_{p+tx}\mathrm{d}t \tag{3.7}$$

for every smooth function f defined on a open ball U around p in $\mathbb{R}^m$ and for all $p+x\in U$. Thus, the set $\{\underline{\pi_j-\pi_j(p)}_p\}$ spans F_p/F_p^2; so F_p/F_p^2 is m-dimensional and the homomorphism (3.5) is an isomorphism.

Note that the inverse of the isomorphism (3.5) is given by

$$\mathrm{Der}\big(\tilde{F}_p,\mathrm{ev}_p\big)\longrightarrow\mathbb{R}^m, \qquad v\longrightarrow\big(v(\underline{\pi_1}_p),\ldots,v(\underline{\pi_m}_p)\big); \tag{3.8}$$

by (3.6), this is a right inverse and thus must be the inverse.

Lemma 3.5. *If $h\colon U \longrightarrow \mathbb{R}$ is a smooth function defined on an open ball U around a point p in $\mathbb{R}^m$, then*

$$h(p+x) = h(p) + \sum_{i=1}^{i=m} x_i \int_0^1 (\partial_i h)_{p+tx} \mathrm{d}t$$

for all $p+x \in U$.

This follows from the Fundamental Theorem of Calculus:

$$\begin{aligned} h(p+x) = h(p) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} h(p+tx)\mathrm{d}t &= h(p) + \int_0^1 \sum_{i=1}^{i=m} (\partial_i h)_{p+tx} x_i \mathrm{d}t \\ &= h(p) + \sum_{i=1}^{i=m} x_i \int_0^1 (\partial_i h)_{p+tx} \mathrm{d}t. \end{aligned}$$

Corollary 3.6. *If M is a smooth m-manifold and $p \in M$, the vector space $\mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p)$ is m-dimensional.*

If $\varphi : U \longrightarrow \mathbb{R}^m$ is a smooth chart around $p \in M$, the map $f \longrightarrow f \circ \varphi$ induces an $\mathbb{R}$-algebra isomorphism

$$\varphi^* \colon \tilde{F}_{\varphi(p)} \longrightarrow \tilde{F}_p, \qquad \underline{f}_{\varphi(p)} \longrightarrow \underline{f \circ \varphi}_p\,. \tag{3.9}$$

Since $\mathrm{ev}_{\varphi(p)} = \mathrm{ev}_p \circ \varphi^*$, φ^* restricts to an isomorphism $F_{\varphi(p)} \longrightarrow F_p$ and descends to an isomorphism

$$F_{\varphi(p)}/F_{\varphi(p)}^2 \longrightarrow F_p/F_p^2\,. \tag{3.10}$$

Thus, Corollary 3.6 follows from Corollary 3.3 and Proposition 3.4.

Definition 3.7. *Let M be a smooth manifold and $p \in M$.*

(1) The tangent space *of M at p is the vector space $T_pM = \mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p)$; a* tangent vector *of M at p is an element of T_pM.*

(2) The cotangent space *of M at p is $T_p^*M \equiv (T_pM)^* \equiv \mathrm{Hom}(T_pM, \mathbb{R})$.*

By Corollary 3.6, T_pM and T_p^*M are m-dimensional vector spaces if M is an m-dimensional manifold. By Proposition 3.4, $T_p\mathbb{R}^m$ is canonically isomorphic to $\mathbb{R}^m$ for every $p \in \mathbb{R}^m$. By Corollary 3.3, $T_p^*M \approx F_p/F_p^2$; an element $\underline{f}_p + F_p^2$ of F_p/F_p^2 determines the vector-space homomorphism

$$T_pM \longrightarrow \mathbb{R}, \qquad v \longrightarrow v\big(\underline{f}_p\big). \tag{3.11}$$

Any smooth function f defined on a neighborhood of p in M defines an element of T^*M in the same way, but this element depends only on

$$\underline{f - f(p)}_p + F_p^2 \in F_p/F_p^2\,.$$

Example 3.8. Let V be an m-dimensional vector space with the canonical smooth structure of Example 1.5. For $p, v \in V$, let

$$\partial_v|_p \colon \tilde{F}_p \longrightarrow \mathbb{R}$$

be the derivation with respect to ev_p defined by (3.2). The homomorphism

$$V \longrightarrow T_pV = \mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p), \qquad v \longrightarrow \partial_v|_p\,, \tag{3.12}$$

is injective because for any linear functional $f\colon V \longrightarrow \mathbb{R}$

$$\partial_v|_p f = \lim_{t\longrightarrow 0}\frac{f(p+tv)-f(p)}{t} = \lim_{t\longrightarrow 0}\frac{f(p)+tf(v)-f(p)}{t} = f(v);$$

so $\partial_v|_p f \neq 0$ on every linear functional f on V not vanishing on v and thus $\partial_v|_p \neq 0 \in T_pV$ if $v \neq 0$ (such functionals exist if $v \neq 0$; they are smooth by Example 2.3). Since the dimension of T_pV is m by Corollary 3.6, the homomorphism (3.12) is an isomorphism of vector spaces. So, for every finite-dimensional vector space V and $p \in T_pV$, (3.12) provides a canonical identification of T_pV with V (but not with $\mathbb{R}^m$); the dual of (3.12) provides a canonical identification T_p^*V with $V^* = \mathrm{Hom}(V, \mathbb{R})$.

4 Differentials of Smooth Maps

Definition 4.1. *Let $h\colon M \longrightarrow N$ be a smooth map between smooth manifolds and $p \in M$.*

(1) The differential of h at p *is the map*

$$\mathrm{d}_ph\colon T_pM \longrightarrow T_{h(p)}N, \qquad \big\{\mathrm{d}_ph(v)\big\}\big(\underline{f}_{h(p)}\big) = v\big(\underline{f \circ h}_p\big) \quad \forall\, v \in T_pM,\ \underline{f}_{h(p)} \in \tilde{F}_{h(p)}\,. \tag{4.1}$$

(2) The pull-back map on the cotangent spaces *is the map*

$$h^* \equiv \big\{\mathrm{d}_ph\big\}^*\colon T^*_{h(p)}N \longrightarrow T^*_pM, \qquad \eta \longrightarrow \eta \circ \mathrm{d}_ph\,. \tag{4.2}$$

The map (4.1) is a vector-space homomorphism, and thus so is h^*. It is immediate from the definition that $\mathrm{d}_p\mathrm{id}_M = \mathrm{id}_{T_pM}$ and thus $\mathrm{id}_M^* = \mathrm{id}_{T_p^*M}$. If $N = \mathbb{R}$, $T_{h(p)}\mathbb{R}$ is canonically isomorphic to $\mathbb{R}$, via the map

$$T_{h(p)}\mathbb{R} \longrightarrow \mathbb{R}, \qquad w \longrightarrow w\big(\mathrm{id}_{\mathbb{R}}\big);$$

see (3.8). In particular, if $v \in T_pM$,

$$\mathrm{d}_ph(v) \longrightarrow \big\{\mathrm{d}_ph(v)\big\}\big(\mathrm{id}_{\mathbb{R}}\big) \equiv v\big(\mathrm{id}_{\mathbb{R}} \circ h\big) = v(h).$$

Thus, under the canonical identification of $T_{h(p)}\mathbb{R}$ with $\mathbb{R}$, the differential d_ph of a smooth map $h\colon M \longrightarrow \mathbb{R}$ is given by

$$\boxed{\mathrm{d}_ph(v) = v(h) \qquad \forall\, v \in T_pM} \tag{4.3}$$

and so corresponds to the same element of T_p^*M as

$$\underline{h - h(p)}_p + F_p^2 \in F_p/F_p^2\,;$$

see (3.11).

Example 4.2. Let V and W be finite-dimensional vector spaces with the canonical smooth structures of Example 1.5. By Example 2.3, every vector-space homomorphism $h\colon V\longrightarrow W$ is smooth. If $p,v\in V$ and $f:U\longrightarrow\mathbb{R}$ is a smooth function defined on a neighborhood of $h(p)$ in W, by (4.1) and (3.2)

$$\begin{aligned}\big\{\mathrm{d}_p h\big(\partial_v|_p\big)\big\}(f)=\partial_v|_p(f\circ h)&=\lim_{t\longrightarrow 0}\frac{f(h(p+tv))-f(h(p))}{t}\\&=\lim_{t\longrightarrow 0}\frac{f(h(p)+th(v))-f(h(p))}{t}=\partial_{h(v)}|_{h(p)}(f)\,.\end{aligned}$$

Thus, under the canonical identifications of T_pV with V and $T_{h(p)}W$ with W as in Example 3.8, the differential $\mathrm{d}_p h$ at p of a vector-space homomorphism $h\colon V\longrightarrow W$ corresponds to the homomorphism h itself; so the diagram

$$\begin{tikzcd} T_pV \arrow[r, "\mathrm{d}_p h"] \arrow[d, equal, "(3.12)"'] & T_{h(p)}W \arrow[d, equal, "(3.12)"] \\ V \arrow[r, "h"] & W \end{tikzcd} \tag{4.4}$$

commutes. In particular, the differentials of an identification $\varphi:V\longrightarrow\mathbb{R}^m$ induce the same identifications on the tangent spaces.

Lemma 4.3. *If $g:M\longrightarrow N$ and $h:N\longrightarrow X$ are smooth maps between smooth manifolds and $p\in M$, then*

$$\mathrm{d}_p(h\circ g)=\mathrm{d}_{g(p)}h\circ\mathrm{d}_p g\colon T_pM\longrightarrow T_{h(g(p))}X. \tag{4.5}$$

Thus, $(h\circ g)^*=g^*\circ h^*\colon T^*_{h(g(p))}X\longrightarrow T^*_pM$ and

$$\boxed{g^*\mathrm{d}_{g(p)}f=\mathrm{d}_p(f\circ g)} \tag{4.6}$$

whenever f is a smooth function on a neighborhood of $g(p)$ in N.

If $v\in T_pM$ and f is a smooth function on a neighborhood of $h(g(p))$ in X, then by (4.1)

$$\begin{aligned}\big\{\{\mathrm{d}_p(h\circ g)\}(v)\big\}(f)=v\big(f\circ h\circ g\big)=\big\{\mathrm{d}_p g(v)\big\}(f\circ h)&=\big\{\mathrm{d}_{g(p)}h\big(\mathrm{d}_p g(v)\big)\big\}(f)\\&=\big\{\{\mathrm{d}_{g(p)}h\circ\mathrm{d}_p g\}(v)\big\}(f);\end{aligned}$$

thus, (4.5) holds. The second claim is the dual of the first. For the last claim, note that

$$g^*\mathrm{d}_{g(p)}f\equiv\mathrm{d}_{g(p)}f\circ\mathrm{d}_p g=\mathrm{d}_p(f\circ g) \tag{4.7}$$

by (4.2) and (4.5). For the purposes of applying (4.2) and (4.5), all expressions in (4.7) are viewed as maps to $T_{f(g(p))}\mathbb{R}$, before its canonical identification with $\mathbb{R}$. The identities of course continue to hold after this identification.

Definition 4.4. *A smooth curve in a smooth manifold M is a smooth map $\gamma\colon(a,b)\longrightarrow M$, where (a,b) is a nonempty open (possibly infinite) interval in $\mathbb{R}$. The tangent vector to a smooth curve $\gamma\colon(a,b)\longrightarrow M$ at $t\in(a,b)$ is the vector*

$$\gamma'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\equiv\mathrm{d}_t\gamma\big(\partial_{e_1}|_t\big)\in T_{\gamma(t)}M, \tag{4.8}$$

where $e_1=1\in\mathbb{R}^1$ is the oriented unit vector.

If $h\colon M\longrightarrow N$ is a smooth map between smooth manifolds and $\gamma\colon (a,b)\longrightarrow M$ is a smooth curve in M, then

$$h\circ\gamma\colon (a,b)\longrightarrow N$$

is a smooth curve on N and by the chain rule (4.5)

$$\begin{aligned}(h\circ\gamma)'(t)&\equiv \mathrm{d}_t\{h\circ\gamma\}\big(\partial_{e_1}|_t\big)=\big\{\mathrm{d}_{\gamma(t)}h\circ\mathrm{d}_t\gamma\big\}\big(\partial_{e_1}|_t\big)=\mathrm{d}_{\gamma(t)}h\big(\mathrm{d}_t\gamma(\partial_{e_1}|_t)\big)\\&=\mathrm{d}_{\gamma(t)}\big(\gamma'(t)\big)\in T_{h(\gamma(t))}N\end{aligned}\tag{4.9}$$

for every $t\in(a,b)$.

Lemma 4.5. *Let V be a finite-dimensional vector space with its canonical smooth structure of Example 1.5. If $\gamma\colon (a,b)\longrightarrow V$ is a smooth curve and $t\in(a,b)$, $\gamma'(t)\in T_{\gamma(t)}V$ corresponds to*

$$\dot\gamma(t)=\lim_{\tau\longrightarrow 0}\frac{\gamma(t+\tau)-\gamma(t)}{\tau}\in V$$

under the canonical isomorphism $T_{\gamma(t)}V\approx V$ provided by (3.12).

If $h:V\longrightarrow W$ is a homomorphism of vectors spaces,

$$\dot{\widehat{h\circ\gamma}}(t)=h(\dot\gamma(t))\tag{4.10}$$

by the linearity of h. Thus, by (4.9) and the commutativity of (4.4), it is sufficient to prove this lemma for $V=\mathbb{R}^m$, which we now assume to be the case. If $f:U\longrightarrow\mathbb{R}$ is a smooth function defined on a neighborhood of $\gamma(t)$ in $\mathbb{R}^m$, by (4.8), (4.1), the usual multi-variable chain rule, and (3.1)

$$\begin{aligned}\big\{\gamma'(t)\big\}(f)=\big\{\mathrm{d}_t\gamma\big(\partial_{e_1}|_t\big)\big\}(f)=\partial_{e_1}|_t(f\circ\gamma)&=\lim_{\tau\longrightarrow 0}\frac{f(\gamma(t+\tau))-f(\gamma(t))}{\tau}\\&=\mathcal{J}(f)_{\gamma(t)}\dot\gamma(t)=\lim_{\tau\longrightarrow 0}\frac{f(\gamma(t)+\tau\dot\gamma(t))-f(\gamma(t))}{\tau}\\&=\partial_{\dot\gamma(t)}|_{\gamma(t)}f\,,\end{aligned}\tag{4.11}$$

where $\mathcal{J}(f)_{\gamma(t)}\colon\mathbb{R}^m\longrightarrow\mathbb{R}$ is the usual Jacobian (matrix of first partials) of the smooth map f from an open subset of $\mathbb{R}^m$ to $\mathbb{R}$ evaluated at $\gamma(t)$. Thus, under the canonical identification of $T_{\gamma(t)}\mathbb{R}^m$ with $\mathbb{R}^m$ provided by (3.5), the tangent vector $\gamma'(t)$ of Definition 4.4 corresponds to the tangent vector $\dot\gamma(t)$ of calculus.

Corollary 4.6. *Let $(M,\mathcal{F}_M)$ be a smooth manifold. For every $p\in M$ and $v\in T_pM$, there exists a smooth curve*

$$\gamma\colon (a,b)\longrightarrow M\qquad \textit{s.t.}\quad \gamma(0)=p,\quad \gamma'(0)=v.$$

If $\varphi\colon U\longrightarrow\mathbb{R}^m$ is a smooth chart around p on M, the homomorphism

$$\mathrm{d}_{\varphi(p)}\varphi^{-1}\colon T_{\varphi(p)}\mathbb{R}^m\longrightarrow T_pM$$

is an isomorphism. Thus, by (4.9), it is sufficient to prove the claim for $V=\mathbb{R}^m$, which we now assume to be the case. By Lemma 4.5 applied with $V=\mathbb{R}^m$, it is to show that for all $p,v\in\mathbb{R}^m$ there exists a smooth curve

$$\gamma\colon (a,b)\longrightarrow\mathbb{R}^m\qquad\text{s.t.}\quad \gamma(0)=p,\quad \dot\gamma(0)=v.$$

An example of such a curve is given by

$$\gamma\colon (-\infty,\infty)\longrightarrow\mathbb{R}^m,\qquad t\longrightarrow p+tv.$$

Example 4.7. By Example 2.4, the map

$$h\colon \mathrm{Mat}_{n\times n}\mathbb{R} \longrightarrow \mathrm{SMat}_n\mathbb{R}, \qquad h(A) = A^{\mathrm{tr}}A,$$

is smooth; we determine the homomorphism $\mathrm{d}_{\mathbb{I}_n}h$. The map

$$\gamma\colon (-\infty,\infty) \longrightarrow \mathrm{Mat}_{n\times n}\mathbb{R}, \qquad t \longrightarrow \mathbb{I}_n+tA,$$

is a smooth curve such that $\gamma(0)=\mathbb{I}_n$ and $\dot\gamma(0)=A$. By (4.10), the homomorphism induced by $\mathrm{d}_{\mathbb{I}_n}h$ via the isomorphisms provided by (3.12) takes $\dot\gamma(0)=A$ to

$$\dot{\overbrace{h\circ\gamma}}(0) \equiv \lim_{t\longrightarrow 0}\frac{h(\gamma(t))-h(\gamma(0))}{t} = \lim_{t\longrightarrow 0}\frac{(\mathbb{I}_n+tA)^{\mathrm{tr}}(\mathbb{I}_n+tA)-\mathbb{I}_n}{t} = A+A^{\mathrm{tr}}.$$

Thus, the homomorphism induced by $\mathrm{d}_{\mathbb{I}_n}h$ via the identifications provided by (3.12) is given by

$$\mathrm{d}_{\mathbb{I}_n}h\colon \mathrm{Mat}_{n\times n}\mathbb{R} \longrightarrow \mathrm{SMat}_n\mathbb{R}, \qquad A \longrightarrow A+A^{\mathrm{tr}}.$$

In particular, $\mathrm{d}_{\mathbb{I}_n}h$ is surjective, because its restriction to the subspace $\mathrm{SMat}_n\mathbb{R}\subset\mathrm{Mat}_{n\times n}\mathbb{R}$ is.

Let $\varphi=(x_1,\ldots,x_m): U \longrightarrow \mathbb{R}^m$ be a smooth chart on a neighborhood of a point p in M; so, $x_i=\pi_i\circ\varphi$, where $\pi_i:\mathbb{R}^m\longrightarrow\mathbb{R}$ is the projection to the i-th component as before. Since the map (3.9) induces the isomorphism (3.10) and $\{\underline{\pi_i-x_i(p)}_{\varphi(p)}\}_i$ is a basis for $F_{\varphi(p)}/F^2_{\varphi(p)}$,

$$\varphi^*\big(\{\underline{\pi_i-x_i(p)}_{\varphi(p)}\}_i\big) \equiv \big\{\underline{(\pi_i-x_i(p))\circ\varphi}_p\big\}_i = \big\{\underline{x_i-x_i(p)}_p\big\}_i$$

is a basis for F_p/F_p^2. Thus, $\{\mathrm{d}_px_i\}_i$ is a basis for T_p^*M, since d_px_i and $\underline{x_i-x_i(p)}_p$ act in the same way on all elements of T_pM; see the paragraph following Definition 4.1. For each $i=1,2,\ldots,m$, let

$$\frac{\partial}{\partial x_i}\bigg|_p = \mathrm{d}_{\varphi(p)}\varphi^{-1}\big(\partial_{e_i}|_{\varphi(p)}\big) \in T_pM. \tag{4.12}$$

By (4.1), for every smooth function f defined on a neighborhood of p in M

$$\begin{aligned}\frac{\partial}{\partial x_i}\bigg|_p(f) &= \big\{\mathrm{d}_{\varphi(p)}\varphi^{-1}\big(\partial_{e_i}|_{\varphi(p)}\big)\big\}(f) = \partial_{e_i}|_{\varphi(p)}\big(f\circ\varphi^{-1}\big)\\ &= \partial_i\big(f\circ\varphi^{-1}\big)|_{\varphi(p)}\end{aligned} \tag{4.13}$$

is the i-th partial derivative of the function $f\circ\varphi^{-1}$ at $\varphi(p)$; this is a smooth function defined on a neighborhood of $\varphi(p)$ in $\mathbb{R}^m$. In particular, for all $i,j=1,2,\ldots,m$

$$\mathrm{d}_px_j\bigg(\frac{\partial}{\partial x_i}\bigg|_p\bigg) = \frac{\partial}{\partial x_i}\bigg|_p(x_j) = \partial_i\big(\pi_j\circ\varphi\circ\varphi^{-1}\big) = \delta_{ij}\,;$$

the first equality above is a special case of (4.3). Thus, $\{\frac{\partial}{\partial x_i}|_p\}_i$ is a basis for T_pM; it is dual to the basis $\{\mathrm{d}_px_i\}_i$ for T_p^*M. The coefficients of other elements of T_pM and T_p^*M with respect to these bases are given by

$$v = \sum_{i=1}^{i=m}\big(\mathrm{d}_px_i(v)\big)\frac{\partial}{\partial x_i}\bigg|_p = \sum_{i=1}^{i=m}v(x_i)\frac{\partial}{\partial x_i}\bigg|_p \qquad \forall\, v\in T_pM; \tag{4.14}$$

$$\eta = \sum_{i=1}^{i=m}\eta\bigg(\frac{\partial}{\partial x_i}\bigg|_p\bigg)\mathrm{d}_px_i \qquad \forall\,\eta\in T_p^*M. \tag{4.15}$$

The first identities in (4.14) and (4.15) are immediate from the two bases being dual to each other (each $\mathrm{d}_p x_j$ gives the same values when evaluated on both sides of the first identity in (4.14); both sides of (4.15) evaluate to the same number on each $\frac{\partial}{\partial x_j}\big|_p$). The second equality in (4.14) follows from (4.3). If f is a smooth function on a neighborhood of p, by (4.15), (4.3), and (4.13)

$$\mathrm{d}_p f = \sum_{i=1}^{i=m} \mathrm{d}_p f\bigg(\frac{\partial}{\partial x_i}\bigg|_p\bigg)\mathrm{d}_p x_i = \sum_{i=1}^{i=m}\bigg(\frac{\partial}{\partial x_i}\bigg|_p(f)\bigg)\mathrm{d}_p x_i = \sum_{i=1}^{i=m}\big(\partial_i(f\circ\varphi^{-1})\big)_{\varphi(p)}\mathrm{d}_p x_i\,. \tag{4.16}$$

If $\psi=(y_1,\ldots,y_m)\colon V\longrightarrow\mathbb{R}^m$ is another smooth chart around p, by (4.14), (4.3), and (4.13)

$$\begin{aligned} &\frac{\partial}{\partial x_j}\bigg|_p = \sum_{i=1}^{i=m}\bigg(\frac{\partial}{\partial x_j}\bigg|_p(y_i)\bigg)\frac{\partial}{\partial y_i}\bigg|_p = \sum_{i=1}^{i=m}\Big(\partial_j(\pi_i\circ\psi\circ\varphi^{-1})_{\varphi(p)}\Big)\frac{\partial}{\partial y_i}\bigg|_p \\ \iff\quad &\bigg(\frac{\partial}{\partial x_1}\bigg|_p,\ldots,\frac{\partial}{\partial x_n}\bigg|_p\bigg) = \bigg(\frac{\partial}{\partial y_1}\bigg|_p,\ldots,\frac{\partial}{\partial y_n}\bigg|_p\bigg)\mathcal{J}\big(\psi\circ\varphi^{-1}\big)_{\varphi(p)}, \end{aligned} \tag{4.17}$$

where $\mathcal{J}(\psi\circ\varphi^{-1})_{\varphi(p)}$ is the Jacobian of the smooth map $\psi\circ\varphi^{-1}$ between the open subsets $\varphi(U\cap V)$ and $\psi(U\cap V)$ of $\mathbb{R}^m$ at $\varphi(p)$; see Figure 1.2 with $\varphi_\alpha=\psi$ and $\varphi_\beta=\varphi$.

Suppose next that $f\colon M\longrightarrow N$ is a map between smooth manifolds and

$$\varphi=(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m \qquad\text{and}\qquad \psi=(y_1,\ldots,y_n)\colon V\longrightarrow\mathbb{R}^n$$

are smooth charts around $p\in M$ and $f(p)\in N$, respectively; see Figure 1.5. By (4.14), (4.1), and (4.13),

$$\begin{aligned} \mathrm{d}_p f\bigg(\frac{\partial}{\partial x_j}\bigg|_p\bigg) &= \sum_{i=1}^{i=n}\bigg\{\mathrm{d}_p f\bigg(\frac{\partial}{\partial x_j}\bigg|_p\bigg)\bigg\}(y_i)\frac{\partial}{\partial y_i}\bigg|_{f(p)} = \sum_{i=1}^{i=n}\bigg(\frac{\partial}{\partial x_j}\bigg|_p(y_i\circ f)\bigg)\frac{\partial}{\partial y_i}\bigg|_{f(p)} \\ &= \sum_{i=1}^{i=n}\big(\partial_j(\pi_i\circ\psi\circ f\circ\varphi^{-1})\big)_{\varphi(p)}\frac{\partial}{\partial y_i}\bigg|_{f(p)}; \end{aligned} \tag{4.18}$$

so the matrix of the linear transformation $\mathrm{d}_p f\colon T_pM\longrightarrow T_{f(p)}N$ with respect to the bases $\{\frac{\partial}{\partial x_j}\big|_p\}_j$ and $\{\frac{\partial}{\partial y_i}\big|_{f(p)}\}_i$ is $\mathcal{J}(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}$, the Jacobian of the smooth map $\psi\circ f\circ\varphi^{-1}$ between the open subsets $\varphi(U\cap f^{-1}(V))$ and $\psi(V)$ of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, evaluated at $\varphi(p)$. In particular, $\mathrm{d}_p f$ is injective (surjective) if and only if $\mathcal{J}(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}$ is. The $f=\mathrm{id}$ case of (4.18) is the change-of-coordinates formula (4.17). If M and N are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, $\varphi=\mathrm{id}_M$, and $\psi=\mathrm{id}_N$, then under the canonical identifications $T_p\mathbb{R}^m=\mathbb{R}^m$ and $T_{f(p)}\mathbb{R}^n=\mathbb{R}^n$ the differential $\mathrm{d}_p f$ is simply the Jacobian $\mathcal{J}(f)_p$ of f at p. The chain-rule formula (4.5) states that the Jacobian of a composition of maps is the (matrix) product of the Jacobians of the maps; if M, N, and X are open subsets of Euclidean spaces, this yields the usual chain rule for smooth maps between open subsets of Euclidean spaces, for free (once it is checked that all definitions above make sense and correspond to the standard ones for Euclidean spaces).

By the above, if $\varphi=(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m$ is a smooth chart around a point p in M, then $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M. A weak converse to this statement is true as well; see Corollary 4.10 below. The key tool in obtaining it is the Inverse Function Theorem for $\mathbb{R}^m$; see [6, Theorem 8.3], for example.

Theorem 4.8 (Inverse Function Theorem)**.** *Let $U' \subset \mathbb{R}^m$ be an open subset and $f: U' \longrightarrow \mathbb{R}^m$ a smooth map. If the Jacobian $\mathcal{J}(f)_p$ of f is non-singular for some $p \in U'$, there exist neighborhoods U of p in U' and V of $f(p)$ in $\mathbb{R}^m$ such that $f: U \longrightarrow V$ is a diffeomorphism.*

Corollary 4.9 (Inverse Function Theorem for Manifolds)**.** *Let $f: M \longrightarrow N$ be a smooth map between smooth manifolds. If the differential $\mathrm{d}_p f: T_pM \longrightarrow T_{f(p)}N$ is an isomorphism for some $p \in M$, then there exist neighborhoods U of p in M and V of $f(p)$ in N such that $f: U \longrightarrow V$ is a diffeomorphism.*

Let $\varphi = (x_1, \ldots, x_m): U' \longrightarrow \mathbb{R}^m$ and $\psi = (y_1, \ldots, y_m): V' \longrightarrow \mathbb{R}^m$ be smooth charts around p in M and $f(p)$ in N, respectively; see Figure 1.5. Then,

$$\psi \circ f \circ \varphi^{-1}: \varphi\big(U' \cap f^{-1}(V')\big) \longrightarrow \varphi(V') \subset \mathbb{R}^m$$

is a smooth map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^m$ such that $\mathcal{J}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$ is non-singular (since by (4.18) this is the matrix of the linear transformation $\mathrm{d}_p f$ with respect to bases $\{\frac{\partial}{\partial x_j|_p}\}_j$ and $\{\frac{\partial}{\partial y_i}|_{f(p)}\}_i$). Since φ and ψ are homeomorphisms onto the open subsets $\varphi(U')$ and $\psi(V')$ of $\mathbb{R}^m$, by Theorem 4.8 there exist open neighborhoods U of p in $U' \cap f^{-1}(V')$ and V of $f(p)$ in V' such that the restriction

$$\psi \circ f \circ \varphi^{-1}: \varphi(U) \longrightarrow \psi(V)$$

is a diffeomorphism. Since $\varphi: U \longrightarrow \varphi(U)$ and $\psi: V \longrightarrow \psi(V)$ are also diffeomorphisms, it follows that so is $f: U \longrightarrow V$ (being composition of $\psi \circ f \circ \varphi^{-1}$ with ψ^{-1} and φ).

Corollary 4.10. *Let M be a smooth m-manifold. If $x_1, \ldots, x_m: U' \longrightarrow \mathbb{R}$ are smooth functions such that $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M for some $p \in U'$, then there exists a neighborhood U of p in U' such that*

$$\varphi = (x_1, \ldots, x_m): U \longrightarrow \mathbb{R}$$

is a smooth chart around p.

Let $f = (x_1, \ldots, x_m): U' \longrightarrow \mathbb{R}^m$. Since $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M, the differential

$$\mathrm{d}_p f = \begin{pmatrix} \mathrm{d}_p x_1 \\ \vdots \\ \mathrm{d}_p x_m \end{pmatrix}: T_pM \longrightarrow \mathbb{R}^m$$

is an isomorphism (for each $v \in T_pM - 0$, there exists i such that $\mathrm{d}_p x_i(v) \neq 0$). Thus, Corollary 4.10 follows immediately from Corollary 4.9 with $M = U'$ and $N = \mathbb{R}^m$.

Corollary 4.11. *Let M be a smooth m-manifold. If $x_1, \ldots, x_n: U' \longrightarrow \mathbb{R}$ are smooth functions such that the set $\{\mathrm{d}_p x_i\}_i$ spans T_p^*M for some $p \in U'$, then there exists a neighborhood U of p in U' such that an m-element subset of $\{x_i\}_i$ determines a smooth chart around p on M.*

This claim follows from Corollary 4.10 by choosing a subset of $\{x_i\}_i$ so that the corresponding subset of $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M.

Corollary 4.12. *Let M be a smooth m-manifold. If $x_1, \ldots, x_k: U' \longrightarrow \mathbb{R}$ are smooth functions such that the set $\{\mathrm{d}_p x_i\}_i$ is linearly independent in T_p^*M for some $p \in U'$, then there exist a neighborhood U of p in U' and a set of smooth functions $x_{k+1}, \ldots, x_m: U \longrightarrow \mathbb{R}$ such that the map*

$$\varphi = \big(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\big): U \longrightarrow \mathbb{R}^m$$

is a smooth chart around p on M.

This claim follows from Corollary 4.10 by choosing a smooth chart $\psi = (y_1, \ldots, y_m) : U'' \longrightarrow \mathbb{R}^m$ on a neighborhood U'' of p in U' and adding some of the functions y_j to the set $\{x_i\}_i$ so that the corresponding set $\{\mathrm{d}_p x_i, \mathrm{d}_p y_j\}$ is a basis for $T_p^* M$.

Remark 4.13. The differential of a smooth map induces a functor from the category of pointed smooth manifolds (smooth manifolds with a choice of a point) and pointed smooth maps (smooth maps taking chosen points to each other) to the category of finite-dimensional vector spaces and vector-space homomorphisms:

$$(M,p) \longrightarrow T_p M, \qquad \big(h\colon (M,p) \longrightarrow (N,q)\big) \longrightarrow \big(\mathrm{d}_p h\colon T_p M \longrightarrow T_q N\big);$$

these mappings take a composition of morphisms to a composition of morphisms by (4.5) and id_M to $\mathrm{id}_{T_p M}$. On the other hand, the pull-back map h^* on the cotangent spaces reverses compositions of morphisms by (4.6) and thus gives rise to a contravariant functor between the same two categories.

5 Immersions and Submanifolds

Definition 5.1. *Let M and N be smooth manifolds.*

(1) A smooth map $f\colon M \longrightarrow N$ is an immersion if the differential $\mathrm{d}_p f\colon T_p M \longrightarrow T_{f(p)} N$ is injective for every $p \in M$.

(2) The manifold M is a submanifold of N if $M \subset N$, M has the subspace topology, and the inclusion map $\iota\colon M \longrightarrow N$ is an immersion.

If $M \subset N$ is a smooth submanifold and $p \in M$, the differential $\mathrm{d}_p \iota\colon T_p M \longrightarrow T_p N$ is an injective homomorphism. In such cases, we will identify $T_p M$ with $\mathrm{Im}\, \mathrm{d}_p \iota \subset T_p N$ via $\mathrm{d}_p \iota$.

Discrete subsets of points (with the unique smooth structure) and open subsets (with the induced smooth structure of Proposition 1.11) of a smooth manifold are submanifolds; see Exercise 25. If M and N are smooth manifolds, the horizontal and vertical slices

$$\mathrm{Im}\, \iota_q,\ \mathrm{Im}\, \iota_p \subset M \times N$$

of Example 2.5 are embedded submanifolds; see Exercise 26. On the other hand, $\mathbb{Q} \subset \mathbb{R}$ does not admit a submanifold structure.

If $f\colon M \longrightarrow N$ is a diffeomorphism between smooth manifolds, then the differential

$$\mathrm{d}_p f\colon T_p M \longrightarrow T_{f(p)} N \tag{5.1}$$

is an isomorphism for every $p \in M$. Thus, a diffeomorphism between two smooth manifolds is a bijective immersion. On the other hand, if $f : M \longrightarrow N$ is an immersion, $\dim M \leq \dim N$. If $\dim M = \dim N$ and $f : M \longrightarrow N$ is an immersion, then the differential (5.1) is an isomorphism for every $p \in M$. Corollary 4.9 then implies that f is a local diffeomorphism. Thus, a bijective immersion $f : M \longrightarrow N$ between smooth manifolds of the same dimension is a diffeomorphism. The assumption that manifolds are second-countable topological spaces turns out to imply that a bijective immersion must be a map between manifolds of the same dimension; see Exercise 31. Thus, a bijective immersion is a diffeomorphism and vice versa.

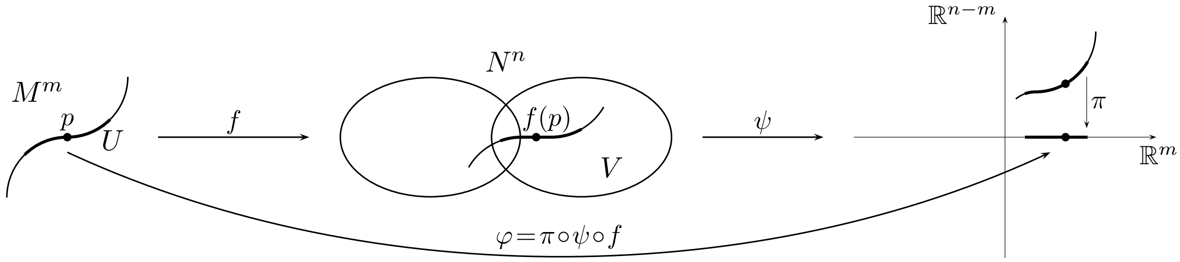


Figure 1.8: An immersion pull-backs a subset of the coordinates on the target to a smooth chart on the domain

A more interesting example of an immersion is the inclusion of $\mathbb{R}^m$ as the coordinate subspace $\mathbb{R}^m \times 0$ into $\mathbb{R}^n$, with $m \le n$. By Proposition 5.3 below, every immersion $f: M \longrightarrow N$ locally (on M and N) looks like the inclusion of $\mathbb{R}^m$ as $\mathbb{R}^m \times 0$ into $\mathbb{R}^n$ and every submanifold $M \subset N$ locally (on N) looks like $\mathbb{R}^m \times 0 \subset \mathbb{R}^n$. We will use the following lemma in the proof of Proposition 5.3.

Lemma 5.2. *Let $f: M^m \longrightarrow N^n$ be a smooth map and $p \in M$. If the differential $\mathrm{d}_p f$ is injective, there exist a neighborhood U of p in M and a smooth chart $\psi = (y_1, \ldots, y_n) : V \longrightarrow \mathbb{R}^n$ around $f(p) \in N$ such that*

$$\varphi = (y_1 \circ f, \ldots, y_m \circ f) : U \longrightarrow \mathbb{R}^m$$

is a smooth chart around $p \in M$.

Since the differential $\mathrm{d}_p f : T_p M \longrightarrow T_{f(p)} N$ is injective, its dual

$$f^* = \{\mathrm{d}_p f\}^* : T^*_{f(p)} N \longrightarrow T^*_p M$$

is surjective. Thus, if $\psi = (y_1, \ldots, y_n) : V \longrightarrow \mathbb{R}^n$ is any smooth chart around $f(p) \in N$, then the set

$$\big\{ f^* \mathrm{d}_{f(p)} y_i = \mathrm{d}_p (y_i \circ f) \big\}_i$$

spans $T^*_p M$ (because the set $\{\mathrm{d}_{f(p)} y_i\}$ is a basis for $T^*_{f(p)} N$). By Corollary 4.11, a subset of $\{y_i \circ f\}_i$ determines a smooth chart around p on M. If this subset is different from $\{y_1 \circ f, \ldots, y_m \circ f\}$, compose ψ with a diffeomorphism of $\mathbb{R}^n$ that switches the coordinates, sending the chosen coordinates (those in the subset) to the first m coordinates.

The statement of Lemma 5.2 is illustrated in Figure 1.8. In summary, if $\mathrm{d}_p f$ is injective, then m of the coordinates of a smooth chart around $f(p)$ give rise to a smooth chart around p. By re-ordering the coordinates around $f(p)$, it can be assumed that it is the first m coordinates that give rise to a smooth chart around p, which is then $\varphi = \pi \circ \psi \circ f$, where $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the projection on the first m coordinates. In particular,

$$\pi : \psi(f(U)) \longrightarrow \varphi(U) \subset \mathbb{R}^m$$

is bijective; so the image of $f(U) \subset V \subset N$ under ψ is the graph of some function $g : \varphi(U) \longrightarrow \mathbb{R}^{n-m}$:

$$\psi(f(U)) = \big\{ (x, g(x)) :\ x \in \varphi(U) \big\}.$$

By construction,

$$\psi(f(p')) = \big(y_1(f(p')), \ldots, y_n(f(p')) \big) = \big(\varphi(p'), g(\varphi(p')) \big) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \qquad \forall\, p' \in U;$$

so $g=(y_{m+1},\ldots,y_n)\circ f\circ\varphi^{-1}$. In the proof of the next proposition, we compose ψ with the diffeomorphism $(x,y)\longrightarrow(x,y-g(x))$ so that the image of $f(U)$ is shifted to $\mathbb{R}^m\times 0$.

Proposition 5.3 (Slice Lemma)**.** *Let $f: M^m\longrightarrow N^n$ be a smooth map and $p\in M$. If d_pf is injective, there exist smooth charts*

$$\varphi\colon U\longrightarrow\mathbb{R}^m \qquad \textit{and} \qquad \psi\colon V\longrightarrow\mathbb{R}^n$$

around $p\in M$ and $f(p)\in N$, respectively, such that the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \mathbb{R}^m \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

commutes, where the bottom arrow is the natural inclusion of $\mathbb{R}^m$ as $\mathbb{R}^m\times 0$, and $f(U)=\psi^{-1}(\mathbb{R}^m\times 0)$.

By Lemma 5.2, there exist a neighborhood U of p in M and a smooth chart $\psi'=(y_1,\ldots,y_n)$: $V'\longrightarrow\mathbb{R}^n$ around $f(p)\in N$ such that

$$\varphi=\pi\circ\psi'\circ f\colon U\longrightarrow\mathbb{R}^m$$

is a smooth chart around $p\in M$, where $\pi\colon\mathbb{R}^n\longrightarrow\mathbb{R}^m$ is the projection on the first m coordinates as before. In particular, $\varphi(U)\subset\mathbb{R}^m$ is an open subset and

$$\psi'\circ f=\big(\varphi,g\circ\varphi\big)\colon U\longrightarrow\mathbb{R}^m\times\mathbb{R}^{n-m},$$

where $g=(y_{m+1},\ldots,y_n)\circ f\circ\varphi^{-1}:\varphi(U)\longrightarrow\mathbb{R}^{n-m}$; this is a smooth function. Thus, the map

$$\Theta\colon\varphi(U)\times\mathbb{R}^{n-m}\longrightarrow\varphi(U)\times\mathbb{R}^{n-m}, \qquad (x,y)\longrightarrow\big(x,y-g(x)\big),$$

is smooth. It is clearly bijective, and

$$\mathcal{J}(\Theta)_{(x,y)}=\begin{pmatrix}\mathbb{I}_m & 0\\ * & \mathbb{I}_{n-m}\end{pmatrix};$$

so Θ is a diffeomorphism. Let $V=\psi'^{-1}(\varphi(U)\times\mathbb{R}^{n-m})$ and

$$\psi=\Theta\circ\psi'\colon V\longrightarrow\mathbb{R}^n.$$

Since $\varphi(U)\times\mathbb{R}^{n-m}$ is open in $\mathbb{R}^n$, V is open in N. Since Θ is a diffeomorphism, ψ is a smooth chart on N. Since $\psi'(V')$ and $\varphi(U)\times\mathbb{R}^{n-m}$ contain $\psi'(f(U))$, $f(U)$ is contained in V. By definition,

$$\begin{aligned}\psi\circ f(p')=\Theta\circ\psi'\circ f(p')=\Theta\big(\varphi(p'),g(\varphi(p'))\big)&=\big(\varphi(p'),g(\varphi(p'))-g(\varphi(p'))\big)\\ &=\big(\varphi(p'),0\big)\in\varphi(U)\times 0 \quad \forall\,p'\in U.\end{aligned}$$

Since $\psi(f(U))=\varphi(U)=\psi(V)\cap\mathbb{R}^m\times 0$, $f(U)=\psi^{-1}(\mathbb{R}^m\times 0)$.

Corollary 5.4. *If $M^m\subset N^n$ is a submanifold, for every $p\in M$ there exists a smooth chart $\psi\equiv(x_1,\ldots,x_n): V\longrightarrow\mathbb{R}^n$ on N around p such that*

$$M\cap V=\psi^{-1}(\mathbb{R}^m\times 0)\equiv\big\{p'\in V\colon\ x_{m+1}(p')=x_{m+2}(p')=\ldots=x_n(p')=0\big\}$$

and $\psi\colon M\cap V\longrightarrow\mathbb{R}^m\times 0=\mathbb{R}^m$ is a smooth chart on M.

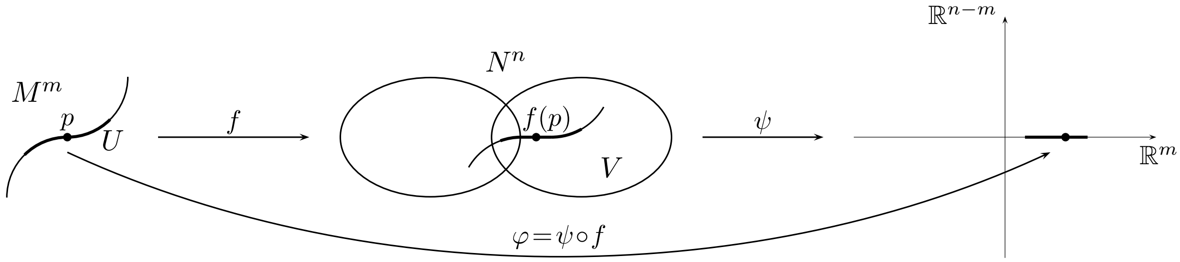


Figure 1.9: The local structure of immersions

Let U be an open neighborhood of p in M and (V,ψ) a smooth chart on N around $p=f(p)$ provided by Proposition 5.3 for the inclusion map $f: M \longrightarrow N$. Since $M\subset N$ has the subspace topology, there exists $W\subset V$ open so that $U=M\cap W$; the smooth chart $(W,\psi|_W)$ then has the desired properties.

According to this corollary, every smooth m-submanifold M of an n-manifold N locally (in a suitably chosen smooth chart) looks like the horizontal slice $\mathbb{R}^m\times 0\subset\mathbb{R}^n$. If $p\in M$ lies in such a chart,

$$\begin{aligned} T_pM &= \mathrm{Span}_{\mathbb{R}}\Big\{\frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, \ldots, \frac{\partial}{\partial x_m}\Big|_p\Big\} \\ &= \big\{v\in T_pN:\ \mathrm{d}_px_{m+1}(v)=\mathrm{d}_px_{m+2}(v)=\ldots=\mathrm{d}_px_n(v)=0\big\}. \end{aligned}$$

Proposition 5.3 completely describes the local structure of immersions, but says nothing about their global structure. Images of 3 different immersions of $\mathbb{R}$ into $\mathbb{R}^2$ are shown in Figure 1.10. Another type phenomena is illustrated by the injective immersion

$$\mathbb{R}\longrightarrow S^1\times S^1, \qquad s\longrightarrow\big(\mathrm{e}^{\mathrm{i}s},\mathrm{e}^{\mathrm{i}\alpha s}\big), \tag{5.2}$$

where $\alpha\in\mathbb{R}-\mathbb{Q}$. The image of this immersion is a dense submanifold of $S^1\times S^1$.

If $\iota: M\longrightarrow N$ is an injective map and $h: X\longrightarrow N$ is any map such that $h(X)\subset\iota(M)$, then there

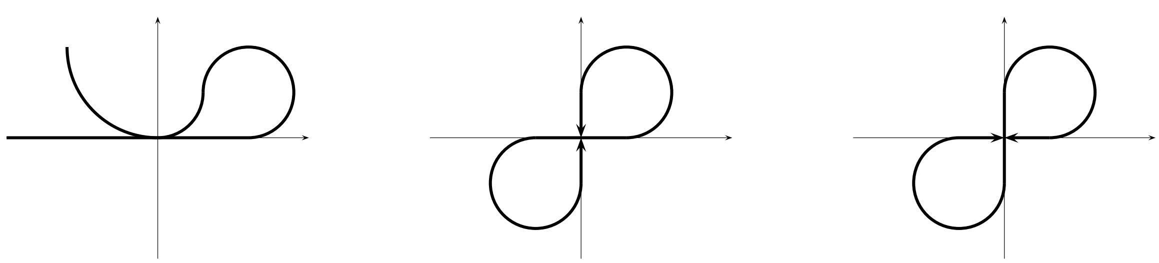


Figure 1.10: Images of some immersions $\mathbb{R}\longrightarrow\mathbb{R}^2$

exists a unique map $h_0\colon X\longrightarrow M$ so that the diagram

$$\begin{tikzcd} & M \arrow[d, "\iota"] \\ X \arrow[ur, dashed, "h_0"] \arrow[r, "h"] & N \end{tikzcd}$$

commutes. If M, N, and X are topological spaces, ι is an embedding, and h is continuous, then h_0 is also continuous [7, Theorem 7.2e]. An analogue of this property holds in the smooth category, as indicated by the next proposition.

Proposition 5.5. *Let $\iota\colon M\longrightarrow N$ be an injective immersion, $h\colon X\longrightarrow N$ a smooth map such that $h(X)\subset\iota(M)$, and $h_0\colon X\longrightarrow M$ the unique map such that $h=\iota\circ h_0$. If h_0 is continuous, then it is smooth; in particular, h_0 is smooth if ι is an embedding (e.g. if M is a submanifold of N).*

It is sufficient to show that every point $q\in X$ has a neighborhood W on which h_0 is smooth. By Proposition 5.3, there exist smooth charts

$$\varphi\colon U\longrightarrow\mathbb{R}^m \qquad\text{and}\qquad \psi\colon V\longrightarrow\mathbb{R}^n$$

around $h_0(q)\in M$ and $h(q)=\iota(h_0(q))\in N$ such that the diagram

$$\begin{tikzcd} & U \arrow[r, "\varphi"] \arrow[d, "\iota"] & \mathbb{R}^m \arrow[d] \\ W \arrow[ur, dashed, "h_0"] \arrow[r, "h"] & V \arrow[r, "\psi"] & \mathbb{R}^n \end{tikzcd}$$

commutes, where $W=h_0^{-1}(U)$ and the right-most arrow is the standard inclusion of $\mathbb{R}^m$ as $\mathbb{R}^m\times 0$ in $\mathbb{R}^n$. Since h_0 is continuous, W is open in X. Since h is smooth and ψ is a smooth chart on N, the map

$$\psi\circ h=\psi\circ\iota\circ h_0=\big(\varphi\circ h_0,0\big)\colon W\longrightarrow\mathbb{R}^m\times\mathbb{R}^{n-m}$$

is smooth. Thus, the map $\varphi\circ h_0:W\longrightarrow\mathbb{R}^m$ is also smooth. Since φ is a smooth chart on M containing the image of $h_0|_W$, it follows that $h_0|_W$ is a smooth map.

It is possible for the map h_0 to be continuous even if $\iota\colon M\longrightarrow N$ is not an embedding (and even if the image of h is not contained in the image of any open subset of M on which ι is an embedding). This is in particular the case for the immersion (5.2), which satisfies the condition of the following definition.

Definition 5.6. *An injective immersion $\iota: M^m\longrightarrow N^n$ is* regular *if for every $q\in N$ there exists a smooth chart $\psi:V\longrightarrow\mathbb{R}^m\times\mathbb{R}^{n-m}$ around q such that the image of every connected subset $U\subset\iota^{-1}(V)$ under ψ is contained in $\psi^{-1}(\mathbb{R}^m\times y)$ for some $y\in\mathbb{R}^{n-m}$ (dependent on U).*

Since the connected components of $\iota^{-1}(V)$ are disjoint open subsets of M and each of them is mapped by ι to one of the horizontal slices $\psi^{-1}(\mathbb{R}^m\times y)$, $\iota^{-1}(V)\subset M$ is contained in at most countably many of the horizontal slices $\psi^{-1}(\mathbb{R}^m\times y)$. In particular, each of the connected components of $\iota^{-1}(V)\subset M$ lies in one of these slices; see Figure 1.11.

Corollary 5.7. *If $\iota: M\longrightarrow N$ is a regular immersion, $h: X\longrightarrow N$ is a smooth map such that $h(X)\subset\iota(M)$, and $h_0\colon X\longrightarrow M$ is the unique map such that $h=\iota\circ h_0$, then h_0 is smooth.*

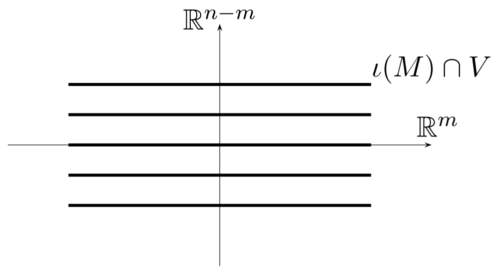


Figure 1.11: Image of a regular immersion $\iota(M)$ in a smooth chart consists of horizontal slices

In light of Proposition 5.5, it is sufficient to show that the map h_0 is continuous. Let U be a connected open subset of M, $x\in h_0^{-1}(U)$, and $p=h_0(x)$. We will show that there is an open subset $W\subset X$ such that $x\in W$ and $h(W)\subset\iota(U)$; since ι is injective, the latter implies that $h_0(W)\subset U$ and so $W\subset h_0^{-1}(U)$. Since ι is a regular immersion, there exists a smooth chart $\psi\colon V\longrightarrow\mathbb{R}^m\times\mathbb{R}^{n-m}$ around $h_0(p)=h(x)\in N$ such that the image of every connected subset $U'\subset\iota^{-1}(V)$ under ψ is contained in $\psi^{-1}(\mathbb{R}^m\times y)$ for some $y\in\mathbb{R}^{n-m}$. Shrinking U and V and shifting ψ, it can be assumed that $\iota(U)=\psi^{-1}(\mathbb{R}^m\times 0)$. Let $W\subset h^{-1}(V)$ be the connected component containing $x\in N$. Since $h(W)\subset\iota(M)\cap V$ is connected, $h(W)$ is contained in one of the horizontal slices $\psi^{-1}(\mathbb{R}^m\times y)$. Since $h(x)\in\psi^{-1}(\mathbb{R}^m\times 0)$, we conclude that $h(W)\subset\psi^{-1}(\mathbb{R}^m\times 0)=\iota(U)$.

On the other hand, h_0 in Proposition 5.5 need not be continuous in general. For example, it is not continuous at $h^{-1}(0)$ if ι and h are immersions described by the middle and right-most diagrams, respectively, in Figure 1.10. A similar example can be obtained from the left diagram in Figure 1.10 if all branches of the curve have infinite contact with the x-axis at the origin (ι and h can then differ by a "branch switch" at the origin).

Corollary 5.8. *Let N be a smooth manifold, $M\subset N$, and $\iota\colon M\longrightarrow N$ the inclusion map.*

(1) If $\mathcal{T}_M$ is a topology on M, there exists at most one smooth structure $\mathcal{F}_M$ on $(M,\mathcal{T}_M)$ with respect to which ι is an immersion.

(2) If $\mathcal{T}_M$ is the subspace topology on M and $(M,\mathcal{T}_M)$ admits a smooth structure $\mathcal{F}_M$ with respect to which ι is an immersion, there exists no other topology $\mathcal{T}'_M$ admitting a smooth structure $\mathcal{F}'_M$ on M with respect to which ι is an immersion.

The first statement of this corollary follows easily from Proposition 5.5. The second statement depends on manifolds being second-countable; its proof makes use of Exercise 31.

Corollary 5.9. *A topological subspace $M\subset N$ admits a smooth structure with respect to which M is a submanifold of N if and only if for every $p\in M$ there exists a neighborhood U of p in N such that the topological subspace $M\cap U$ of N admits a smooth structure with respect to which $M\cap U$ is a submanifold of N.*

By Corollary 5.8, the smooth structures on the overlaps of such open subsets must agree.

The middle and right-most diagrams in Figure 1.10 are examples of a subset M of a smooth manifold N that admits two different manifold structures $(M,\mathcal{T}_M,\mathcal{F}_M)$, in different topologies, with respect to which the inclusion map $\iota\colon M\longrightarrow N$ is an embedding. In light of the second statement of Proposition 5.8, this is only possible because M does not admit such a smooth structure in the subspace topology. On the other hand, if manifolds were not required to be second-countable, the discrete topology on $\mathbb{R}$ would provide a second manifold structure with respect to which the identity map $\mathbb{R} \longrightarrow \mathbb{R}$, with the target $\mathbb{R}$ having the standard manifold structure, would be an immersion.

6 Submersions and Submanifolds

This section is in a sense dual to Section 5. It describes ways of constructing new immersions and submanifolds by studying properties of submersions (smooth maps with surjective differentials), rather than studying properties of immersions and submanifolds. While Section 5 exploits Corollary 4.11, this section makes use of Corollary 4.12, as well as of the *Slice Lemma* (Proposition 5.3).

If M and N are smooth manifolds, the component projection maps

$$\pi_1\colon M\times N \longrightarrow M, \qquad \pi_2\colon M\times N \longrightarrow N,$$

are submersions; see Exercise 26. By the following lemma, every submersion locally has this form.

Lemma 6.1. *Let $h\colon M^m \longrightarrow Z^k$ be a smooth map and $p\in M$. If the differential $\mathrm{d}_p h$ is surjective, there exist smooth charts*

$$\varphi\colon U \longrightarrow \mathbb{R}^m \qquad \text{and} \qquad \psi\colon V \longrightarrow \mathbb{R}^k$$

around $p\in M$ and $h(p)\in Z$, respectively, such that the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "h"'] & \mathbb{R}^m \arrow[d] \\ V \arrow[r, "\psi"] & \mathbb{R}^k \end{tikzcd}$$

commutes, where the right arrow is the natural projection map from $\mathbb{R}^m$ to $\mathbb{R}^k\times 0\subset\mathbb{R}^m$.

Let $\psi=(y_1,\ldots,y_k)\colon V \longrightarrow \mathbb{R}^k$ be a smooth chart on Z around $f(p)$. Since the differential $\mathrm{d}_p h$ is surjective, its dual map

$$h^* = \{\mathrm{d}_p h\}^*\colon T^*_{h(p)}N \longrightarrow T^*_p M$$

is injective. Since $\{\mathrm{d}_{h(p)} y_i\}$ is a basis for $T^*_{h(p)}N$, it follows that the set

$$\big\{h^*\mathrm{d}_{h(p)} y_i = \mathrm{d}_p(y_i\circ h)\big\}$$

is linearly independent in $T^*_p M$. By Corollary 4.12, it can be extended to a smooth chart

$$\varphi\colon \big(y_1\circ h,\ldots,y_k\circ h, x_{k+1},\ldots,x_m\big)\colon U \longrightarrow \mathbb{R}^k\times\mathbb{R}^{m-k}$$

on M, where U is a neighborhood of p in $h^{-1}(V)$.

Lemma 6.1 can be seen as a counter-part of the *Slice Lemma* (Proposition 5.3). While an immersion locally looks like the inclusion

$$\mathbb{R}^m \longrightarrow \mathbb{R}^m\times 0 \subset \mathbb{R}^n, \qquad m\le n,$$

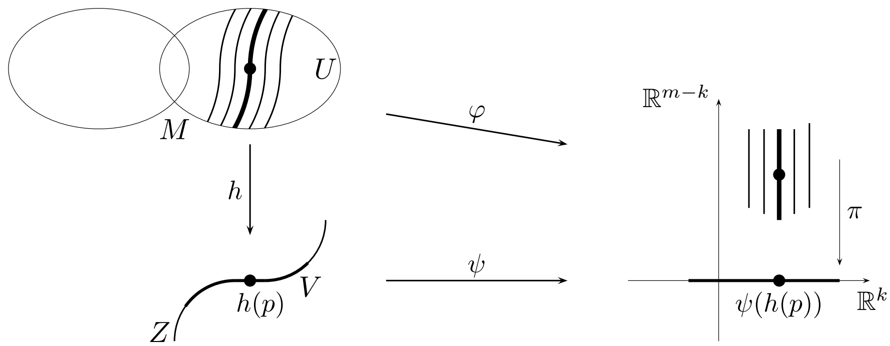


Figure 1.12: The local structure of submersions

a submersion locally looks like the projection

$$\mathbb{R}^m \longrightarrow \mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^m, \qquad k \le m.$$

Thus, an immersion can locally be represented by a horizontal slice in a smooth chart, while the pre-image of a point in the target of a submersion is locally a vertical slice (it is customary to present projections vertically, as in Figure 1.12).

Corollary 6.2. *Let $h\colon M \longrightarrow Z$ be a smooth map and $p \in M$. If the differential $\mathrm{d}_p h$ is surjective, there exist a neighborhood U of p in M and a smooth structure on the subspace $h^{-1}(h(p)) \cap U$ of M so that $h^{-1}(h(p)) \cap U$ is a submanifold of M and*

$$\mathrm{codim}_M\big(h^{-1}(h(p)) \cap U\big) \equiv \dim M - \dim\big(h^{-1}(h(p)) \cap U\big) = \dim Z.$$

If $\psi\colon V \longrightarrow \mathbb{R}^k$ and $\varphi = (\psi\circ h, \phi) : U \longrightarrow \mathbb{R}^k \times \mathbb{R}^{m-k}$ are smooth charts on Z around $h(p)$ and on M around p, respectively, provided by Lemma 6.1,

$$h^{-1}(h(p)) \cap U = \{\psi\circ h\}^{-1}\big(\psi(h(p))\big) \cap U = \{\pi\circ\varphi\}^{-1}\big(\psi(h(p))\big) = \varphi^{-1}\big(\psi(h(p)) \times \mathbb{R}^{m-k}\big).$$

Since $\varphi\colon U \longrightarrow \varphi(U)$ is a homeomorphism, so is the map

$$\varphi\colon h^{-1}(h(p)) \cap U \longrightarrow \psi(h(p)) \times \mathbb{R}^{m-k} \cap \varphi(U)$$

in the subspace topologies. Thus,

$$\phi\colon h^{-1}(h(p)) \cap U \longrightarrow \mathbb{R}^{m-k}$$

induces a smooth structure on $h^{-1}(h(p)) \cap U \subset M$ in the subspace topology with respect to which the inclusion $h^{-1}(h(p)) \cap U \longrightarrow M$ is an immersion because so is the inclusion

$$\psi(h(p)) \times \mathbb{R}^{m-k} \longrightarrow \mathbb{R}^k \times \mathbb{R}^{m-k}.$$

Theorem 6.3 (Implicit Function Theorem for Manifolds)**.** *Let $f: M \longrightarrow N$ be a smooth map and $Y \subset N$ an embedded submanifold. If*

$$T_{f(p)}N = \mathrm{Im}\,\mathrm{d}_p f + T_{f(p)}Y \qquad \forall\, p \in f^{-1}(Y), \tag{6.1}$$

then $f^{-1}(Y)$ admits the structure of an embedded submanifold of M,

$$\mathrm{codim}_M f^{-1}(Y) = \mathrm{codim}_N Y, \qquad T_p\big(f^{-1}(Y)\big) = \{\mathrm{d}_p f\}^{-1}\big(T_{f(p)}Y\big) \;\; \forall\, p \in f^{-1}(Y).$$

In order to prove the first two statements, it is sufficient to show that for every $p\in f^{-1}(Y)$ there exists a neighborhood U of p in M such that $f^{-1}(Y)\cap U$ admits the structure of an embedded submanifold of M of the claimed dimension; see Corollary 5.9. As provided by Corollary 5.4, let $\psi\colon V\longrightarrow\mathbb{R}^n$ be a smooth chart on N around $f(p)\in Y$ such that $Y\cap V=\psi^{-1}(\mathbb{R}^l\times 0)$, where $l=\dim Y$. Let $\tilde\pi\colon\mathbb{R}^n\longrightarrow 0\times\mathbb{R}^{n-l}$ be the projection map and

$$h=\tilde\pi\circ\psi\circ f : f^{-1}(V)\longrightarrow V\longrightarrow\mathbb{R}^n\longrightarrow\mathbb{R}^{n-l}.$$

Since $\mathbb{R}^l\times 0=\tilde\pi^{-1}(0)$, $Y\cap V=\psi^{-1}(\tilde\pi^{-1}(0))$ and

$$f^{-1}(Y)\cap f^{-1}(V)=f^{-1}(Y\cap V)=f^{-1}\big(\psi^{-1}(\tilde\pi^{-1}(0))\big)=h^{-1}(0). \tag{6.2}$$

On the other hand, by the chain rule (4.5)

$$\mathrm{d}_p h=\mathrm{d}_{\psi(f(p))}\tilde\pi\circ\mathrm{d}_{f(p)}\psi\circ\mathrm{d}_p f\colon T_pM\longrightarrow T_{f(p)}N\longrightarrow T_{\psi(f(p))}\mathbb{R}^n\longrightarrow T_0(0\times\mathbb{R}^{n-l}). \tag{6.3}$$

The homomorphism $\mathrm{d}_{\psi(f(p))}\tilde\pi$ is onto, as is the homomorphism $\mathrm{d}_{f(p)}\psi$. On the other hand,

$$\mathrm{d}_{\psi(f(p))}\tilde\pi\circ\mathrm{d}_{f(p)}\psi=\mathrm{d}_{f(p)}(\tilde\pi\circ\psi)\colon T_{f(p)}N\longrightarrow T_{\psi(f(p))}\mathbb{R}^n\longrightarrow T_0(0\times\mathbb{R}^{n-l})$$

by the chain rule (4.5) and thus vanishes on $T_{f(p)}Y$ (since $\tilde\pi\circ\psi$ maps Y to 0 in $0\times\mathbb{R}^{n-l}$). So, by (6.1), the restriction

$$\mathrm{d}_{\psi(f(p))}\tilde\pi\circ\mathrm{d}_{f(p)}\psi\colon\mathrm{Im}\,\mathrm{d}_pf\longrightarrow T_0(0\times\mathbb{R}^{n-l})$$

is onto, i.e. the homomorphism (6.3) is surjective. By Corollary 6.2 and (6.2), there exists a neighborhood U of p in $f^{-1}(V)$ such that

$$f^{-1}(Y)\cap U=f^{-1}(Y)\cap f^{-1}(V)\cap U=h^{-1}(0)\cap U$$

admits the structure of an embedded submanifold of M of codimension l, as required. For the last statement, note that

$$T_p\big(f^{-1}(Y)\big)\subset\{\mathrm{d}_pf\}^{-1}\big(T_{f(p)}Y\big)\qquad\forall\,p\in f^{-1}(Y),$$

since $f(f^{-1}(Y))\subset Y$; the opposite inclusion holds for dimensional reasons.

Corollary 6.4. *Let $f\colon M\longrightarrow N$ be a smooth map and $q\in N$. If*

$$\mathrm{d}_pf\colon T_pM\longrightarrow T_qN\quad\textit{is onto}\quad\forall\ p\in f^{-1}(q), \tag{6.4}$$

then $f^{-1}(q)$ admits the structure of an embedded submanifold of M of codimension equal to the dimension of N and

$$T_p\big(f^{-1}(q)\big)=\ker\big(\mathrm{d}_pf\colon T_pM\longrightarrow T_qN\big)\qquad\forall\,p\in f^{-1}(q).$$

This is just the $Y=\{q\}$ case of Theorem 6.3.

Example 6.5. Let $f:\mathbb{R}^{m+1}\longrightarrow\mathbb{R}$ be given by $f(x)=|x|^2$. This is a smooth map, and its differential at $x\in\mathbb{R}^{m+1}$ with respect to the standard bases for $T_x\mathbb{R}^{m+1}$ and $T_{f(x)}\mathbb{R}$ is

$$\mathcal{J}(f)_x=\big(2x_1\ 2x_2\ \ldots\ 2x_{m+1}\big)\colon\mathbb{R}^{m+1}\longrightarrow\mathbb{R}.$$

Thus, d_xf is surjective if and only if $x\neq 0$, i.e. $f(x)\neq 0$. By Corollary 6.4, $f^{-1}(q)$ with $q\neq 0$ then admits the structure of an embedded submanifold of $\mathbb{R}^{m+1}$ and its codimension is 1 (so the dimension is m). This is indeed the case, since $f^{-1}(q)$ is the sphere of radius $\sqrt{q}$ centered at the origin if $q>0$ and the empty set (which is a smooth manifold of any dimension) if $q<0$. If $q=0$, $f^{-1}(q)=\{0\}$; this happens to be a smooth submanifold of $\mathbb{R}^{m+1}$, but of the wrong dimension.

Example 6.6. Corollary 6.4 can be used to show that the group SO_n is a smooth submanifold of $\mathrm{Mat}_{n\times n}\mathbb{R}$, while U_n and SU_n are smooth submanifolds of $\mathrm{Mat}_{n\times n}\mathbb{C}$. For example, with $\mathrm{SMat}_n\mathbb{R}$ denoting the space of symmetric $n\times n$ real matrices, define

$$f\colon \mathrm{Mat}_{n\times n}\mathbb{R} \longrightarrow \mathrm{SMat}_{n\times n}\mathbb{R}, \qquad \text{by} \quad f(A) = AA^{\mathrm{tr}}\,.$$

Then, $O(n)=f^{-1}(\mathbb{I}_n)$. It is then sufficient to show that the differential $\mathrm{d}_A f$ is onto for all $A\in O(n)$. Since $f=f\circ R_A$ for every $A\in O(n)$, where the diffeomorphism

$$R_A\colon \mathrm{Mat}_{n\times n}\mathbb{R} \longrightarrow \mathrm{Mat}_{n\times n}\mathbb{R} \qquad \text{is given by} \quad R_A(B) = BA,$$

it is sufficient to establish that $\mathrm{d}_{\mathbb{I}} f$ is surjective. This is done in Example 4.7.

Corollary 6.7 (Implicit Function Theorem for Maps)**.** *Let $f\colon X\longrightarrow M$ and $g\colon Y\longrightarrow M$ be smooth maps. If*

$$T_{f(x)}M = \mathrm{Im}\,\mathrm{d}_x f + \mathrm{Im}\,\mathrm{d}_y g \qquad \forall\ (x,y)\in X\times Y \ \text{s.t.}\ f(x)=g(y), \tag{6.5}$$

then the space

$$X\times_M Y \equiv \big\{(x,y)\in X\times Y\colon\ f(x)=g(y)\big\}$$

admits the structure of an embedded submanifold of $X\times Y$ of codimension equal to the dimension of M and

$$T_{(p,q)}\big(X\times_M Y\big) = \big\{(v,w)\in T_pX\oplus T_qY\colon\ \mathrm{d}_p f(v)=\mathrm{d}_q g(w)\big\} \quad \forall\,(p,q)\in X\times_M Y$$

under the identification of Exercise 26. Furthermore, the projection map $\pi_1=\pi_X\colon X\times_M Y\longrightarrow X$ is injective (an immersion) if $g\colon Y\longrightarrow M$ is injective (an immersion).

This corollary is obtained by applying Theorem 6.3 to the smooth map

$$h = (f,g)\colon X\times Y \longrightarrow M\times M.$$

Its last statement immediately implies Warner's Theorem 1.39. The commutative diagram

$$\begin{tikzcd} X\times_M Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & M \end{tikzcd}$$

is known as a fibered square.

Corollary 6.8 (Implicit Function Theorem for Intersections)**.** *Let $X,Y\subset M$ be embedded submanifolds. If*

$$T_pM = T_pX + T_pY \qquad \forall\ p\in X\cap Y, \tag{6.6}$$

then $X\cap Y$ is a smooth submanifold of X, Y, and M,

$$\dim X\cap Y = \dim X + \dim Y - \dim M, \qquad T_p(X\cap Y) = T_pX\cap T_pY \subset T_pM \quad \forall\,p\in X\cap Y.$$

This corollary is a special case of Corollary 6.7.

Remark 6.9. Submanifolds $X, Y \subset M$ satisfying (6.6) are said to be transverse (in M); this is written as $X \pitchfork Y$ or $X \pitchfork_M Y$ to be specific. For example, two distinct lines in the plane are transverse, but two intersecting lines in $\mathbb{R}^3$ are not. Similarly, smooth maps $f: X \longrightarrow M$ and $g: Y \longrightarrow M$ satisfying (6.5) are called transverse; this is written as $f \pitchfork g$ or $f \pitchfork_M g$. If $f: M \longrightarrow N$ satisfies (6.1) with respect to a submanifold $Y \subset N$, f is said to transverse to Y; this is written as $f \pitchfork Y$ or $f \pitchfork_N Y$. Finally, if $f: M \longrightarrow N$ satisfies (6.4) with respect to $q \in N$, q is said to be a regular value of f. By Corollary 6.4, the pre-image of a regular value is a smooth submanifold in the domain of codimension equal to the dimension of the target. By Sard's Theorem [4, §2], the set of a regular values is dense in the target (in fact, its complement is a set of measure 0); so the pre-images of most points in the target of a smooth map are smooth submanifolds of the domain, though in some cases they may all be empty (e.g. if the dimension of the domain is lower than the dimension of the target).

The standard version of the Implicit Function Theorem for $\mathbb{R}^m$, Corollary 6.10 below, says that under certain conditions a system of k equations in m variables has a locally smooth $(m-k)$-dimensional space of solutions which can be described as a graph of a function $g: \mathbb{R}^{m-k} \longrightarrow \mathbb{R}^k$. It is normally obtained as an application of the Inverse Function Theorem for $\mathbb{R}^m$, Theorem 4.8 above. It can also be deduced from the proof of Lemma 6.1 and by itself implies Corollary 6.2.

Corollary 6.10 (Implicit Function Theorem for $\mathbb{R}^m$)**.** *Let $U \subset \mathbb{R}^{m-k} \times \mathbb{R}^k$ be an open subset and $f: U \longrightarrow \mathbb{R}^k$ a smooth function. If $(x_0, y_0) \in f^{-1}(0)$ is such that the right $k \times k$ submatrix of $\mathcal{J}(f)_{(x_0,y_0)}$, $\frac{\partial f}{\partial y}|_{(x_0,y_0)}$, is non-singular, then there exist open neighborhoods V of x_0 in $\mathbb{R}^{m-k}$ and W of y_0 in $\mathbb{R}^k$ and a smooth function $g: V \longrightarrow W$ such that*

$$f^{-1}(0) \cap V \times W = \big\{(x, g(x)) \colon x \in V\big\}.$$

Exercises

1. Show that the collection (1.1) is indeed a smooth structure on M, according to Definition 1.3.

2. Show that the maps $\varphi_\pm: U_\pm \longrightarrow \mathbb{R}^m$ described after Example 1.7 are indeed charts on S^m and the overlap map between them is

$$\varphi_+ \circ \varphi_-^{-1} \colon \varphi_-(U_+ \cap U_-) = \mathbb{R}^m - 0 \longrightarrow \varphi_+(U_+ \cap U_-) = \mathbb{R}^m - 0, \qquad x \longrightarrow \frac{x}{|x|^2}.$$

3. Show that the map $\varphi_{1/2}$ in Example 1.8 is well-defined and is indeed a homeomorphism.

4. With notation as in Example 1.10, show that

(a) the map $S^{2n+1}/S^1 \longrightarrow (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$ induced by inclusions $S^{2n+1} \longrightarrow \mathbb{C}^{2n+1}$ and $S^1 \longrightarrow \mathbb{C}^*$ is a homeomorphism with respect to the quotient topologies;

(b) the quotient topological space, $\mathbb{C}P^n$, is a compact topological $2n$-manifold which admits a structure of a *complex* (in fact, *algebraic*) n-manifold, i.e. it can be covered by charts whose overlap maps, $\varphi_\alpha \circ \varphi_\beta^{-1}$, are holomorphic maps between open subsets of $\mathbb{C}^n$ (and rational functions on $\mathbb{C}^n$);

(c) $\mathbb{C}P^n$ contains $\mathbb{C}^n$ (with its complex structure) as a dense open subset.

5. Let V and W be finite-dimensional vector spaces with the canonical smooth structures of Example 1.5. Show that the canonical smooth structure on the vector space $V\oplus W = V\times W$ is the same as the product smooth structure.

6. Show that a composition of two smooth maps (local diffeomorphisms, diffeomorphisms) is again smooth (a local diffeomorphism, a diffeomorphism).

7. Let $f\colon M\longrightarrow N$ be a map between smooth manifolds. Show that f is a smooth map if and only if for every smooth function $h\colon N\longrightarrow\mathbb{R}$ the function $h\circ f\colon M\longrightarrow\mathbb{R}$ is also smooth.

8. Verify Lemma 2.2.

9. Let V be a finite-dimensional vector space with the canonical smooth structure of Example 1.5. Show that the vector space operations,

$$\begin{aligned} V\times V &\longrightarrow V, & (v_1,v_2) &\longrightarrow v_1+v_2\,,\\ \mathbb{R}\times V &\longrightarrow V, & (r,v) &\longrightarrow rv\,, \end{aligned}$$

are smooth maps.

10. Let V_1,V_2,W be finite-dimensional vector spaces with their canonical smooth structures of Example 1.5. Show that any bilinear map

$$\cdot\colon V_1\times V_2\longrightarrow W, \qquad v_1\otimes v_2\longrightarrow v_1\cdot v_2\,,$$

is smooth.

11. Show that the two smooth structures $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{R}^1$ in Example 1.6 are not the same, but $(\mathbb{R}^1,\mathcal{F})$ and $(\mathbb{R}^1,\mathcal{F}')$ are diffeomorphic smooth manifolds.

12. Let $S^1\subset\mathbb{C}$ and MB be the unit circle and the infinite Mobius band with the smooth structures of Examples 1.7 and 1.8, respectively. Show that the map

$$\mathrm{MB}=\big([0,1]\times\mathbb{R}\big)/\!\sim\longrightarrow S^1, \qquad [s,t]\longrightarrow \mathrm{e}^{2\pi\mathrm{i}s}\,,$$

is well-defined and smooth.

13. Let $(M,\mathcal{F})$ be a smooth m-manifold and $U\subset M$ an open subset. Show that $\mathcal{F}|_U$ is the unique smooth structure on the topological subspace U of M satisfying either of the following two properties:

(SSM1) the inclusion map $\iota\colon U\longrightarrow M$ is a local diffeomorphism;

(SSM2) if N is a smooth manifold, a continuous map $f\colon N\longrightarrow U$ is smooth if and only if the map $\iota\circ f\colon N\longrightarrow M$ is smooth.

14. Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth manifolds and $\mathcal{F}_{M\times N}$ the product smooth structure on $M\times N$ of Proposition 1.11. Show that $\mathcal{F}_{M\times N}$ is the unique smooth structure on the product topological space $M\times N$ satisfying either of the following two properties:

(PSM1) the slice inclusion maps $\iota_q\colon M\longrightarrow M\times N$, with $q\in N$, and $\iota_p\colon M\longrightarrow M\times N$, with $p\in M$, and the projection maps $\pi_M,\pi_N\colon M\times N\longrightarrow M,N$ are smooth;

(PSM2) if X is a smooth manifold, continuous maps $f\colon X\longrightarrow M$ and $g\colon X\longrightarrow N$ are smooth if and only if the map $f\times g\colon X\longrightarrow M\times N$ is smooth.

15. Verify Lemmas 2.6 and Corollary 2.7.

16. Verify Proposition 2.8.

17. Show that the actions (2.4), (2.5), and (1.3) satisfy the assumptions of Proposition 2.8 and that the quotient smooth structures on

$$S^1=\mathbb{R}/\mathbb{Z}, \qquad \mathrm{MB}=(\mathbb{R}\times\mathbb{R})/\mathbb{Z}, \quad \text{and} \quad \mathbb{R}P^n=S^n/\mathbb{Z}_2,$$

are the same as the smooth structures of Examples 1.7, 1.8, and 1.9, respectively.

18. Verify that the addition and product operations on $\tilde{F}_p$ described after Definition 3.1 are well-defined and make $\tilde{F}_p$ into an $\mathbb{R}$-algebra with valuation map ev_p.

19. Deduce (3.7) from Lemma 3.5.

20. Verify that the map (3.9) is well-defined and is indeed an $\mathbb{R}$-algebra homomorphism.

21. Verify that the differential $\mathrm{d}_p h$ of a smooth map $h: M\longrightarrow N$, as defined in (4.1), is indeed well-defined. In other words, $\mathrm{d}_p h(v)$ is a derivation on $\tilde{F}_{h(p)}$ for all $v\in T_pM$. Show that $\mathrm{d}_p h\colon T_pM\longrightarrow T_{h(p)}N$ is a vector-space homomorphism.

22. Let M be a smooth connected manifold and $f\colon M\longrightarrow N$ a smooth map. Show that $\mathrm{d}_p f=0$ for all $p\in M$ if and only if f is a constant map.

23. Let M be a smooth manifold, V a finite-dimensional vector space with the canonical smooth structure of Example 1.5, and $f,g\colon M\longrightarrow V$ smooth maps. Show that

$$\mathrm{d}_p(f+g)=\mathrm{d}_p f+\mathrm{d}_p g\colon T_pM\longrightarrow V \qquad \forall\, p\in M,$$

under the canonical identifications $T_{f(p)}V, T_{g(p)}, T_{f(p)+g(p)}V=V$ of Example 3.8.

24. Let $f,g\colon M\longrightarrow\mathbb{R}$ be smooth maps. Show that

$$\mathrm{d}_p(fg)=f(p)\mathrm{d}_p g+g(p)\mathrm{d}_p f\colon T_pM\longrightarrow\mathbb{R} \qquad \forall\, p\in M.$$

More generally, suppose V_1,V_2,W are finite-dimensional vector spaces with their canonical smooth structures of Example 1.5,

$$\cdot\colon V_1\otimes V_2\longrightarrow W, \qquad v_1\otimes v_2\longrightarrow v_1\cdot v_2\,,$$

is a bilinear map, and $f_1\colon M\longrightarrow V_1$ and $f_2\colon M\longrightarrow V_2$ are smooth maps. Show that

$$\mathrm{d}_p(f_1\cdot f_2)=f_1(p)\cdot\mathrm{d}_p f_2+\mathrm{d}_p f_1\cdot f_2(p)\colon T_pM\longrightarrow W \qquad \forall\, p\in M,$$

under the canonical identifications $T_{f_1(p)}V_1=V_1$, $T_{f_2(p)}V_2=V_2$, and $T_{f_1(p)\cdot f_2(p)}W=W$ of Example 3.8.

25. Let $(M,\mathcal{F})$ be a smooth manifold, $U\subset M$ an open subset with the smooth structure induced from M as in Proposition 1.11, and $\iota\colon U\longrightarrow M$ the inclusion map. Show that the differential

$$\mathrm{d}_p\iota\colon T_pU \longrightarrow T_pM$$

is an isomorphism for all $p\in U$.

26. Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth manifolds and $M\times N$ their Cartesian product with the product smooth structure of Proposition 1.11. With notation as in Example 2.5, show that the homomorphisms

$$\begin{aligned}
&\mathrm{d}_p\iota_q+\mathrm{d}_q\iota_p\colon T_pM\oplus T_qN \longrightarrow T_{(p,q)}(M\times N), && (v_1,v_2)\longrightarrow \mathrm{d}_p\iota_q(v_1)+\mathrm{d}_q\iota_p(v_2)\\
&\mathrm{d}_{(p,q)}\pi_1\oplus\mathrm{d}_{(p,q)}\pi_2\colon T_{(p,q)}(M\times N)\longrightarrow T_pM\oplus T_qN, && w\longrightarrow \big(\mathrm{d}_{(p,q)}\pi_1(w),\mathrm{d}_{(p,q)}\pi_2(w)\big),
\end{aligned}$$

are isomorphisms and mutual inverses for all $p\in M$ and $q\in N$.

27. Let M be a non-empty compact m-manifold. Show that there exists no immersion $f\colon M\longrightarrow\mathbb{R}^m$.

28. Show that there exists no immersion $f\colon S^1\times S^1\longrightarrow\mathbb{R}P^2$.

29. Let M be a smooth manifold and $p\in M$ a fixed point of a smooth map $f\colon M\longrightarrow M$, i.e. $f(p)=p$. Show that if all eigenvalues of the linear transformation

$$\mathrm{d}_pf\colon T_pM \longrightarrow T_pM$$

are different from 1 (so $\mathrm{d}_pf(v)\neq v$ for all $v\in T_pM-0$), then p is an isolated fixed point (has a neighborhood that contains no other fixed point).

30. Let M be an embedded submanifold in a smooth manifold N and $\iota\colon M\longrightarrow N$ the inclusion map. Show that for every $p\in M$ the image of the differential

$$\mathrm{d}_p\iota\colon T_pM \longrightarrow T_pN$$

is the subspace of T_pN consisting of the vectors $\alpha'(0)$, where $\alpha\colon(-\epsilon,\epsilon)\longrightarrow N$ is a smooth map such that $\mathrm{Im}\,\alpha\subset M$ and $\alpha(0)=p$.

31. Show that a bijective immersion $f\colon M\longrightarrow N$ between two smooth manifolds is a diffeomorphism. *Hint:* you'll need to use that M is second-countable, along with either

(i) if $f\colon U\longrightarrow\mathbb{R}^n$ is a smooth map from an open subset of $\mathbb{R}^m$ with $m<n$, the measure of $f(U)\subset\mathbb{R}^n$ is 0;

(ii) the *Slice Lemma* (Proposition 5.3) and the *Baire Category Theorem* [7, Theorem 7.2].

32. Show that the map (5.2) is an injective immersion and that its image is dense in $S^1\times S^1$.

33. Verify Corollary 5.8.

34. Show that the smooth structures on S^m of Example 6.5 and Exercise 2 are the same.

35. Show that the topological subspace

$$\big\{(x,y)\in\mathbb{R}^2 : x^3+xy+y^3=1\big\}$$

of $\mathbb{R}^2$ is a smooth curve (i.e. admits a natural structure of smooth 1-manifold with respect to which it is a submanifold of $\mathbb{R}^2$).

36. (a) For what values of $t \in \mathbb{R}$, is the subspace

$$\big\{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} \colon x_1^2+\ldots+x_n^2-x_{n+1}^2 = t\big\}$$

a smooth embedded submanifold of $\mathbb{R}^{n+1}$?

(b) For such values of t, determine the diffeomorphism type of this submanifold (i.e. show that it is diffeomorphic to something rather standard). *Hint:* Draw some pictures.

37. Show that the special unitary group

$$SU_n = \big\{A \in \mathrm{Mat}_n\mathbb{C} \colon \bar{A}^{\mathrm{tr}}A = \mathbb{I}_n,\ \ \det\, A = 1\big\}$$

is a smooth compact manifold. What is its dimension?

38. Verify Corollary 6.7.

39. With notation as in Corollary 6.7, show that every pair of continuous maps $p\colon Z \longrightarrow X$ and $q\colon Z \longrightarrow Y$ such that $f \circ p = g \circ q$ factors through a unique continuous map $r\colon Z \longrightarrow X \times_M Y$,

$$\begin{tikzcd}
Z \arrow[dr, dashed, "r"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& X \times_M Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & M
\end{tikzcd}$$

and that $X \times_M Y$ is the unique (up to homeomorphism) topological space possessing this property for all (p,q) as above. If in addition the assumption (6.5) holds and p and q are smooth, then r is also smooth, and $X \times_M Y$ is the unique (up to diffeomorphism) smooth manifold possessing this property for all (p,q) as above.

40. Verify Corollary 6.8.

41. Let M be a smooth manifold and $p \in M$ a fixed point of a smooth map $f\colon M \longrightarrow M$, i.e. $f(p) = p$. Show that if all eigenvalues of the linear transformation

$$\mathrm{d}_p f \colon T_pM \longrightarrow T_pM$$

are different from 1 (so $\mathrm{d}_p f(v) \neq v$ for all $v \in T_pM - 0$), then p is an isolated fixed point (has a neighborhood that contains no other fixed point).

42. Deduce Corollary 6.10 from the proof of Lemma 6.1 and Corollary 6.2 from Corollary 6.10.

Chapter 2

Smooth Vector Bundles

7 Definitions and Examples

A *(smooth) real vector bundle* V of rank k over a smooth manifold M is a smoothly varying family of k-dimensional real vector spaces which is locally trivial. Formally, it is a triple (M,V,π), where M and V are smooth manifolds and

$$\pi\colon V \longrightarrow M$$

is a surjective submersion. For each $p\in M$, the fiber $V_p\equiv\pi^{-1}(p)$ of V over p is a real k-dimensional vector space; see Figure 2.1. The vector-space structures in V_p vary smoothly with $p\in M$. This means that the scalar multiplication map

$$\mathbb{R}\times V \longrightarrow V, \qquad (c,v)\longrightarrow c\cdot v, \tag{7.1}$$

and the addition map

$$V\times_M V \equiv \big\{(v_1,v_2)\in V\times V\colon \pi(v_1)=\pi(v_2)\in M\big\} \longrightarrow V, \qquad (v_1,v_2)\longrightarrow v_1+v_2, \tag{7.2}$$

are smooth. Note that we can add $v_1,v_2\in V$ only if they lie in the same fiber over M, i.e.

$$\pi(v_1)=\pi(v_2) \qquad\Longleftrightarrow\qquad (v_1,v_2)\in V\times_M V.$$

The space $V\times_M V$ is a smooth submanifold of $V\times V$ by the *Implicit Function Theorem for Maps* (Corollary 6.7). The local triviality condition means that for every point $p\in M$ there exist a neighborhood U of p in M and a diffeomorphism

$$h\colon V|_U \equiv \pi^{-1}(U) \longrightarrow U\times\mathbb{R}^k$$

such that h takes every fiber of π to the corresponding fiber of the projection map $\pi_1\colon U\times\mathbb{R}^k\longrightarrow U$, i.e. $\pi_1\circ h=\pi$ on $V|_U$ so that the diagram

$$\begin{tikzcd} V|_U\equiv\pi^{-1}(U) \arrow[rr, "h", "\approx"'] \arrow[dr, "\pi"'] & & U\times\mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

commutes, and the restriction of h to each fiber is linear:

$$h(c_1v_1+c_2v_2) = c_1h(v_1)+c_2h(v_2)\in x\times\mathbb{R}^k \qquad \forall\; c_1,c_2\in\mathbb{R},\; v_1,v_2\in V_x,\; x\in U.$$

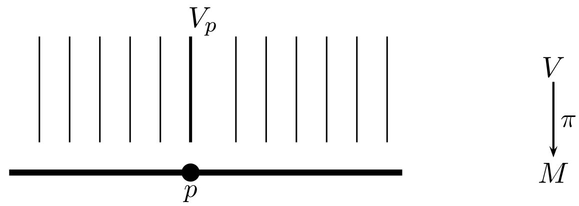


Figure 2.1: Fibers of a vector-bundle projection map are vector spaces of the same rank.

These conditions imply that the restriction of h to each fiber V_x of π is an isomorphism of vector spaces. In summary, V locally (and not just pointwise) looks like bundles of $\mathbb{R}^k$'s over open sets in M glued together. This is in a sense analogous to an m-manifold being open subsets of $\mathbb{R}^m$ glued together in a nice way. Here is a formal definition.

Definition 7.1. *A real vector bundle of rank k is a tuple $(M, V, \pi, \cdot, +)$ such that*

($\mathbb{R}$VB1) *M and V are smooth manifolds and $\pi\colon V \longrightarrow M$ is a smooth map;*

($\mathbb{R}$VB2) *$\cdot\colon \mathbb{R}\times V \longrightarrow V$ is a map s.t. $\pi(c\cdot v)=\pi(v)$ for all $(c,v)\in\mathbb{R}\times V$;*

($\mathbb{R}$VB3) *$+\colon V\times_M V \longrightarrow V$ is a map s.t. $\pi(v_1+v_2)=\pi(v_1)=\pi(v_2)$ for all $(v_1,v_2)\in V\times_M V$;*

($\mathbb{R}$VB4) *for every point $p \in M$ there exist a neighborhood U of p in M and a diffeomorphism $h\colon V|_U \longrightarrow U\times\mathbb{R}^k$ such that*

- ($\mathbb{R}$VB4-a) *$\pi_1\circ h=\pi$ on $V|_U$ and*
- ($\mathbb{R}$VB4-b) *the map $h|_{V_x}\colon V_x \longrightarrow x\times\mathbb{R}^k$ is an isomorphism of vector spaces for all $x\in U$.*

The spaces M and V are called the base and the total space of the vector bundle (M, V, π). It is customary to call $\pi : V \longrightarrow M$ a vector bundle and V a vector bundle over M. If M is an m-manifold and $V \longrightarrow M$ is a real vector bundle of rank k, then V is an $(m+k)$-manifold. Its smooth charts are obtained by restricting the trivialization maps h for V, as above, to small coordinate charts in M.

Example 7.2. If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1\colon M\times\mathbb{R}^k \longrightarrow M$$

is a real vector bundle of rank k over M. It is called the trivial rank k real vector bundle over M and denoted $\pi\colon \tau_k^{\mathbb{R}} \longrightarrow M$ or simply $\pi\colon \tau_k \longrightarrow M$ if there is no ambiguity.

Example 7.3. Let $M=S^1$ be the unit circle and $V=\mathrm{MB}$ the infinite Mobius band of Example 1.8. With notation as in Example 1.8, the map

$$\pi\colon V \longrightarrow M, \qquad [s,t] \longrightarrow \mathrm{e}^{2\pi\mathrm{i}s},$$

defines a real line bundle (i.e. rank 1 bundle) over S^1. Trivializations of this vector bundle can be constructed as follows. With $U_\pm=S^1-\{\pm1\}$, let

$$h_+\colon V|_{U_+} \longrightarrow U_+\times\mathbb{R}, \qquad [s,t] \longrightarrow \big(\mathrm{e}^{2\pi\mathrm{i}s}, t\big);$$
$$h_-\colon V|_{U_-} \longrightarrow U_-\times\mathbb{R}, \qquad [s,t] \longrightarrow \begin{cases} (\mathrm{e}^{2\pi\mathrm{i}s}, t), & \text{if } s\in(1/2,1];\\ (\mathrm{e}^{2\pi\mathrm{i}s}, -t), & \text{if } s\in[0,1/2). \end{cases}$$

Both maps are diffeomorphisms, with respect to the smooth structures of Example 1.8 on MB and of Example 1.7 on S^1. Furthermore, $\pi_1\circ h_\pm=\pi$ and the restriction of $h_\pm$ to each fiber of π is a linear map to $\mathbb{R}$.

Example 7.4. Let $\mathbb{R}P^n$ be the real projective space of dimension n described in Example 1.9 and

$$\gamma_n=\big\{(\ell,v)\in\mathbb{R}P^n\times\mathbb{R}^{n+1}\colon v\in\ell\big\},$$

where $\ell\subset\mathbb{R}^{n+1}$ denotes a one-dimensional linear subspace. If $U_i\subset\mathbb{R}P^n$ is as in Example 1.9, the map

$$h_i\colon \gamma_n\cap U_i\times\mathbb{R}^{n+1}\longrightarrow U_i\times\mathbb{R},\qquad \big(\ell,(v_0,\ldots,v_n)\big)\longrightarrow(\ell,v_i),$$

is a homeomorphism. The overlap maps,

$$h_i\circ h_j^{-1}\colon U_i\cap U_j\times\mathbb{R}\longrightarrow U_i\cap U_j\times\mathbb{R},\qquad (\ell,c)\longrightarrow\big(\ell,(X_i/X_j)c\big),$$

are smooth. By Lemma 2.6, the collection $\{(\gamma_n\cap U_i\times\mathbb{R}^{n+1},h_i)\}$ of generalized smooth charts then induces a smooth structure on the topological subspace $\gamma_n\subset\mathbb{R}P^n\times\mathbb{R}^{n+1}$. With this smooth structure, γ_n is an embedded submanifold of $\mathbb{R}P^n\times\mathbb{R}^{n+1}$ and the projection on the first component,

$$\pi=\pi_1:\gamma_n\longrightarrow\mathbb{R}P^n\,,$$

defines a smooth real line bundle. The fiber over a point $\ell\in\mathbb{R}P^n$ is the *one-dimensional subspace* ℓ *of* $\mathbb{R}^{n+1}$! For this reason, γ_n is called the tautological line bundle over $\mathbb{R}P^n$. Note that $\gamma_1\longrightarrow S^1$ is the infinite Mobius band of Example 7.3.

Example 7.5. If M is a smooth m-manifold, let

$$TM=\bigsqcup_{p\in M}T_pM,\qquad \pi\colon TM\longrightarrow M,\quad \pi(v)=p\ \text{ if } v\in T_pM.$$

If $\varphi_\alpha\colon U_\alpha\longrightarrow\mathbb{R}^m$ is a smooth chart on M, let

$$\tilde\varphi_\alpha\colon TM|_{U_\alpha}\equiv\pi^{-1}(U_\alpha)\longrightarrow U_\alpha\times\mathbb{R}^m,\qquad \tilde\varphi_\alpha(v)=\big(\pi(v),\mathrm{d}_{\pi(v)}\varphi_\alpha v\big).\tag{7.3}$$

If $\varphi_\beta\colon U_\beta\longrightarrow\mathbb{R}^m$ is another smooth chart, the overlap map

$$\tilde\varphi_\alpha\circ\tilde\varphi_\beta^{-1}\colon U_\alpha\cap U_\beta\times\mathbb{R}^m\longrightarrow U_\alpha\cap U_\beta\times\mathbb{R}^m$$

is a smooth map between open subsets of $\mathbb{R}^{2m}$. By Corollary 2.7, the collection of generalized smooth charts

$$\big\{(\pi^{-1}(U_\alpha),\tilde\varphi_\alpha)\colon(U_\alpha,\varphi_\alpha)\in\mathcal{F}_M\big\},$$

where $\mathcal{F}_M$ is the smooth structure of M, then induces a manifold structure on the set TM. With this smooth structure on TM, the projection $\pi\colon TM\longrightarrow M$ defines a smooth real vector bundle of rank m, called the tangent bundle of M.

Definition 7.6. *A complex vector bundle of rank k is a tuple $(M,V,\pi,\cdot,+)$ such that*

($\mathbb{C}$VB1) *M and V are smooth manifolds and $\pi\colon V\longrightarrow M$ is a smooth map;*

($\mathbb{C}$VB2) *$\cdot\colon\mathbb{C}\times V\longrightarrow V$ is a map s.t. $\pi(c\cdot v)=\pi(v)$ for all $(c,v)\in\mathbb{C}\times V$;*

(ℂVB3) $+: V\times_M V\longrightarrow V$ *is a map s.t.* $\pi(v_1+v_2)=\pi(v_1)=\pi(v_2)$ *for all* $(v_1,v_2)\in V\times_M V$*;*

(ℂVB4) *for every point* $p\in M$ *there exists a neighborhood* U *of* p *in* M *and a diffeomorphism* $h\colon V|_U\longrightarrow U\times\mathbb{C}^k$ *such that*

  (ℂVB4-a) $\pi_1\circ h=\pi$ *on* $V|_U$ *and*

  (ℂVB4-b) *the map* $h|_{V_x}\colon V_x\longrightarrow x\times\mathbb{C}^k$ *is an isomorphism of complex vector spaces for all* $x\in U$*.*

Similarly to a real vector bundle, a complex vector bundle over M locally looks like bundles of $\mathbb{C}^k$'s over open sets in M glued together. If M is an m-manifold and $V\longrightarrow M$ is a complex vector bundle of rank k, then V is an $(m+2k)$-manifold. A complex vector bundle of rank k is also a real vector bundle of rank $2k$, but a real vector bundle of rank $2k$ need not in general admit a complex structure.

Example 7.7. If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1\colon M\times\mathbb{C}^k\longrightarrow M$$

is a complex vector bundle of rank k over M. It is called the trivial rank-k complex vector bundle over M and denoted $\pi\colon\tau_k^{\mathbb{C}}\longrightarrow M$ or simply $\pi\colon\tau_k\longrightarrow M$ if there is no ambiguity.

Example 7.8. Let $\mathbb{C}P^n$ be the complex projective space of dimension n described in Example 1.10 and

$$\gamma_n=\big\{(\ell,v)\in\mathbb{C}P^n\times\mathbb{C}^{n+1}\colon v\in\ell\big\}.$$

The projection $\pi\colon\gamma_n\longrightarrow\mathbb{C}P^n$ defines a smooth complex line bundle. The fiber over a point $\ell\in\mathbb{C}P^n$ is the *one-dimensional complex subspace* ℓ *of* $\mathbb{C}^{n+1}$. For this reason, γ_n is called the tautological line bundle over $\mathbb{C}P^n$.

Example 7.9. If M is a complex m-manifold, the tangent bundle TM of M is a complex vector bundle of rank m over M.

8 Sections and Homomorphisms

Definition 8.1. (1) *A (smooth)* section of *a (real or complex) vector bundle* $\pi: V\longrightarrow M$ *is a (smooth) map* $s\colon M\longrightarrow V$ *such that* $\pi\circ s=\mathrm{id}_M$*, i.e.* $s(x)\in V_x$ *for all* $x\in M$*.*

(2) *A* vector field *on a smooth manifold is a section of the tangent bundle* $TM\longrightarrow M$*.*

If $\pi: V=M\times\mathbb{R}^k\longrightarrow M$ is the trivial bundle of rank k, a section of π is a map $s: M\longrightarrow V$ of the form

$$s=(\mathrm{id}_M,f)\colon M\longrightarrow M\times\mathbb{R}^k$$

for some map $f: M\longrightarrow\mathbb{R}^k$. This section is smooth if and only if f is a smooth map. Thus, a (smooth) section of the trivial vector bundle of rank k over M is essentially a (smooth) map $M\longrightarrow\mathbb{R}^k$.

If s is a smooth section, then $s(M)$ is an embedded submanifold of V: the injectivity of s and $\mathrm{d}s$ is immediate from $\pi\circ s=\mathrm{id}_M$, while the embedding property follows from the continuity of π. Every fiber V_x of V is a vector space and thus has a distinguished element, the zero vector in V_x, which

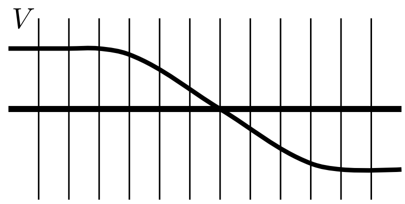


Figure 2.2: The image of a vector-bundle section is an embedded submanifold of the total space.

we denote by 0_x. It follows that every vector bundle admits a canonical section, called the zero section,

$$s_0(x) = (x, 0_x) \in V_x.$$

This section is smooth, since on a trivialization of V over an open subset U of M it is given by the inclusion of U as $U\times 0$ into $U\times\mathbb{R}^k$ or $U\times\mathbb{C}^k$. Thus, M can be thought of as sitting inside of V as the zero section, which is a deformation retract of V; see Figure 2.2.

If $s\colon M\longrightarrow V$ is a section of a vector bundle $V\longrightarrow M$ and $h\colon V|_U\longrightarrow U\times\mathbb{R}^k$ is a trivialization of V over an open subset $U\subset M$, then

$$h\circ s = (\mathrm{id}_U, s_h)\colon U \longrightarrow U\times\mathbb{R}^k \tag{8.1}$$

for some $s_h\colon U\longrightarrow\mathbb{R}^k$. Since the trivializations h cover V and each trivialization h is a diffeomorphism, a section $s\colon M\longrightarrow V$ is smooth if and only if the induced functions $s_h\colon U\longrightarrow\mathbb{R}^k$ are smooth in all trivializations $h\colon V|_U\longrightarrow U\times\mathbb{R}^k$ of V.

Every trivialization $h: V|_U\longrightarrow U\times\mathbb{R}^k$ of a vector bundle $V\longrightarrow M$ over an open subset $U\subset M$ corresponds to a k-tuple $(s_1,\ldots,s_k)$ of smooth sections of V over U such that the set $\{s_i(x)\}_i$ forms a basis for $V_x\equiv\pi^{-1}(x)$ for all $x\in U$. Let $e_1,\ldots,e_k$ be the standard coordinate vectors in $\mathbb{R}^k$. If $h\colon V|_U\longrightarrow U\times\mathbb{R}^k$ is a trivialization of V, then each section

$$s_i = h^{-1}\circ(\mathrm{id}_U, e_i)\colon U \longrightarrow V|_U, \qquad s_i(x) = h^{-1}(x, e_i),$$

is smooth. Since $\{e_i\}$ is a basis for $\mathbb{R}^k$ and $h\colon V_x\longrightarrow x\times\mathbb{R}^k$ is a vector-space isomorphism, $\{s_i(x)\}_i$ is a basis for V_x for all $x\in U$. Conversely, if $s_1,\ldots,s_k: U\longrightarrow V|_U$ are smooth sections such that $\{s_i(x)\}_i$ is a basis for V_x for all $x\in U$, then the map

$$\psi\colon U\times\mathbb{R}^k \longrightarrow V|_U, \qquad (x, c_1,\ldots,c_k) \longrightarrow c_1s_1(x)+\ldots+c_ks_k(x), \tag{8.2}$$

is a diffeomorphism commuting with the projection maps; its inverse, $h=\psi^{-1}$, is thus a trivialization of V over U. If in addition $s\colon M\longrightarrow V$ is any bundle section and

$$s_h \equiv \big(s_{h,1},\ldots,s_{h,k}\big)\colon U \longrightarrow \mathbb{R}^k$$

is as in (8.1), then

$$s(x) = h^{-1}\big(x, s_{h,1}(x),\ldots,s_{h,k}(x)\big) = s_{h,1}(x)s_1(x)+\ldots+s_{h,k}(x)s_k(x) \quad \forall\, x\in U.$$

Thus, a bundle section $s: M\longrightarrow V$ is smooth if and only if for every open subset $U\subset M$ and a k-tuple of smooth sections $s_1,\ldots,s_k: U\longrightarrow V|_U$ such that $\{s_i(x)\}_i$ is a basis for V_x for all $x\in U$ the coefficient functions

$$c_1,\ldots,c_k\colon U \longrightarrow \mathbb{R}, \qquad s(x) \equiv c_1(x)s_1(x)+\ldots+c_k(x)s_k(x) \quad \forall\, x\in U,$$

are smooth.

For example, let $\pi\colon V=TM\longrightarrow M$ be the tangent bundle of a smooth m-manifold M. If $\tilde{\varphi}_\alpha$ is a trivialization of TM over $U_\alpha\subset M$ as in (7.3),

$$s_i(x)\equiv\tilde{\varphi}_\alpha^{-1}(x,e_i)=\frac{\partial}{\partial x_i}\bigg|_x \qquad \forall\, x\in U_\alpha$$

is the i-th coordinate vector field. Thus, a vector field $X\colon M\longrightarrow TM$ is smooth if and only if for every smooth chart $\varphi_\alpha=(x_1,\ldots,x_m)\colon U_\alpha\longrightarrow\mathbb{R}^m$ the coefficient functions

$$c_1,\ldots,c_m\colon U\longrightarrow\mathbb{R}, \qquad X(p)\equiv c_1(p)\frac{\partial}{\partial x_1}\bigg|_p+\ldots+c_m(p)\frac{\partial}{\partial x_m}\bigg|_p \quad \forall\, p\in U,$$

are smooth. If $X\colon M\longrightarrow TM$ is a vector field on M and $p\in M$, sometimes it will be convenient to denote the value $X(p)\in T_pM$ of X at p by X_p. If in addition $f\in C^\infty(M)$, define

$$Xf\colon M\longrightarrow\mathbb{R} \qquad \text{by} \qquad \{Xf\}(p)=X_p(f) \qquad \forall\, p\in M.$$

A vector field X on M is smooth if and only if $Xf\in C^\infty(M)$ for every $f\in C^\infty(M)$.

The set of all smooth sections of a vector bundle $\pi\colon V\longrightarrow M$ is denoted by $\Gamma(M;V)$. This is naturally a module over the ring $C^\infty(M)$ of smooth functions on M, since $fs\in\Gamma(M;V)$ whenever $f\in C^\infty(M)$ and $s\in\Gamma(M;V)$. We will denote the set $\Gamma(M;TM)$ of smooth vector fields on M by $\mathrm{VF}(M)$. It carries a canonical structure of Lie algebra over $\mathbb{R}$, with the Lie bracket defined by

$$\begin{gathered}[\cdot,\cdot]\colon \mathrm{VF}(M)\times\mathrm{VF}(M)\longrightarrow\mathrm{VF}(M),\\ [X,Y]_p(f)=X_p(Yf)-Y_p(Xf) \qquad \forall\, p\in M,\ f\in C^\infty(U),\ U\subset M \text{ open},\ p\in U;\end{gathered} \tag{8.3}$$

see Exercise 5.

Definition 8.2. (1) *Suppose $\pi\colon V\longrightarrow M$ and $\pi'\colon V'\longrightarrow N$ are real (or complex) vector bundles. A (smooth) map $\tilde{f}\colon V\longrightarrow V'$ is a (smooth)* vector-bundle homomorphism *if $\tilde{f}$ descends to a map $f\colon M\longrightarrow N$, i.e. the diagram*

$$\begin{tikzcd} V \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & V' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{8.4}$$

commutes, and the restriction $\tilde{f}\colon V_x\longrightarrow V_{f(x)}$ is linear (or $\mathbb{C}$-linear, respectively) for all $x\in M$.

(2) *If $\pi\colon V\longrightarrow M$ and $\pi'\colon V'\longrightarrow M$ are vector bundles, a smooth vector-bundle homomorphism $\tilde{f}\colon V\longrightarrow V'$ is an* isomorphism of vector bundles *if $\pi'\circ\tilde{f}=\pi$, i.e. the diagram*

$$\begin{tikzcd} V \arrow[rr, "\tilde{f}"] \arrow[dr, "\pi"'] & & V' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd} \tag{8.5}$$

commutes, and $\tilde{f}$ is a diffeomorphism (or equivalently, its restriction to each fiber is an isomorphism of vector spaces). If such an isomorphism exists, then V and V' are said to be isomorphic vector bundles.

Let $\tilde{f}\colon V \longrightarrow V'$ be a vector-bundle homomorphism between vector bundles over the same space M that covers id_M as in (8.5). If

$$h\colon V|_U \longrightarrow U\times\mathbb{R}^k \qquad \text{and} \qquad h'\colon V'|_U \longrightarrow U\times\mathbb{R}^{k'}$$

are trivializations of V and V' over the same open subset $U\subset M$, then there exists

$$\tilde{f}_{h'h}\colon U \longrightarrow \mathrm{Mat}_{k'\times k}\mathbb{R} \qquad \text{s.t.} \qquad h'\circ\tilde{f}\circ h^{-1}(x,v) = \big(x, \tilde{f}_{h'h}(x)v\big) \quad \forall\, x\in U,\, v\in\mathbb{R}^k\,. \tag{8.6}$$

Since the trivializations h and h' are diffeomorphisms that cover V and V', respectively, a vector-bundle homomorphism as in (8.5) is smooth if and only if the induced function

$$\tilde{f}_{hh'}\colon U \longrightarrow \mathrm{Mat}_{k'\times k}\mathbb{R}$$

is smooth for every pair, $h\colon V|_U \longrightarrow U\times\mathbb{R}^k$ and $h'\colon V'|_U \longrightarrow U\times\mathbb{R}^{k'}$, of trivializations of V and V' over U.

Example 8.3. The tangent bundle $\pi\colon T\mathbb{R}^n \longrightarrow \mathbb{R}^n$ of $\mathbb{R}^n$ is canonically trivial. The map

$$T\mathbb{R}^n \longrightarrow \mathbb{R}^n\times\mathbb{R}^n\,, \qquad v \longrightarrow \big(\pi(v); v(\pi_1),\ldots,v(\pi_n)\big),$$

where $\pi_i\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ are the component projection maps, is a vector-bundle isomorphism.

Lemma 8.4. *The real line bundle $V \longrightarrow S^1$ given by the infinite Mobius band of Example 7.3 is not isomorphic to the trivial line bundle $S^1\times\mathbb{R} \longrightarrow S^1$.*

Proof: In fact, (V,S^1) is not even homeomorphic to $(S^1\times\mathbb{R},S^1)$. Since

$$S^1\times\mathbb{R} - s_0(S^1) \equiv S^1\times\mathbb{R} - S^1\times 0 = S^1\times\mathbb{R}^- \sqcup S^1\times\mathbb{R}^+,$$

the space $S^1\times\mathbb{R} - S^1$ is not connected. On the other hand, $V - s_0(S^1)$ is connected. If MB is the standard Mobius Band and $S^1\subset MB$ is the central circle, $MB - S^1$ is a deformation retract of $V - S^1$. On the other hand, the boundary of MB has only one connected component (this is the primary feature of MB) and is a deformation retract of $MB - S^1$. Thus, $V - S^1$ is connected as well.

Lemma 8.5. *If $\pi\colon V \longrightarrow M$ is a real (or complex) vector bundle of rank k, V is isomorphic to the trivial real (or complex) vector bundle of rank k over M if and only if V admits k sections $s_1,\ldots,s_k$ such that the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent over $\mathbb{R}$ (or over $\mathbb{C}$, respectively) in V_x for all $x\in M$.*

Proof: We consider the real case; the proof in the complex case is nearly identical.
(1) Suppose $\psi\colon M\times\mathbb{R}^k \longrightarrow V$ is an isomorphism of vector bundles over M. Let $e_1,\ldots,e_k$ be the standard coordinate vectors in $\mathbb{R}^k$. Define sections $s_1,\ldots,s_k$ of V over M by

$$s_i(x) = \psi\big(x, e_i\big) \qquad \forall\, i = 1,\ldots,k,\; x\in M.$$

Since the maps $x \longrightarrow (x,e_i)$ are sections of $M\times\mathbb{R}^k$ over M and ψ is a bundle homomorphism, the maps s_i are sections of V. Since the vectors (x,e_i) are linearly independent in $x\times\mathbb{R}^k$ and ψ is an isomorphism on every fiber, the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x for all $x\in M$, as needed.

(2) Suppose $s_1,\ldots,s_k$ are sections of V such that the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x for all $x\in M$. Define the map

$$\psi\colon M\times\mathbb{R}^k \longrightarrow V \qquad \text{by} \qquad \psi(x,c_1,\ldots,c_k)=c_1s_1(x)+\ldots+c_ks_k(x)\in V_x.$$

Since the sections $s_1,\ldots,s_k$ and the vector space operations on V are smooth, the map h is smooth. It is immediate that $\pi(\psi(x,c))=x$ and the restriction of ψ to $x\times\mathbb{R}^k$ is linear; thus, ψ is a vector-bundle homomorphism. Since the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x, the homomorphism ψ is injective and thus an isomorphism on every fiber. We conclude that ψ is an isomorphism between vector bundles over M.

9 Transition Data

Suppose $\pi\colon V\longrightarrow M$ is a real vector bundle of rank k. By Definition 7.1, there exists a collection $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for V such that $\bigcup_{\alpha\in\mathcal{A}}U_\alpha=M$. Since (U_α,h_α) is a trivialization for V,

$$h_\alpha\colon V|_{U_\alpha}\longrightarrow U_\alpha\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_\alpha=\pi$ and the restriction $h_\alpha\colon V_x\longrightarrow x\times\mathbb{R}^k$ is linear for all $x\in U_\alpha$. Thus, for all $\alpha,\beta\in\mathcal{A}$,

$$h_{\alpha\beta}\equiv h_\alpha\circ h_\beta^{-1}\colon \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k\longrightarrow\big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_{\alpha\beta}=\pi_1$, i.e. $h_{\alpha\beta}$ maps $x\times\mathbb{R}^k$ to $x\times\mathbb{R}^k$, and the restriction of $h_{\alpha\beta}$ to $x\times\mathbb{R}^k$ defines an isomorphism of $x\times\mathbb{R}^k$ with itself. Such a diffeomorphism must be given by

$$(x,v)\longrightarrow\big(x,g_{\alpha\beta}(x)v\big)\qquad\forall\,v\in\mathbb{R}^k,$$

for a unique element $g_{\alpha\beta}(x)\in\mathrm{GL}_k\mathbb{R}$ (the general linear group of $\mathbb{R}^k$). The map $h_{\alpha\beta}$ is then given by

$$h_{\alpha\beta}(x,v)=\big(x,g_{\alpha\beta}(x)v\big)\qquad\forall\,x\in U_\alpha\cap U_\beta,\ v\in\mathbb{R}^k,$$

and is completely determined by the map $g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}$ (and $g_{\alpha\beta}$ is determined by $h_{\alpha\beta}$). Since $h_{\alpha\beta}$ is smooth, so is $g_{\alpha\beta}$.

Example 9.1. Let $\pi\colon V\longrightarrow S^1$ be the Mobius band line bundle of Example 7.3. If $\{(U_\pm,h_\pm)\}$ is the pair of trivializations described in Example 7.3, then

$$h_-\circ h_+^{-1}\colon U_+\cap U_-\times\mathbb{R}\longrightarrow U_+\cap U_-\times\mathbb{R},\quad (p,v)\longrightarrow\big(p,g_{-+}(p)v\big)=\begin{cases}(p,v), & \text{if } \mathrm{Im}\,p<0,\\ (p,-v), & \text{if } \mathrm{Im}\,p>0,\end{cases}$$

$$\text{where}\qquad g_{-+}\colon U_+\cap U_-=S^1-\{\pm1\}\longrightarrow\mathrm{GL}_1\mathbb{R}=\mathbb{R}^*,\quad g_{-+}(p)=\begin{cases}-1, & \text{if } \mathrm{Im}\,p>0;\\ 1, & \text{if } \mathrm{Im}\,p<0.\end{cases}$$

In this case, the transition maps $g_{\alpha\beta}$ are locally constant, which is rarely the case.

Suppose $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations of a rank k vector bundle $\pi: V\longrightarrow M$ covering M. Any (smooth) section $s: M\longrightarrow V$ of π determines a collection of (smooth) maps $\{s_\alpha\colon U_\alpha\longrightarrow\mathbb{R}^k\}_{\alpha\in\mathcal{A}}$ such that

$$h_\alpha\circ s(x)=\big(x,s_\alpha(x)\big)\quad\forall\,x\in U_\alpha\quad\Longrightarrow\quad s_\alpha(x)=g_{\alpha\beta}(x)s_\beta(x)\quad\forall\,x\in U_\alpha\cap U_\beta\,,\alpha,\beta\in\mathcal{A},\tag{9.1}$$

where $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ is the transition data for the collection of trivializations $\{h_\alpha\}_{\alpha\in\mathcal{A}}$ of V. Conversely, a collection of (smooth) maps $\{s_\alpha\colon U_\alpha\longrightarrow\mathbb{R}^k\}_{\alpha\in\mathcal{A}}$ satisfying the second condition in (9.1) induces a well-defined (smooth) section of π via the first equation in (9.1). Similarly, suppose $\{(U_\alpha,h'_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations of a rank k' vector bundle $\pi'\colon V'\longrightarrow M$ covering M. A (smooth) vector-bundle homomorphism $\tilde{f}: V\longrightarrow V'$ covering id_M as in (8.5) determines a collection of (smooth) maps

$$\{\tilde{f}_\alpha\colon U_\alpha\longrightarrow\mathrm{Mat}_{k'\times k}\mathbb{R}\}_{\alpha\in\mathcal{A}} \quad \text{s.t.} \quad h'_\alpha\circ\tilde{f}\circ h_\alpha^{-1}(x,v)=\big(x,\tilde{f}_\alpha(x)v\big) \quad \forall\,(x,v)\in U_\alpha\times\mathbb{R}^k \tag{9.2}$$

$$\Longrightarrow \qquad \tilde{f}_\alpha(x)g_{\alpha\beta}(x)=g'_{\alpha\beta}(x)\tilde{f}_\beta(x) \quad x\in U_\alpha\cap U_\beta\,,\alpha,\beta\in\mathcal{A}, \tag{9.3}$$

where $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ is the transition data for the collection of trivializations $\{h'_\alpha\}_{\alpha\in\mathcal{A}}$ of V'. Conversely, a collection of (smooth) maps as in (9.2) satisfying (9.3) induces a well-defined (smooth) vector-bundle homomorphism $\tilde{f}\colon V\longrightarrow V'$ covering id_M as in (8.5) via the equation in (9.2).

By the above, starting with a real rank k vector bundle $\pi\colon V\longrightarrow M$, we can obtain an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth transition maps

$$\big\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}.$$

These transition maps satisfy:

(VBT1) $g_{\alpha\alpha}\equiv\mathbb{I}_k$, since $h_{\alpha\alpha}\equiv h_\alpha\circ h_\alpha^{-1}=\mathrm{id}$;

(VBT2) $g_{\alpha\beta}g_{\beta\alpha}\equiv\mathbb{I}_k$, since $h_{\alpha\beta}\circ h_{\beta\alpha}=\mathrm{id}$;

(VBT3) $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\equiv\mathbb{I}_k$, since $h_{\alpha\beta}\circ h_{\beta\gamma}\circ h_{\gamma\alpha}=\mathrm{id}$.

The last condition is known as the (Čech) cocycle condition (more details in Chapter 5 of Warner). It is sometimes written as

$$g_{\alpha_1\alpha_2}g_{\alpha_0\alpha_2}^{-1}g_{\alpha_0\alpha_1}\equiv\mathbb{I}_k \qquad \forall\,\alpha_0,\alpha_1,\alpha_2\in\mathcal{A}.$$

In light of (VBT2), the two versions of the cocycle condition are equivalent.

Conversely, given an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\big\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfy (VBT1)-(VBT3), we can assemble a rank k vector bundle $\pi'\colon V'\longrightarrow M$ as follows. Let

$$V'=\Big(\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times U_\alpha\times\mathbb{R}^k\Big)\Big/\sim, \qquad \text{where}$$
$$(\beta,x,v)\sim\big(\alpha,x,g_{\alpha\beta}(x)v\big) \quad \forall\ \alpha,\beta\in\mathcal{A},\ x\in U_\alpha\cap U_\beta,\ v\in\mathbb{R}^k.$$

The relation $\sim$ is reflexive by (VBT1), symmetric by (VBT2), and transitive by (VBT3) and (VBT2). Thus, $\sim$ is an equivalence relation, and V' carries the quotient topology. Let

$$q\colon\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times U_\alpha\times\mathbb{R}^k\longrightarrow V' \qquad \text{and} \qquad \pi'\colon V'\longrightarrow M, \quad [\alpha,x,v]\longrightarrow x,$$

be the quotient map and the natural projection map (which is well-defined). If $\beta\in\mathcal{A}$ and W is a subset of $U_\beta\times\mathbb{R}^k$, then

$$q^{-1}\big(q(\beta\times W)\big)=\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times h_{\alpha\beta}(W),\qquad\text{where}$$
$$h_{\alpha\beta}\colon \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k\longrightarrow\big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k,\qquad h_{\alpha\beta}(x,v)=\big(x,g_{\alpha\beta}(x)v\big).$$

In particular, if $\beta\times W$ is an open subset of $\beta\times U_\beta\times\mathbb{R}^k$, then $q^{-1}\big(q(\beta\times W)\big)$ is an open subset of $\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times U_\alpha\times\mathbb{R}^k$. Thus, q is an open continuous map. Since its restriction

$$q_\alpha\equiv q|_{\alpha\times U_\alpha\times\mathbb{R}^k}$$

is injective, $(q_\alpha(\alpha\times U_\alpha\times\mathbb{R}^k),q_\alpha^{-1})$ is a smooth chart on V' in the sense of Lemma 2.6. The overlap maps between these charts are the maps $h_{\alpha\beta}$ and thus smooth.[1] Thus, by Lemma 2.6, these charts induce a smooth structure on V'. The projection map $\pi'\colon V'\longrightarrow M$ is smooth with respect to this smooth structure, since it induces projection maps on the charts. Since

$$\pi_1=\pi'\circ q_\alpha\colon\alpha\times U_\alpha\times\mathbb{R}^k\longrightarrow U_\alpha\subset M,$$

the diffeomorphism q_α induces a vector-space structure in V'_x for each $x\in U_\alpha$ such that the restriction of q_α to each fiber is a vector-space isomorphism. Since the restriction of the overlap map $h_{\alpha\beta}$ to $x\times\mathbb{R}^k$, with $x\in U_\alpha\cap U_\beta$, is a vector-space isomorphism, the vector space structures defined on V'_x via the maps q_α and q_β are the same. We conclude that $\pi'\colon V'\longrightarrow M$ is a real vector bundle of rank k.

If $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\big\{g_{\alpha\beta}: U_\alpha\cap U_\beta\longrightarrow \mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$ are transition data arising from a vector bundle $\pi\colon V\longrightarrow M$, then the vector bundle V' constructed in the previous paragraph is isomorphic to V. Let $\{(U_\alpha,h_\alpha)\}$ be the trivializations as above, giving rise to the transition functions $g_{\alpha\beta}$. We define

$$\tilde f\colon V\longrightarrow V'\qquad\text{by}\qquad \tilde f(v)=\big[\alpha,h_\alpha(v)\big]\quad\text{if}\quad \pi(v)\in U_\alpha.$$

If $\pi(v)\in U_\alpha\cap U_\beta$, then

$$\big[\beta,h_\beta(v)\big]=\big[\alpha,h_{\alpha\beta}(h_\beta(v))\big]=\big[\alpha,h_\alpha(v)\big]\in V',$$

i.e. the map $\tilde f$ is well-defined (depends only on v and not on α). It is immediate that $\pi'\circ\tilde f=\pi$. Since the map

$$q_\alpha^{-1}\circ\tilde f\circ h_\alpha^{-1}\colon U_\alpha\times\mathbb{R}^k\longrightarrow\alpha\times U_\alpha\times\mathbb{R}^k$$

is the identity (and thus smooth), $\tilde f$ is a smooth map. Since the restrictions of q_α and h_α to every fiber are vector-space isomorphisms, it follows that so is $\tilde f$. We conclude that $\tilde f$ is a vector-bundle isomorphism.

In summary, a real rank k vector bundle over M determines a set of transition data with values in $\mathrm{GL}_k\mathbb{R}$ satisfying (VBT1)-(VBT3) above (many such sets, of course) and a set of transition data satisfying (VBT1)-(VBT3) determines a real rank-k vector bundle over M. These two processes are well-defined and are inverses of each other when applied to the set of equivalence classes of vector bundles and the set of equivalence classes of transition data satisfying (VBT1)-(VBT3).

[1] Formally, the overlap map is $(\beta\longrightarrow\alpha)\times h_{\alpha\beta}$.

Two vector bundles over M are defined to be equivalent if they are isomorphic as vector bundles over M. Two sets of transition data

$$\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}} \qquad \text{and} \qquad \{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}},$$

with $\mathcal{A}$ consisting of *all* sufficiently small open subsets of M, are said to be equivalent if there exists a collection of smooth functions $\{f_\alpha\colon U_\alpha\longrightarrow \mathrm{GL}_k\mathbb{R}\}_{\alpha\in\mathcal{A}}$ such that

$$g'_{\alpha\beta}=f_\alpha g_{\alpha\beta}f_\beta^{-1}, \qquad \forall\alpha,\beta\in\mathcal{A},$$ [2]

i.e. the two sets of transition data differ by the action of a Čech 0-chain (more in Chapter 5 of Warner). Along with the cocycle condition on the gluing data, this means that isomorphism classes of real rank k vector bundles over M can be identified with $\check{H}^1(M;\mathrm{GL}_k\mathbb{R})$, the quotient of the space of Čech cocycles of degree one by the subspace of Čech boundaries.

Remark 9.2. In Chapter 5 of Warner, Čech cohomology groups, $\check{H}^m$, are defined for (sheafs of) abelian groups. However, the first two groups, $\check{H}^0$ and $\check{H}^1$, generalize to non-abelian groups as well.

If $\pi\colon V\longrightarrow M$ is a complex rank k vector bundle over M, we can similarly obtain transition data for V consisting of an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow \mathrm{GL}_k\mathbb{C}\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfies (VBT1)-(VBT3). Conversely, given such transition data, we can construct a complex rank k vector bundle over M. The set of isomorphism classes of complex rank k vector bundles over M can be identified with $\check{H}^1(M;\mathrm{GL}_k\mathbb{C})$.

10 Operations on Vector Bundles

Vector bundles can be restricted to smooth submanifolds and pulled back by smooth maps. All natural operations on vector spaces, such as taking quotient vector space, dual vector space, direct sum of vector spaces, tensor product of vector spaces, and exterior powers also carry over to vector bundles via transition functions.

Restrictions and pullbacks

If N is a smooth manifold, $M\subset N$ is an embedded submanifold, and $\pi\colon V\longrightarrow N$ is a vector bundle of rank k (real or complex) over N, then its restriction to M,

$$\pi\colon V|_M\equiv\pi^{-1}(M)\longrightarrow M,$$

is a vector bundle of rank k over M. It inherits a smooth structure from V by the *Slice Lemma* (Proposition 5.3) or the *Implicit Function Theorem for Manifolds* (Theorem 6.3). If $\{(U_\alpha,h_\alpha)\}$ is a collection of trivializations for $V\longrightarrow N$, then $\{(M\cap U_\alpha, h_\alpha|_{\pi^{-1}(M\cap U_\alpha)})\}$ is a collection of trivializations for $V|_M\longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V\longrightarrow N$, then $\{g_{\alpha\beta}|_{M\cap U_\alpha\cap U_\beta}\}$ is

[2]According to the discussion around (9.3), such a collection $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ corresponds, via trivializations, to an isomorphism between the vector bundles determined by $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ and $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$.

transition data for $V|_M \longrightarrow M$.

If $f: M \longrightarrow N$ is a smooth map and $\pi: V \longrightarrow N$ is a vector bundle of rank k, there is a pullback bundle over M:

$$f^*V \equiv M \times_N V \equiv \big\{(p,v) \in M \times V : f(p) = \pi(v)\big\} \xrightarrow{\pi_1} M. \tag{10.1}$$

Note that f^*V is the maximal subspace of $M \times V$ so that the diagram

$$\begin{tikzcd} f^*V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & V \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. By the *Implicit Function Theorem for Maps* (Corollary 6.7), f^*V is a smooth submanifold of $M \times V$. By construction, the fiber of π_1 over $p \in M$ is $p \times V_{f(p)} \subset M \times V$, i.e. the fiber of π over $f(p) \in N$:

$$\big(f^*V\big)_p = p \times V_{f(p)} \qquad \forall\, p \in M. \tag{10.2}$$

If $\{(U_\alpha, h_\alpha)\}$ is a collection of trivializations for $V \longrightarrow N$, then $\{(f^{-1}(U_\alpha), h_\alpha \circ f)\}$ is a collection of trivializations for $f^*V \longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V \longrightarrow N$, then $\{g_{\alpha\beta} \circ f\}$ is transition data for $f^*V \longrightarrow M$. The case discussed in the previous paragraph corresponds to f being the inclusion map.

Lemma 10.1. *If $\tilde{f}: V \longrightarrow V'$ is a vector-bundle homomorphism covering a smooth map $f: M \longrightarrow N$ as in (8.4), there exists a bundle homomorphism $\phi: V \longrightarrow f^*V'$ so that the diagram*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[rr, bend left, "\tilde{f}"] \arrow[dr, "\pi"'] & f^*V' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & V' \arrow[d, "\pi'"] \\ & M \arrow[r, "f"] & N \end{tikzcd}$$

commutes.

The map ϕ is defined by

$$\phi: V \longrightarrow M \times V', \qquad \phi(v) = \big(\pi(v), \tilde{f}(v)\big).$$

Since $f \circ \pi = \pi' \circ \tilde{f}$,

$$\phi(v) \in f^*V' \equiv M \times_N V' \equiv \big\{(p, v') \in M \times V' : f(p) = \pi'(v')\big\}.$$

Since $f^*V' \subset M \times V'$ is a smooth embedded submanifold, the map $\phi: V \longrightarrow f^*V'$ obtained by restricting the range is smooth; see Proposition 5.5. The above diagram commutes by the construction of ϕ. Since $\tilde{f}$ is linear on each fiber of V, so is ϕ.

If $f: M \longrightarrow N$ is a smooth map, then $\mathrm{d}_p f: T_pM \longrightarrow T_{f(p)}N$ is a linear map which varies smoothly with p. It thus gives rises to a smooth map,

$$\mathrm{d}f: TM \longrightarrow TN, \qquad v \longrightarrow \mathrm{d}_{\pi(v)} f(v). \tag{10.3}$$

However, this description of $\mathrm{d}f$ gives no indication that $\mathrm{d}f$ maps $v \in T_pM$ to $T_{f(p)}N$ or that this map is linear on each T_pM. One way to fix this defect is to state that (10.3) is a bundle homomorphism covering the map $f\colon M \longrightarrow N$, i.e. that the diagram

$$\begin{tikzcd} TM \arrow[r, "\mathrm{d}f"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{10.4}$$

commutes. By Lemma 10.1, $\mathrm{d}f$ then induces a vector-bundle homomorphism from TM to f^*TN so that the diagram

$$\begin{tikzcd} TM \arrow[rr, "\mathrm{d}f"] \arrow[dr, "\pi"'] & & f^*TN \arrow[dl, "\pi_1"] \arrow[r, dashed, "\pi_2"] & TN \arrow[d, dashed, "\pi'"] \\ & M \arrow[rr, dashed, "f"] & & N \end{tikzcd} \tag{10.5}$$

commutes. The triangular part of (10.5) is generally the preferred way of describing $\mathrm{d}f$. The description (10.4) factors through the triangular part of (10.5), as indicated by the dashed arrows. The triangular part of (10.5) also leads to a more precise statement of the *Implicit Function Theorem*, which is rather useful in topology of manifolds; see Theorem 10.11 below.

If $\pi\colon V \longrightarrow N$ is a smooth vector bundle, $f\colon M \longrightarrow N$ is a smooth map, and $s\colon N \longrightarrow V$ is a bundle section of V, then

$$f^*s\colon M \longrightarrow f^*V, \qquad \{f^*s\}(p) = \big(p, s(f(p))\big) \in f^*V \equiv M\times_N V \subset M\times V,$$

is a bundle section of $f^*V \longrightarrow M$. If s is smooth, then $f^*s\colon M \longrightarrow M\times V$ is a smooth map with the image in $M\times_N V$. Since $M\times_N V \subset M\times V$ is an embedded submanifold, $f^*s\colon M \longrightarrow f^*V$ is a smooth map by Proposition 5.5. Thus, a smooth map $f\colon M \longrightarrow N$ induces a homomorphism of vector spaces

$$f^*\colon \Gamma(N;V) \longrightarrow \Gamma(M;f^*V), \qquad s \longrightarrow f^*s, \tag{10.6}$$

which is also a homomorphism of modules with respect to the ring homomorphism

$$f^*\colon C^\infty(N) \longrightarrow C^\infty(M), \qquad g \longrightarrow g\circ f\,.$$

In the case of tangent bundles, the homomorphism (10.6) is compatible with the Lie algebra structures on the spaces of vector fields, as described by the following lemma.

Lemma 10.2. *Let $f: M \longrightarrow N$ be a smooth map. If $X_1, X_2 \in \mathrm{VF}(M)$ and $Y_1, Y_2 \in \mathrm{VF}(N)$ are smooth vector fields on M and N, respectively, such that $\mathrm{d}f(X_i) = f^*Y_i \in \Gamma(M; f^*TN)$ for $i=1,2$, then*

$$\mathrm{d}f\big([X_1,X_2]\big) = f^*[Y_1,Y_2].$$

This is checked directly from the relevant definitions.

The pullback operation on vector bundles also extends to homomorphisms. Let $f : M \longrightarrow N$ be a smooth map and $\pi_V : V \longrightarrow N$ and $\pi_W : W \longrightarrow N$ be vector bundles. Any vector-bundle homomorphism $\varphi\colon V \longrightarrow W$ over N induces a vector-bundle homomorphism $f^*\varphi\colon f^*V \longrightarrow f^*W$ over M so that the diagram

$$\begin{tikzcd}
f^*V \arrow[rrr, "\pi_2"] \arrow[drr, "f^*\varphi"] \arrow[ddr, "\pi_1"'] & & & V \arrow[dr, "\varphi"] \arrow[dd, "\pi_V"'] & \\
& & f^*W \arrow[rr, "\pi_2"] \arrow[dl, "\pi_1"'] & & W \arrow[dl, "\pi_W"] \\
& M \arrow[rr, "f"] & & N &
\end{tikzcd} \tag{10.7}$$

commutes. The vector-bundle homomorphism $f^*\varphi$ is given by

$$(f^*\varphi)_p = \mathrm{id}\times\varphi_{f(p)}\colon (f^*V)_p = p\times V_{f(p)} \longrightarrow (f^*W)_p = p\times W_{f(p)}, \qquad (p,v) \longrightarrow \big(p, \varphi_p(v)\big),$$

where φ_p is the restriction of φ to the fiber $V_{f(p)} = \pi_V^{-1}(f(p))$ over $f(p)\in N$.

Subbundles and foliations by submanifolds

Definition 10.3. *Let M be a smooth manifold.*

(1) A rank k' subbundle of a vector bundle *$\pi\colon V \longrightarrow M$ is a smooth submanifold V' of V such that $\pi|_{V'}\colon V' \longrightarrow M$ is a vector bundle of rank k'.*

(2) A rank k distribution on *M is a rank k subbundle of $TM \longrightarrow M$.*

A subbundle of course cannot have a larger rank than the ambient bundle; so $\mathrm{rk}\, V' \le \mathrm{rk}\, V$ in Definition 10.3 and the equality holds if and only if $V' = V$. By Exercise 17, the requirement that $\pi|_{V'}\colon V' \longrightarrow M$ is a vector bundle of rank k' can be replaced by the condition that $V'_p \equiv V_p \cap V'$ is a k'-dimensional linear subspace of V_p for all $p\in M$.

If $f: M \longrightarrow N$ is an immersion, the bundle homomorphism $\mathrm{d}f$ as in (10.5) is injective and the image of $\mathrm{d}f$ in f^*TN is a subbundle of f^*TN. In the case $M\subset N$ is an embedded submanifold and f is the inclusion map, we identify TM with the image of $\mathrm{d}\iota$ in $f^*TN = TN|_M$. By Lemma 10.2, if $Y_1, Y_2 \in \mathrm{VF}(N)$ are smooth vector fields on N, then

$$Y_1\big|_M, Y_2\big|_M \in \mathrm{VF}(M) \subset \Gamma(M; TN|_M) \quad\Longrightarrow\quad [Y_1, Y_2]\big|_M \in \mathrm{VF}(M) \subset \Gamma(M; TN|_M).$$

Definition 10.4. *Let N be a smooth manifold.*

(1) A collection $\{\iota_\alpha : M_\alpha \longrightarrow N\}_{\alpha\in\mathcal{A}}$ of injective immersions from m-manifolds is a foliation of *N^n if the collection $\{\mathrm{Im}\,\iota_\alpha\}_{\alpha\in\mathcal{A}}$ covers N and for every $q\in N$ there exists a smooth chart $\psi\colon V \longrightarrow \mathbb{R}^m\times\mathbb{R}^{n-m}$ around q such that the image under ι_α of every connected subset $U\subset \iota_\alpha^{-1}(V)$ under ψ is contained in $\psi^{-1}(\mathbb{R}^m\times y)$ for some $y\in\mathbb{R}^{n-m}$ (dependent on U).*

(2) A foliation $\{\iota_\alpha : M_\alpha \longrightarrow N\}_{\alpha\in\mathcal{A}}$ of N is proper *if ι_α is an embedding and the images of ι_α partition N (their union covers M and any two of them are either disjoint or the same).*

Thus, a foliation of N consists of regular immersions that cover N and are regular in a systematic way (all of them correspond to horizontal slices in a single coordinate chart); see Figure 2.3. Since manifolds are second-countable and the subset $\iota_\alpha^{-1}(V)\subset M_\alpha$ in Definition 10.4 is open, $\iota_\alpha(\iota_\alpha^{-1}(V))$

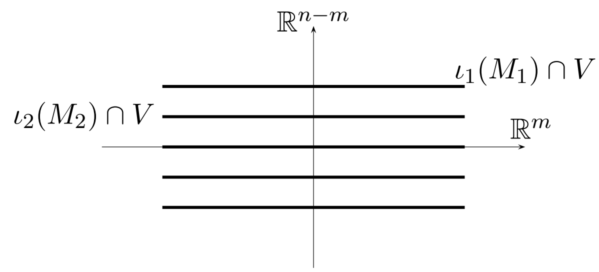


Figure 2.3: A foliation of N in a smooth chart V.

is contained in at most countably many of the horizontal slices $\psi^{-1}(\mathbb{R}^m\times y)$. The images of $\mathrm{d}\iota_\alpha$ in TN determine a rank m distribution $\mathcal{D}$ on N. By Lemma 10.2, if $Y_1,Y_2\in\mathrm{VF}(N)$ are vector fields on N, then

$$Y_1,Y_2\in\Gamma(N;\mathcal{D})\subset\mathrm{VF}(N)\quad\Longrightarrow\quad[Y_1,Y_2]\in\Gamma(N;\mathcal{D})\subset\mathrm{VF}(N).\tag{10.8}$$

Definition 10.5. *Let $\mathcal{D}\subset TN$ be a distribution on a smooth manifold N. An injective immersion $\iota\colon M\longrightarrow N$ is integral for $\mathcal{D}$ if*

$$\mathrm{Im}\,\mathrm{d}_p\iota=\mathcal{D}_{\iota(p)}\subset T_{\iota(p)}N\qquad\forall\,p\in M.$$

If $\iota\colon M\longrightarrow N$ is an integrable injective immersion for a distribution $\mathcal{D}$ on N, then in particular

$$\dim M=\mathrm{rk}\,\mathcal{D}.$$

If N admits a foliation $\{\iota_\alpha\colon M_\alpha\longrightarrow N\}_{\alpha\in\mathcal{A}}$ by injective immersions integral to a distribution $\mathcal{D}$ on N, then $\Gamma(N;\mathcal{D})\subset\mathrm{VF}(N)$ is a Lie subalgebra. By *Frobenius Theorem*, the converse is also true.

Example 10.6. The collection of embeddings

$$\iota_\alpha\colon\mathbb{R}^m\longrightarrow\mathbb{R}^n=\mathbb{R}^m\times\mathbb{R}^{n-m},\quad\iota_\alpha(x)=(x,\alpha),\qquad\alpha\in\mathbb{R}^{n-m},$$

is a proper foliation of $\mathbb{R}^n$ by m-manifolds. The corresponding distribution $\mathcal{D}\subset T\mathbb{R}^n$ is described by

$$\mathcal{D}=\mathbb{R}^n\times(\mathbb{R}^m\times 0)\subset\mathbb{R}^n\times\mathbb{R}^n=T\mathbb{R}^n.$$

Example 10.7. The collection of embeddings

$$\iota_\alpha\colon S^1\longrightarrow S^{2n+1}\subset\mathbb{C}^{n+1},\quad\iota_\alpha(\mathrm{e}^{\mathrm{i}\theta})=\mathrm{e}^{\mathrm{i}\theta}\alpha,\qquad\alpha\in S^{2n+1},$$

is a proper foliation of S^{2n+1} by circles. The corresponding distribution $\mathcal{D}\subset TS^{2n+1}$ is described by

$$\mathcal{D}=\big\{(p,\mathrm{i}rp)\colon\,p\in S^{2n+1},\,r\in\mathbb{R}\big\}\subset TS^{2n+1}\subset T\mathbb{C}^{n+1}\big|_{S^{2n+1}}=S^{2n+1}\times\mathbb{C}^{n+1}.$$

The embedded submanifolds of this foliations are the fibers of the quotient projection map

$$\pi\colon S^{2n+1}\longrightarrow S^{2n+1}/S^1=\mathbb{C}P^n$$

of Example 1.10. This is an S^1-bundle over $\mathbb{C}P^n$. In general, the fibers of the projection map $\pi\colon N\longrightarrow B$ of any smooth fiber bundle form a proper foliation of the total space N of the bundle. The corresponding distribution $\mathcal{D}\subset TN$ is then the vertical tangent bundle of π:

$$\mathcal{D}_p=\ker\mathrm{d}_p\pi\subset T_pN\qquad\forall\,p\in N.$$

Example 10.8. An example of a foliation, which is not proper, is provided by the skew lines on the torus of the same irrational slope η:

$$\iota_\alpha\colon \mathbb{R} \longrightarrow S^1 \times S^1, \quad \iota_\alpha(s) = \big(\alpha \mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}\eta s}\big), \qquad \alpha \in S^1 \subset \mathbb{C}.$$

If $\eta \in \mathbb{Q}$, this foliation is proper. In either case, the corresponding distribution $\mathcal{D}$ on $S^1 \times S^1$ is described by

$$\mathcal{D}_{(\mathrm{e}^{\mathrm{i}t_1}, \mathrm{e}^{\mathrm{i}t_2})} = \mathrm{d}_{(t_1,t_2)} q\big(\{(r, \eta r) \in \mathbb{R}^2 = T_{(t_1,t_2)}\mathbb{R}^2 \colon\ r \in \mathbb{R}\}\big),$$

where $q\colon \mathbb{R}^2 \longrightarrow S^1 \times S^1$ the usual covering map.

Quotient and normal bundles

If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and $V' \subset V$ is a linear subspace, then we can form the quotient vector space, V/V'. If W is another vector space, $W' \subset W$ is a linear subspace, and $g\colon V \longrightarrow W$ is a linear map such that $g(V') \subset W'$, then g descends to a linear map between the quotient spaces:

$$\bar{g}\colon V/V' \longrightarrow W/W'.$$

If we choose bases for V and W such that the first few vectors in each basis form bases for V' and W', then the matrix for g with respect to these bases is of the form:

$$g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

The matrix for $\bar{g}$ is then D. If g is an isomorphism from V to W that restricts to an isomorphism from V' to W', then $\bar{g}$ is an isomorphism from V/V' to W/W'. Any vector-space homomorphism $\varphi\colon V \longrightarrow W$ such that $V' \subset \ker \varphi$ descends to a homomorphism $\bar{\varphi}$ so that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "q"'] & W \\ V/V' \arrow[ur, dashed, "\bar{\varphi}"'] & \end{tikzcd}$$

commutes.

If $V' \subset V$ is a subbundle, we can form a quotient bundle, $V/V' \longrightarrow M$, such that

$$(V/V')_p = V_p/V'_p \qquad \forall\, p \in M.$$

The topology on V/V' is the quotient topology for the natural surjective map $q : V \longrightarrow V/V'$. The vector-bundle structure on V/V' is determined from those of V and V' by requiring that q be a smooth vector-bundle homomorphism. Thus, if s is a smooth section of V, then $q \circ s$ is a smooth section of V/V'; so, there is a homomorphism

$$\Gamma(M; V) \longrightarrow \Gamma(M; V/V'), \qquad s \longrightarrow q \circ s,$$

of $C^\infty(M)$-modules. There is also a short exact sequence[3] of vector bundles over M,

$$0 \longrightarrow V' \longrightarrow V \stackrel{q}{\longrightarrow} V/V' \longrightarrow 0,$$

[3] exact means that at each position the kernel of the outgoing vector-bundle homomorphism equals the image of the incoming one; short means that it consists of five terms with zeros (rank 0 vector bundles) at the ends

where the zeros denote the zero vector bundle $M\times 0 \longrightarrow M$. We can choose a system of trivializations $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ of V such that

$$h_\alpha\big(V'|_{U_\alpha}\big) = U_\alpha \times (\mathbb{R}^{k'}\times 0) \subset U_\alpha\times\mathbb{R}^k \qquad \forall\,\alpha\in\mathcal{A}. \tag{10.9}$$

Let $q_{k'}: \mathbb{R}^k \longrightarrow \mathbb{R}^{k-k'}$ be the projection onto the last $(k-k')$ coordinates. The trivializations for V/V' are then given by $\{(U_\alpha, \{\mathrm{id}\times q_{k'}\}\circ h_\alpha)\}$. Alternatively, if $\{g_{\alpha\beta}\}$ is transition data for V such that the upper-left $k'\times k'$-submatrices of $g_{\alpha\beta}$ correspond to V' (as is the case for the above trivializations h_α) and $\bar{g}_{\alpha\beta}$ is the lower-right $(k-k')\times(k-k')$ matrix of $g_{\alpha\beta}$, then $\{\bar{g}_{\alpha\beta}\}$ is transition data for V/V'. Any vector-bundle homomorphism $\varphi: V \longrightarrow W$ over M such that $\varphi(v)=0$ for all $v\in V'$ descends to a vector-bundle homomorphism $\bar{\varphi}$ so that $\varphi=\bar{\varphi}\circ q$. We leave proofs of the following lemmas as an exercise.

Lemma 10.9. *If $f: M\longrightarrow N$ is a smooth map and $W, W'\longrightarrow N$ are smooth vector bundles,*

$$f^*\big(W/W'\big) \approx (f^*W)/(f^*W')$$

as vector bundles over M.

Lemma 10.10. *Let $V\longrightarrow M$ and $W\longrightarrow N$ be vector bundles over smooth manifolds and $f: M\longrightarrow N$ a smooth map. A vector-bundle homomorphism $\tilde{f}: V\longrightarrow W$ covering f as in (8.4) and vanishing on a subbundle $V'\subset V$ induces a vector-bundle homomorphism*

$$\bar{f}: V/V' \longrightarrow W$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde{f}$ is smooth.

If $\iota: Y\longrightarrow N$ is an immersion, the image of $\mathrm{d}\iota$ in ι^*TN is a subbundle of ι^*TN. In this case, the quotient bundle,

$$\mathcal{N}_N\iota \equiv \iota^*TN\big/\mathrm{Im}\,\mathrm{d}\iota \longrightarrow Y,$$

is called the normal bundle for the immersion ι. If Y is an embedded submanifold and ι is the inclusion map, TY is a subbundle of $\iota^*TN = TN|_Y$ and the quotient subbundle,

$$\mathcal{N}_NY \equiv \mathcal{N}_N\iota = \iota^*TN\big/\mathrm{Im}\,\mathrm{d}\iota = TN|_Y\big/TY \longrightarrow Y,$$

is called the normal bundle of Y in N; its rank is the codimension of Y in N. If $f: M\longrightarrow N$ is a smooth map and $X\subset M$ is an embedded submanifold, the vector-bundle homomorphism $\mathrm{d}f$ in (10.5) restricts (pulls back by the inclusion map) to a vector-bundle homomorphism

$$\mathrm{d}f|_X: TM|_X \longrightarrow (f^*TN)\big|_X$$

over X, which can be composed with the inclusion homomorphism $TX\longrightarrow TM|_X$,

$$TX \longrightarrow TM|_X \stackrel{\mathrm{d}f|_X}{\longrightarrow} (f^*TN)\big|_X\,.$$

If in addition $f(X)\subset Y$, then the above sequence can be composed with the f^*-pullback of the projection homomorphism $q: TN|_Y\longrightarrow\mathcal{N}_NY$,

$$TX \longrightarrow TM|_X \stackrel{\mathrm{d}f|_X}{\longrightarrow} (f^*TN)\big|_X \stackrel{f^*q}{\longrightarrow} f^*\mathcal{N}_NY\,. \tag{10.10}$$

This composite vector-bundle homomorphism is 0, since $\mathrm{d}_x f(v) \in T_{f(x)}Y$ for all $x \in X$. Thus, it descends to a vector-bundle homomorphism

$$\mathrm{d}f \colon \mathcal{N}_M X \longrightarrow f^*\mathcal{N}_N Y \tag{10.11}$$

over X. If $f \pitchfork_N Y$ as in (6.1), then the map $TM|_X \longrightarrow f^*\mathcal{N}_N Y$ in (10.10) is onto and thus the vector-bundle homomorphism (10.11) is surjective on every fiber. Finally, if $X = f^{-1}(Y)$, the ranks of the two bundles in (10.11) are the same by the last statement in Theorem 6.3, and so (10.11) is an isomorphism of vector bundles over X. Combining this observation with Theorem 6.3, we obtain a more precise statement of the latter.

Theorem 10.11. *Let $f : M \longrightarrow N$ be a smooth map and $Y \subset N$ an embedded submanifold. If $f \pitchfork_N Y$ as in (6.1), then $X \equiv f^{-1}(Y)$ is an embedded submanifold of M and the differential $\mathrm{d}f$ induces a vector-bundle isomorphism*

$$\begin{tikzcd} \mathcal{N}_M X \arrow[rr, "\mathrm{d}f"] \arrow[dr, "\pi"'] & & f^*(\mathcal{N}_N Y) \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd} \tag{10.12}$$

Since the ranks of $\mathcal{N}_M X$ and $f^*(\mathcal{N}_N Y)$ are the codimensions of X in M and Y in N, respectively, this theorem implies Theorem 6.3. If $Y = \{q\}$ for some $q \in N$, then $\mathcal{N}_N Y$ is a trivial vector bundle and thus so is $\mathcal{N}_M X \approx f^*(\mathcal{N}_N Y)$. For example, the unit sphere $S^m \subset \mathbb{R}^{m+1}$ has trivial normal bundle, because

$$S^m = f^{-1}(1), \qquad \text{where} \quad f \colon \mathbb{R}^{m+1} \longrightarrow \mathbb{R}, \quad f(x) = |x|^2.$$

A trivialization of the normal bundle to S^m is given by

$$T\mathbb{R}^{m+1}/TS^m \longrightarrow S^m \times \mathbb{R}, \qquad (x, v) \longrightarrow (x, x \cdot v).$$

Corollary 10.12. *Let $f : X \longrightarrow M$ and $g \colon Y \longrightarrow M$ be smooth maps. If $f \pitchfork_M g$ as in (6.5), then the space*

$$X \times_M Y \equiv \big\{(x, y) \in X \times Y \colon\, f(x) = g(y)\big\}$$

is an embedded submanifold of $X \times Y$ and the differential $\mathrm{d}f$ induces a vector-bundle isomorphism

$$\begin{tikzcd} \mathcal{N}_{X\times Y}(X \times_M Y) \arrow[rr, "\mathrm{d}(f\circ\pi_X)+\mathrm{d}(g\circ\pi_Y)"] \arrow[dr, "\pi"'] & & \pi_X^* f^* TM = \pi_Y^* g^* TM \arrow[dl] \\ & X \times_M Y & \end{tikzcd} \tag{10.13}$$

Furthermore, the projection map $\pi_1 = \pi_X \colon X \times_M Y \longrightarrow X$ is injective (immersion) if $g \colon Y \longrightarrow M$ is injective (immersion).

This corollary is obtained by applying Theorem 10.11 to the smooth map

$$f \times g \colon X \times Y \longrightarrow M \times M.$$

All other versions of the *Implicit Function Theorem* stated in these notes are special cases of this corollary.

Direct sums

If V and V' are two vector spaces, we can form a new vector space, $V\oplus V'=V\times V'$, the direct sum of V and V'. There are natural inclusions $V,V'\longrightarrow V\oplus V'$ and projections $V\oplus V'\longrightarrow V,V'$. Linear maps $f\colon V\longrightarrow W$ and $f'\colon V'\longrightarrow W'$ induce a linear map

$$f\oplus f'\colon V\oplus V'\longrightarrow W\oplus W'.$$

If we choose bases for V, V', W, and W' so that f and f' correspond to some matrices A and D, then with respect to the induced bases for $V\oplus V'$ and $W\oplus W'$,

$$f\oplus g=\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}.$$

If $\pi\colon V\longrightarrow M$ and $\pi'\colon V'\longrightarrow M$ are smooth vector bundles, we can form their direct sum, $V\oplus V'$, so that

$$(V\oplus V)_p=V_p\oplus V'_p \qquad \forall\, p\in M.$$

The vector-bundle structure on $V\oplus V'$ is determined from those of V and V' by requiring that either the natural inclusion maps $V,V'\longrightarrow V\oplus V'$ or the projections $V\oplus V'\longrightarrow V,V'$ be smooth vector-bundle homomorphisms over M. Thus, if s and s' are sections of V and V', then $s\oplus s'$ is a smooth section of $V\oplus V'$ if and only if s and s' are smooth. So, the map

$$\begin{gathered}\Gamma(M;V)\oplus\Gamma(M;V')\longrightarrow\Gamma(M;V\oplus V'),\\ (s,s')\longrightarrow s\oplus s', \qquad \big\{s\oplus s'\big\}(p)=s(p)\oplus s'(p)\ \ \forall\, p\in M,\end{gathered}$$

is an isomorphism of $C^\infty(M)$-modules. If $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and V', transition data for $V\oplus V'$ is given by $\{g_{\alpha\beta}\oplus g'_{\alpha\beta}\}$, i.e. we put the first matrix in the top left corner and the second matrix in the bottom right corner. Alternatively,

$$\pi\times\pi'\colon V\times V'\longrightarrow M\times M$$

is a smooth vector bundle with respect to the product structures and

$$V\oplus V'=d^*(V\times V'), \tag{10.14}$$

where $d\colon M\longrightarrow M\times M$, $\quad d(p)=(p,p)$ is the diagonal embedding.

The operation $\oplus$ is easily seen to be commutative and associative (the resulting vector bundles are isomorphic). If $\tau_0=M\longrightarrow M$ is trivial rank 0 bundle,

$$\tau_0\oplus V\approx V$$

for every vector bundle $V\longrightarrow M$. If $n\in\mathbb{Z}^{\geq 0}$, let

$$nV=\underbrace{V\oplus\ldots\oplus V}_{n};$$

by convention; $0V=\tau_0$. We leave proofs of the following lemmas as an exercise.

Lemma 10.13. *If $f\colon M \longrightarrow N$ is a smooth map and $W, W' \longrightarrow N$ are smooth vector bundles,*

$$f^*\big(W \oplus W'\big) \approx (f^*W) \oplus (f^*W')$$

as vector bundles over M.

Lemma 10.14. *Let $V, V' \longrightarrow M$ and $W, W' \longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M \longrightarrow N$ a smooth map. Vector-bundle homomorphisms*

$$\tilde{f}\colon V \longrightarrow W \qquad \textit{and} \qquad \tilde{f}'\colon V' \longrightarrow W'$$

covering f as in (8.4) induce a vector-bundle homomorphism

$$\tilde{f}\oplus\tilde{f}'\colon V\oplus V' \longrightarrow W\oplus W'$$

covering f; this induced homomorphism is smooth if and only if $\tilde{f}$ and $\tilde{f}'$ are smooth.

If $V, V' \longrightarrow M$ are vector bundles, then V and V' are vector subbundles of $V\oplus V'$. It is immediate that

$$\big(V\oplus V'\big)/V = V' \qquad \text{and} \qquad \big(V\oplus V'\big)/V' = V.$$

These equalities hold in the holomorphic category as well (i.e. when the bundles and the base manifold carry complex structures and all trivializations and transition maps are holomorphic). Conversely, if V' is a subbundle of V, by Section 11 below

$$V \approx (V/V') \oplus V'$$

as smooth vector bundles, real or complex. However, if V and V' are holomorphic bundles, V may not have the same holomorphic structure as $(V/V')\oplus V'$ (i.e. the two bundles are isomorphic as smooth vector bundles, but not as holomorphic ones).

Dual bundles

If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$), the dual vector space is the space of the linear homomorphisms to the field ($\mathbb{R}$ or $\mathbb{C}$, respectively):

$$V^* = \mathrm{Hom}_{\mathbb{R}}\big(V, \mathbb{R}\big) \qquad \text{or} \qquad V^* = \mathrm{Hom}_{\mathbb{C}}\big(V, \mathbb{C}\big).$$

A linear map $g\colon V \longrightarrow W$ between two vector spaces induces a dual map in the "opposite" direction:

$$g^*\colon W^* \longrightarrow V^*, \qquad \big\{g^*(L)\big\}(v) = L\big(g(v)\big) \quad \forall\ L \in W^*,\ v \in V.$$

If $V = \mathbb{R}^k$ and $W = \mathbb{R}^n$, then g is given by an $n\times k$-matrix, and its dual is given by the transposed $k\times n$-matrix.

If $\pi\colon V \longrightarrow M$ is a smooth vector bundle of rank k (say, over $\mathbb{R}$), the dual bundle of V is a vector bundle $V^* \longrightarrow M$ such that

$$(V^*)_p = V_p^* \qquad \forall\, p\in M.$$

The vector-bundle structure on V^* is determined from that of V by requiring that the natural map

$$V\oplus V^* = V \times_M V^* \longrightarrow \mathbb{R}\ (\text{or } \mathbb{C}), \qquad (v, L) \longrightarrow L(v), \tag{10.15}$$

be smooth. Thus, if s and ψ are smooth sections of V and V^*,

$$\psi(s)\colon M \longrightarrow \mathbb{R}, \qquad \{\psi(s)\}(p) = \{\psi(p)\}\big(s(p)\big),$$

is a smooth function on M. So, the map

$$\Gamma(M;V) \times \Gamma(M;V^*) \longrightarrow C^\infty(M), \qquad (s,\psi) \longrightarrow \psi(s),$$

is a nondegenerate pairing of $C^\infty(M)$-modules. If $\{g_{\alpha\beta}\}$ is transition data for V, i.e. the transitions between smooth trivializations are given by

$$h_\alpha \circ h_\beta^{-1}\colon U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p,v) \longrightarrow \big(p, g_{\alpha\beta}(p)v\big),$$

the dual transition maps are then given by

$$U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p,v) \longrightarrow \big(p, g_{\alpha\beta}(p)^{\mathrm{tr}} v\big).$$

However, these maps reverse the direction, i.e. they go from the α-side to the β-side. To fix this problem, we simply take the inverse of $g_{\alpha\beta}(p)^{\mathrm{tr}}$:

$$U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p,v) \longrightarrow \big(p, \{g_{\alpha\beta}(p)^{\mathrm{tr}}\}^{-1} v\big).$$

So, transition data for V^* is $\{(g_{\alpha\beta}^{\mathrm{tr}})^{-1}\}$. As an example, if V is a line bundle, then $g_{\alpha\beta}$ is a smooth nowhere-zero function on $U_\alpha \cap U_\beta$ and $(g^*)_{\alpha\beta}$ is the smooth function given by $1/g_{\alpha\beta}$. We leave proofs of the following lemmas as an exercise.

Lemma 10.15. *If $f\colon M \longrightarrow N$ is a smooth map and $W \longrightarrow N$ is a smooth vector bundle,*

$$f^*(W^*) \approx (f^*W)^*$$

as vector bundles over M.

Lemma 10.16. *Let $V \longrightarrow M$ and $W \longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M \longrightarrow N$ a diffeomorphism. A vector-bundle homomorphism $\tilde{f}\colon V \longrightarrow W$ covering f as in (8.4) induces a vector-bundle homomorphism*

$$\tilde{f}^*\colon W^* \longrightarrow V^*$$

covering f^{-1}; this induced homomorphism is smooth if and only if the homomorphism $\tilde{f}$ is.

The cotangent bundle of a smooth manifold M, $\pi\colon T^*M \longrightarrow M$, is the dual of its tangent bundle, $TM \longrightarrow M$, i.e. $T^*M = (TM)^*$. For each $p \in M$, the fiber of the cotangent bundle over p is the cotangent space T_p^*M of M at p; see Definition 3.7. A section $\alpha\colon M \longrightarrow T^*M$ of T^*M is called a 1-form on M; it assigns to each $p \in M$ a linear map

$$\alpha_p \equiv \alpha(p)\colon T_pM \longrightarrow \mathbb{R}.$$

If in addition X is a vector field, then

$$\alpha(X)\colon M \longrightarrow \mathbb{R}, \qquad \{\alpha(X)\}(p) = \alpha_p\big(X(p)\big),$$

is a function on M. The section α is smooth if and only if $\alpha(X) \in C^\infty(M)$ for every smooth vector field X on M. If $\varphi = (x_1, \ldots, x_m)\colon U \longrightarrow \mathbb{R}^m$ is a smooth chart, the sections

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \in \mathrm{VF}(U)$$

form a basis for $\mathrm{VF}(U)$ as a $C^\infty(U)$-module. Since

$$\mathrm{d}_p x_i\bigg(\frac{\partial}{\partial x_j}\bigg) = \delta_{ij} \qquad \forall\, i,j=1,2,\ldots,m,$$

$\mathrm{d}x_i(X)\in C^\infty(U)$ for all $X\in\mathrm{VF}(U)$ and $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M for all $p\in U$. Thus, $\mathrm{d}x_i$ is a smooth section of T^*M over U and the inverse of the map

$$U\times\mathbb{R}^m \longrightarrow T^*M|_U, \qquad (p,c_1,\ldots,c_m)\longrightarrow c_1\mathrm{d}_p x_1+\ldots+c_m\mathrm{d}_p x_m\,,$$

is a trivialization of T^*M over U; see Section 8. By (4.16), this inverse is given by

$$T^*M|_U \longrightarrow U\times\mathbb{R}^m, \qquad u\longrightarrow \bigg(\pi(u), u\bigg(\frac{\partial}{\partial x_1}\bigg),\ldots,u\bigg(\frac{\partial}{\partial x_m}\bigg)\bigg),$$

where $\pi\colon T^*M\longrightarrow M$ is the projection map. Thus, a 1-form α on M is smooth if and only if for every smooth chart $\varphi_\alpha=(x_1,\ldots,x_m):U_\alpha\longrightarrow\mathbb{R}^m$ the coefficient functions

$$c_1=\alpha\bigg(\frac{\partial}{\partial x_1}\bigg),\ldots,c_m=\bigg(\frac{\partial}{\partial x_m}\bigg):U\longrightarrow\mathbb{R}, \qquad \alpha_p\equiv c_1(p)\mathrm{d}_p x_1+\ldots+c_m(p)\mathrm{d}_p x_m \quad \forall\,p\in U,$$

are smooth. The $C^\infty(M)$-module of 1-forms on M is denoted by $E^1(M)$.

Tensor products

If V and V' are two vector spaces, we can form a new vector space, $V\otimes V'$, the tensor product of V and V'. If $g\colon V\longrightarrow W$ and $g'\colon V'\longrightarrow W'$ are linear maps, they induce a linear map

$$g\otimes g'\colon V\otimes V'\longrightarrow W\otimes W'.$$

If we choose bases $\{e_j\}$, $\{e'_n\}$, $\{f_i\}$, and $\{f'_m\}$ for V, V', W, and W', respectively, then $\{e_j\otimes e'_n\}_{(j,n)}$ and $\{f_i\otimes f'_m\}_{(i,m)}$ are bases for $V\otimes V'$ and $W\otimes W'$. If the matrices for g and g' with respect to the chosen bases for V, V', W, and W' are $(g_{ij})_{i,j}$ and $(g'_{mn})_{m,n}$, then the matrix for $g\otimes g'$ with respect to the induced bases for $V\otimes V'$ and $W\otimes W'$ is $(g_{ij}g'_{mn})_{(i,m),(j,n)}$. The rows of this matrix are indexed by the pairs (i,m) and the columns by the pairs (j,n). In order to actually write down the matrix, we need to order all pairs (i,m) and (j,n). If the vector spaces V and W are one-dimensional, g corresponds to a single number g_{ij}, while $g\otimes g'$ corresponds to the matrix $(g_{mn})_{m,n}$ multiplied by this number.

If $\pi: V\longrightarrow M$ and $\pi': V'\longrightarrow M$ are smooth vector bundles, we can form their tensor product, $V\otimes V'$, so that

$$(V\otimes V')_p = V_p\otimes V'_p \qquad \forall\,p\in M.$$

The topology and smooth structure on $V\otimes V'$ are determined from those of V and V' by requiring that if s and s' are smooth sections of V and V', then $s\otimes s'$ is a smooth section of $V\otimes V'$. So, the map

$$\begin{gathered}\Gamma(M;V)\otimes\Gamma(M;V')\longrightarrow\Gamma(M;V\otimes V'),\\ (s,s')\longrightarrow s\otimes s', \qquad \big\{s\otimes s'\big\}(p)=s(p)\otimes s'(p)\ \ \forall\,p\in M,\end{gathered}$$

is a homomorphism of $C^\infty(M)$-modules (but not an isomorphism). If $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and V', then transition data for $V\otimes V'$ is given by $\{g_{\alpha\beta}\otimes g'_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $g_{\alpha\beta}\otimes g'_{\alpha\beta}$ from $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ as in the previous paragraph. If V and V' are line bundles, then $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are smooth nowhere-zero functions on $U_\alpha\cap U_\beta$ and $(g\otimes g')_{\alpha\beta}$ is the smooth function given by $g_{\alpha\beta}g'_{\alpha\beta}$.

The operation $\otimes$ is easily seen to be commutative and associative (the resulting vector bundles are isomorphic). If $\tau_1\longrightarrow M$ is the trivial line bundle,

$$\tau_1\otimes V\approx V$$

for every vector bundle $V\longrightarrow M$ is a vector bundle. If $n\in\mathbb{Z}^+$, let

$$V^{\otimes n}=\underbrace{V\otimes\ldots\otimes V}_{n},\qquad V^{\otimes(-n)}=(V^*)^{\otimes n}\equiv\underbrace{V^*\otimes\ldots\otimes V^*}_{n};$$

by convention, $V^{\otimes 0}=\tau_1$. We leave proofs of the following lemmas as an exercise.

Lemma 10.17. *If $f\colon M\longrightarrow N$ is a smooth map and $W,W'\longrightarrow N$ are smooth vector bundles,*

$$f^*\big(W\otimes W'\big)\approx(f^*W)\otimes(f^*W')$$

as vector bundles over M.

Lemma 10.18. *Let $V,V'\longrightarrow M$ and $W,W'\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. Vector-bundle homomorphisms*

$$\tilde f\colon V\longrightarrow W\qquad\textit{and}\qquad\tilde f\colon V'\longrightarrow W'$$

covering f as in (8.4) induce a vector-bundle homomorphism

$$\tilde f\otimes\tilde f'\colon V\otimes V'\longrightarrow W\otimes W'$$

covering f; this induced homomorphism is smooth if $\tilde f$ and $\tilde f'$ are smooth.

Lemma 10.19. *Let $V,V'\longrightarrow M$ and $W\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. A bundle map*

$$\tilde f\colon V\oplus V'=V\times_M V\longrightarrow W$$

covering f as in (8.4) such that the restriction of $\tilde f$ to each fiber $V_p\times V_p$ is linear in each component induces a vector-bundle homomorphism

$$\bar f\colon V\otimes V'\longrightarrow W$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde f$ is.

Exterior products

If V is a vector space and k is a nonnegative integer, we can form the k-th exterior power, $\Lambda^k V$, of V. A linear map $g\colon V \longrightarrow W$ induces a linear map

$$\Lambda^k g\colon \Lambda^k V \longrightarrow \Lambda^k W.$$

If n is a nonnegative integer, let $S_k(n)$ be the set of increasing k-tuples of integers between 1 and n:

$$S_k(n) = \big\{(i_1, \ldots, i_k) \in \mathbb{Z}^k \colon 1 \le i_1 < i_2 < \ldots < i_k \le n\big\}.$$

If $\{e_j\}_{j=1,\ldots,n}$ and $\{f_i\}_{i=1,\ldots,m}$ are bases for V and W, then $\{e_\eta\}_{\eta \in S_k(n)}$ and $\{f_\mu\}_{\mu \in S_k(m)}$ are bases for $\Lambda^k V$ and $\Lambda^k W$, where

$$e_{(\eta_1,\ldots,\eta_k)} = e_{\eta_1} \wedge \ldots \wedge e_{\eta_k} \qquad \text{and} \qquad f_{(\mu_1,\ldots,\mu_k)} = f_{\mu_1} \wedge \ldots \wedge f_{\mu_k}.$$

If $(g_{ij})_{i=1,\ldots,m, j=1,\ldots,n}$ is the matrix for g with respect to the chosen bases for V and W, then

$$\big(\det\big((g_{\mu_r \eta_s})_{r,s=1,\ldots,k}\big)\big)_{(\mu,\eta) \in I_k(m) \times I_k(n)}$$

is the matrix for $\Lambda^k g$ with respect to the induced bases for $\Lambda^k V$ and $\Lambda^k W$. The rows and columns of this matrix are indexed by the sets $S_k(m)$ and $S_k(n)$, respectively. The (μ, η)-entry of the matrix is the determinant of the $k \times k$-submatrix of $(g_{ij})_{i,j}$ with the rows and columns indexed by the entries of μ and η, respectively. In order to actually write down the matrix, we need to order the sets $S_k(m)$ and $S_k(n)$. If $k = m = n$, then $\Lambda^k V$ and $\Lambda^k W$ are one-dimensional vector spaces, called the top exterior power of V and W, with bases

$$\big\{e_1 \wedge \ldots \wedge e_k\big\} \qquad \text{and} \qquad \big\{f_1 \wedge \ldots \wedge f_k\big\}.$$

With respect to these bases, the homomorphism $\Lambda^k g$ corresponds to the number $\det(g_{ij})_{i,j}$. If $k > n$ (or $k > m$), then $\Lambda^k V$ (or $\Lambda^k W$) is the zero vector space and the corresponding matrix is empty.

If $\pi\colon V \longrightarrow M$ is a smooth vector bundle, we can form its k-th exterior power, $\Lambda^k V$, so that

$$(\Lambda^k V)_p = \Lambda^k V_p \qquad \forall\, p \in M.$$

The topology and smooth structure on $\Lambda^k V$ are determined from those of $\Lambda^k V$ by requiring that if $s_1, \ldots, s_k$ are smooth sections of V, then $s_1 \wedge \ldots \wedge s_k$ is a smooth section of $\Lambda^k V$. Thus, the map

$$\Lambda^k\big(\Gamma(M; V)\big) \longrightarrow \Gamma(M; \Lambda^k V),$$
$$(s_1, \ldots, s_k) \longrightarrow s_1 \wedge \ldots \wedge s_k, \qquad \{s_1 \wedge \ldots \wedge s_k\}(p) = s_1(p) \wedge \ldots \wedge s_k(p) \quad \forall\, p \in M,$$

is a homomorphism of $C^\infty(M)$-modules (but not an isomorphism). If $\{g_{\alpha\beta}\}$ is transition data for V, then transition data for $\Lambda^k V$ is given by $\{\Lambda^k g_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $\Lambda^k g_{\alpha\beta}$ from each matrix $g_{\alpha\beta}$ as in the previous paragraph. As an example, if the rank of V is k, then the transition data for the line bundle $\Lambda^k V$, called the top exterior power of V, is $\{\det g_{\alpha\beta}\}$. By definition, $\Lambda^0 V = \tau_1^{\mathbb{R}}$ is the trivial line bundle over M.

It follows directly from the definitions that if $V \longrightarrow M$ is a vector bundle of rank k and $L \longrightarrow M$ is a line bundle (vector bundle of rank one), then

$$\Lambda^{\rm top}(V \oplus L) \equiv \Lambda^{k+1}(V \oplus L) = \Lambda^k V \otimes L \equiv \Lambda^{\rm top} V \otimes L.$$

More generally, if $V,W\longrightarrow M$ are any two vector bundles, then

$$\Lambda^{\rm top}(V\oplus W)=(\Lambda^{\rm top}V)\otimes(\Lambda^{\rm top}W)\qquad\text{and}\qquad \Lambda^k(V\oplus W)=\bigoplus_{i+j=k}(\Lambda^iV)\otimes(\Lambda^jW).$$

We leave proofs of the following lemmas as exercises.

Lemma 10.20. *If $f\colon M\longrightarrow N$ is a smooth map, $W\longrightarrow N$ is a smooth vector bundle, and $k\in\mathbb{Z}^{\ge0}$,*

$$f^*\big(\Lambda^kW\big)\approx\Lambda^k(f^*W)$$

as vector bundles over M.

Lemma 10.21. *Let $V\longrightarrow M$ be a vector bundle. If $k,l\in\mathbb{Z}^{\ge0}$, the map*

$$\begin{gathered}\Gamma(M;\Lambda^kV)\otimes\Gamma(M;\Lambda^lV)\longrightarrow\Gamma(M;\Lambda^{k+l}V)\\ (s_1,s_2)\longrightarrow s_1\wedge s_2,\qquad \{s_1\wedge s_2\}(p)=s_1(p)\wedge s_2(p)\quad\forall\,p\in M,\end{gathered}$$

is a well-defined homomorphism of $C^\infty(M)$-modules.

Lemma 10.22. *Let $V\longrightarrow M$ and $W\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. A vector-bundle homomorphism $\tilde f\colon V\longrightarrow W$ covering f as in (8.4) induces a vector-bundle homomorphism*

$$\Lambda^k\tilde f\colon\Lambda^kV\longrightarrow\Lambda^kW$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde f$ is.

Lemma 10.23. *Let $V\longrightarrow M$ and $W\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. A bundle homomorphism*

$$\tilde f\colon kV\equiv\underbrace{V\times_M\ldots\times_MV}_{k}\longrightarrow W$$

covering f as in (8.4) such that the restriction of $\tilde f$ to each fiber V_p^k is linear in each component and alternating induces a vector-bundle homomorphism

$$\bar f\colon\Lambda^kV\longrightarrow W$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde f$ is.

Remark: For complex vector bundles, the above constructions (exterior power, tensor product, direct sum, etc.) are always done over $\mathbb{C}$, unless specified otherwise. So if V is a complex vector bundle of rank k over M, the top exterior power of V is the complex line bundle Λ^kV over M (could also be denoted as $\Lambda^k_{\mathbb{C}}V$). In contrast, if we forget the complex structure of V (so that it becomes a real vector bundle of rank $2k$), then its top exterior power is the real line bundle $\Lambda^{2k}V$ (could also be denoted as $\Lambda^{2k}_{\mathbb{R}}V$).

If M is a smooth manifold, a section of the bundle $\Lambda^k(T^*M)\longrightarrow M$ is called a k-form on M. A smooth nowhere-vanishing section s of $\Lambda^{\rm top}(T^*M)$, i.e.

$$s(p)\in\Lambda^{\rm top}(T_p^*M)-0\qquad\forall\,p\in M,$$

is called a volume form on M; Corollary 12.2 below provides necessary and sufficient conditions for such a section to exist. The space of smooth k-forms on M is often denoted by $E^k(M)$, rather than $\Gamma(M;\Lambda^k(T^*M))$.

11 Metrics on Fibers

If V is a vector space over $\mathbb{R}$, a positive-definite inner-product on V is a symmetric bilinear map

$$\langle\cdot,\cdot\rangle\colon V\times V\longrightarrow\mathbb{R},\quad (v,w)\longrightarrow\langle v,w\rangle,\qquad\text{s.t.}\qquad \langle v,v\rangle>0\;\;\forall\,v\in V-0.$$

If $\langle,\rangle$ and $\langle,\rangle'$ are positive-definite inner-products on V and $a,a'\in\bar{\mathbb{R}}^+$ are not both zero, then

$$a\langle,\rangle+a'\langle,\rangle'\colon V\times V\longrightarrow\mathbb{R},\qquad \big\{a\langle,\rangle+a'\langle,\rangle'\big\}(v,w)=a\langle v,w\rangle+a'\langle v,w\rangle',$$

is also a positive-definite inner-product. If W is a subspace of V and $\langle,\rangle$ is a positive-definite inner-product on V, let

$$W^\perp=\big\{v\in V\colon\langle v,w\rangle=0\;\forall\,w\in W\big\}$$

be the orthogonal complement of W in V. In particular,

$$V=W\oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V\longrightarrow V/W$$

induces an isomorphism from $W^\perp$ to V/W so that

$$V\approx W\oplus(V/W).$$

If M is a smooth manifold and $V\longrightarrow M$ is a smooth real vector bundle of rank k, a Riemannian metric on V is a positive-definite inner-product in each fiber $V_x\approx\mathbb{R}^k$ of V that varies smoothly with $x\in M$. Formally, the smoothness requirement is one of the following equivalent conditions:

(a) the map $\langle,\rangle\colon V\times_M V\longrightarrow\mathbb{R}$ is smooth;

(b) the section $\langle,\rangle$ of the vector bundle $(V\otimes V)^*\longrightarrow M$ is smooth;

(c) if s_1,s_2 are smooth sections of the vector bundle $V\longrightarrow M$, then the map

$$\big\langle s_1,s_2\big\rangle\colon M\longrightarrow\mathbb{R},\qquad p\longrightarrow\big\langle s_1(p),s_2(p)\big\rangle,$$

is smooth;

(d) if $h\colon V|_U\longrightarrow U\times\mathbb{R}^k$ is a trivialization of V, then the matrix-valued function,

$$B\colon U\longrightarrow\mathrm{Mat}_k\mathbb{R}\quad\text{s.t.}\quad\big\langle h^{-1}(p,v),h^{-1}(p,w)\big\rangle=v^tB(p)w\quad\forall\,p\in U,\,v,w\in\mathbb{R}^k,$$

is smooth.

Every real vector bundle admits a Riemannian metric. Such a metric can be constructed by covering M by a locally finite collection of trivializations for V and patching together positive-definite inner-products on each trivialization using a partition of unity; see Definition 11.1 below. If W is a subspace of V and $\langle,\rangle$ is a Riemannian metric on V, let

$$W^\perp=\big\{v\in V\colon\langle v,w\rangle=0\;\forall\,w\in W\big\}$$

be the orthogonal complement of W in V. Then $W^\perp \longrightarrow M$ is a vector subbundle of V and

$$V = W \oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V \longrightarrow V/W$$

induces a vector bundle isomorphism from $W^\perp$ to V/W so that

$$V \approx W \oplus (V/W).$$

Definition 11.1. *A smooth partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of a smooth manifold M is a collection $\{\eta_\alpha\}_{\alpha\in\mathcal{A}}$ of smooth functions on M with values in $[0,1]$ such that*

(PU1) *the collection $\{\operatorname{supp}\eta_\alpha\}_{\alpha\in\mathcal{A}}$ is locally finite;*

(PU2) $\operatorname{supp}\eta_\alpha \subset U_\alpha$ *for every $\alpha\in\mathcal{A}$;*

(PU3) $\sum\limits_{\alpha\in\mathcal{A}} \eta_\alpha \equiv 1.$

If V is a vector space over $\mathbb{C}$, a nondegenerate Hermitian inner-product on V is a map

$$\langle\cdot,\cdot\rangle\colon V\times V \longrightarrow \mathbb{C}, \quad (v,w) \longrightarrow \langle v,w\rangle,$$

which is $\mathbb{C}$-antilinear in the first input, $\mathbb{C}$-linear in the second input,

$$\langle w,v\rangle = \overline{\langle v,w\rangle} \qquad \text{and} \qquad \langle v,v\rangle > 0 \;\; \forall\, v\in V-0.$$

If $\langle,\rangle$ and $\langle,\rangle'$ are nondegenerate Hermitian inner-products on V and $a,a'\in\bar{\mathbb{R}}^+$ are not both zero, then $a\langle,\rangle+a'\langle,\rangle'$ is also a nondegenerate Hermitian inner-product on V. If W is a complex subspace of V and $\langle,\rangle$ is a nondegenerate Hermitian inner-product on V, let

$$W^\perp = \big\{v\in V\colon \langle v,w\rangle = 0 \;\forall\, w\in W\big\}$$

be the orthogonal complement of W in V. In particular,

$$V = W \oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V \longrightarrow V/W$$

induces an isomorphism from $W^\perp$ to V/W so that

$$V \approx W \oplus (V/W).$$

If M is a smooth manifold and $V \longrightarrow M$ is a smooth complex vector bundle of rank k, a Hermitian metric on V is a nondegenerate Hermitian inner-product in each fiber $V_x \approx \mathbb{C}^k$ of V that varies smoothly with $x\in M$. Formally, the smoothness requirement is one of the following equivalent conditions:

(a) the map $\langle,\rangle\colon V\times_M V\longrightarrow\mathbb{C}$ is smooth;

(b) the section $\langle,\rangle$ of the vector bundle $(V\otimes_{\mathbb{R}}V)^*\longrightarrow M$ is smooth;

(c) if s_1,s_2 are smooth sections of the vector bundle $V\longrightarrow M$, then the function $\langle s_1,s_2\rangle$ on M is smooth;

(d) if $h\colon V|_U\longrightarrow U\times\mathbb{C}^k$ is a trivialization of V, then the matrix-valued function,

$$B\colon U\longrightarrow \mathrm{Mat}_k\mathbb{C} \quad \text{s.t.} \quad \big\langle h^{-1}(p,v),h^{-1}(p,w)\big\rangle = \bar{v}^tB(p)w \quad \forall\, p\in M,\, v,w\in\mathbb{C}^k,$$

is smooth.

Similarly to the real case, every complex vector bundle admits a Hermitian metric. If W is a subspace of V and $\langle,\rangle$ is a Hermitian metric on V, let

$$W^\perp = \big\{v\in V\colon \langle v,w\rangle=0\ \forall\, w\in W\big\}$$

be the orthogonal complement of W in V. Then $W^\perp\longrightarrow M$ is a complex vector subbundle of V and

$$V = W\oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V\longrightarrow V/W$$

induces an isomorphism of complex vector bundles over M so that

$$V\approx W\oplus(V/W).$$

If $V\longrightarrow M$ is a real vector bundle of rank k with a Riemannian metric $\langle,\rangle$ or a complex vector bundle of rank k with a Hermitian metric $\langle,\rangle$, let

$$SV\equiv\big\{v\in V\colon\langle v,v\rangle=1\big\}\longrightarrow M$$

be the sphere bundle of V. In the real case, the fiber of SV over every point of M is S^{k-1}. Furthermore, if U is a small open subset of M, then $SV|_U\approx U\times S^{k-1}$ as bundles over U, i.e. SV is an S^{k-1}-fiber bundle over M. In the complex case, SV is an S^{2k-1}-fiber bundle over M. If $V\longrightarrow M$ is a real line bundle (vector bundle of rank one) with a Riemannian metric $\langle,\rangle$, then $SV\longrightarrow M$ is an S^0-fiber bundle. In particular, if U is a small open subset of M, $SV|_U$ is diffeomorphic to $U\times\{\pm1\}$. Thus, $SV\longrightarrow M$ is a $2:1$-covering map. If M is connected, the covering space SV is connected if and only if V is *not* orientable; see Section 12 below.

12 Orientations

If V is a real vector space of dimension k, the top exterior power of V, i.e.

$$\Lambda^{\mathrm{top}}V\equiv\Lambda^kV$$

is a one-dimensional vector space. Thus, $\Lambda^{\mathrm{top}}V-0$ has exactly two connected components. An orientation on V is a component $\mathcal{C}$ of $\Lambda^{\mathrm{top}}V-0$. If $\mathcal{C}$ is an orientation on V, then a basis $\{e_i\}$ for V is called oriented (with respect to $\mathcal{C}$) if

$$e_1\wedge\ldots\wedge e_k \in \mathcal{C}.$$

If $\{f_j\}$ is another basis for V and A is the change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, i.e.

$$\big(f_1,\ldots,f_k\big)=\big(e_1,\ldots,e_k\big)A \qquad\Longleftrightarrow\qquad f_j=\sum_{i=1}^{i=k}A_{ij}e_i,$$

then

$$f_1\wedge\ldots\wedge f_k=(\det A)e_1\wedge\ldots\wedge e_k.$$

Thus, two different bases for V belong to the same orientation on V if and only if the determinant of the corresponding change-of-basis matrix is positive.

Suppose $V\longrightarrow M$ is a real vector bundle of rank k. An orientation for V is an orientation for each fiber $V_x\approx\mathbb{R}^k$, which varies smoothly (or continuously, or is locally constant) with $x\in M$. This means that if

$$h\colon V|_U\longrightarrow U\times\mathbb{R}^k$$

is a trivialization of V and U is connected, then h is either orientation-preserving or orientation-reversing (with respect to the standard orientation of $\mathbb{R}^k$) on every fiber. If V admits an orientation, V is called orientable.

Lemma 12.1. *Suppose $V\longrightarrow M$ is a smooth real vector bundle.*

(1) V is orientable if and only if V^* is orientable.

(2) V is orientable if and only if there exists a collection $\{U_\alpha,h_\alpha\}$ of trivializations that covers M such that

$$\det g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathbb{R}^+,$$

where $\{g_{\alpha\beta}\}$ is the corresponding transition data.

(3) V is orientable if and only if the line bundle $\Lambda^{\mathrm{top}}V\longrightarrow M$ is orientable.

(4) If V is a line bundle, V is orientable if and only if V is (isomorphic to) the trivial line bundle $M\times\mathbb{R}$.

(5) If M is connected and V is a line bundle, V is orientable if and only if the sphere bundle SV (with respect to any Riemann metric on V) is not connected.

Proof: (1) Since $\Lambda^{\mathrm{top}}(V^*)\approx(\Lambda^{\mathrm{top}}V)^*$ and a line bundle L is trivial if and only if L^* is trivial, this claim follows from (3) and (4).

(2) If V has an orientation, we can choose a collection $\{U_\alpha,h_\alpha\}$ of trivializations that covers M such that the restriction of h_α to each fiber is orientation-preserving (if a trivialization is orientation-reversing, simply multiply its first component by -1). Then, the corresponding transition data $\{g_{\alpha\beta}\}$ is orientation-preserving, i.e.

$$\det g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathbb{R}^+.$$

Conversely, suppose $\{U_\alpha, h_\alpha\}$ is a collection of trivializations that covers M such that

$$\det g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^+.$$

Then, if $x \in U_\alpha$ for some α, define an orientation on V_x by requiring that

$$h_\alpha\colon V_x \longrightarrow x \times \mathbb{R}^k$$

is orientation-preserving. Since $\det g_{\alpha\beta}$ is $\mathbb{R}^+$-valued, the orientation on V_x is independent of α such that $x \in U_\alpha$. Each of the trivializations h_α is then orientation-preserving on each fiber.

(3) An orientation for V is the same as an orientation for $\Lambda^{\mathrm{top}} V$, since

$$\Lambda^{\mathrm{top}} V = \Lambda^{\mathrm{top}}\big(\Lambda^{\mathrm{top}} V\big).$$

Furthermore, if $\{(U_\alpha, h_\alpha)\}$ is a collection of trivializations for V such that the corresponding transition functions $g_{\alpha\beta}$ have positive determinant, then $\{(U_\alpha, \Lambda^{\mathrm{top}} h_\alpha)\}$ is a collection of trivializations for $\Lambda^{\mathrm{top}} V$ such that the corresponding transition functions $\Lambda^{\mathrm{top}} g_{\alpha\beta} = \det(g_{\alpha\beta})$ have positive determinant as well.

(4) The trivial line bundle $M \times \mathbb{R}$ is orientable, with an orientation determined by the standard orientation on $\mathbb{R}$. Thus, if V is isomorphic to the trivial line bundle, then V is orientable. Conversely, suppose V is an oriented line bundle. For each $x \in M$, let

$$\mathcal{C}_x \subset \Lambda^{\mathrm{top}} V = V$$

be the chosen orientation of the fiber. Choose a Riemannian metric on V and define a section s of V by requiring that for all $x \in M$

$$\big\langle s(x), s(x)\big\rangle = 1 \qquad \text{and} \qquad s(x) \in \mathcal{C}_x.$$

This section is well-defined and smooth (as can be seen by looking on a trivialization). Since it does not vanish, the line bundle V is trivial by Lemma 8.5.

(5) If V is orientable, then V is isomorphic to $M \times \mathbb{R}$, and thus

$$SV = S(M \times \mathbb{R}) = M \times S^0 = M \sqcup M$$

is not connected. Conversely, if M is connected and SV is not connected, let SV^+ be one of the components of SV. Since $SV \longrightarrow M$ is a covering projection, so is $SV^+ \longrightarrow M$. Since the latter is one-to-one, it is a diffeomorphism, and its inverse determines a nowhere-zero section of V. Thus, V is isomorphic to the trivial line bundle by Lemma 8.5.

If V is a complex vector space of dimension k, V has a canonical orientation as a real vector space of dimension $2k$. If $\{e_i\}$ is a basis for V over $\mathbb{C}$, then

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\}$$

is a basis for V over $\mathbb{R}$. The orientation determined by such a basis is the canonical orientation for the underlying real vector space V. If $\{f_j\}$ is another basis for V over $\mathbb{C}$, B is the complex change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, A is the real change-of-basis matrix from

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\} \qquad \text{to} \qquad \big\{f_1, \mathrm{i}f_1, \ldots, f_k, \mathrm{i}f_k\big\},$$

then

$$\det A = (\det B)\overline{\det B} \in \mathbb{R}^+.$$

Thus, the two bases over $\mathbb{R}$ induced by complex bases for V determine the same orientation for V. This implies that every complex vector bundle $V \longrightarrow M$ is orientable as a real vector bundle.

A smooth manifold M is called orientable if its tangent bundle, $TM \longrightarrow M$, is orientable.

Corollary 12.2. *Let M be a smooth manifold. The following statements are equivalent:*

(1) M is orientable;

(2) the bundle $T^*M \longrightarrow M$ is orientable;

(3) M admits a volume form;

(4) there exists a collection of smooth charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ that covers M such that

$$\det \mathcal{J}(\varphi_\alpha \circ \varphi_\beta^{-1})_x > 0 \qquad \forall\, x \in \varphi_\beta(U_\alpha \cap U_\beta),\ \alpha, \beta \in \mathcal{A}.$$

The equivalence of the first three conditions follows immediately from Lemma 12.1. If $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of charts as in (4), then

$$h_\alpha = \tilde{\varphi}_\alpha : TM|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^m, \qquad v \longrightarrow \big(\pi(v), v(\varphi_\alpha)\big),$$

is a collection of trivializations of TM as in Lemma 12.1-(2) for $V = TM$, since

$$\begin{aligned} \tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1} &: U_\alpha \cap U_\beta \times \mathbb{R}^m \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^m, \quad (p, v) \longrightarrow \big(p, \mathcal{J}(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(p)} v\big), \\ h_\alpha \circ h_\beta^{-1} &: U_\alpha \cap U_\beta \times \mathbb{R}^m \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^m, \quad (p, v) \longrightarrow \big(p, g_{\alpha\beta}(p) v\big). \end{aligned}$$

In particular, if such a collection of charts exists, then TM is orientable. Conversely, suppose $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations of TM as in Lemma 12.1-(2), $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is any collection of smooth charts on M, and U_α is connected. In particular,

$$\tilde{\varphi}_\alpha \circ h_\alpha^{-1} : U_\alpha \times \mathbb{R}^m \longrightarrow U_\alpha \times \mathbb{R}^m, \qquad (p, v) \longrightarrow \big(p, \{h_\alpha^{-1}(p, v)\}(\varphi_\alpha)\big),$$

is a smooth vector-bundle isomorphism. Thus, there is a smooth map

$$A_\alpha : U_\alpha \longrightarrow \mathrm{GL}_m\mathbb{R} \qquad \text{s.t.} \qquad \{h_\alpha^{-1}(p, v)\}(\varphi_\alpha) = A_\alpha(p) v \ \ \forall\, v \in \mathbb{R}^m.$$

Since U_α is connected, $\det A_\alpha$ does not change sign on U_α. By changing the sign of the first component of φ_α if necessary, it can be assumed that $\det A_\alpha(p) > 0$ for all $p \in U_\alpha$ and $\alpha \in \mathcal{A}$. Thus,

$$\det \mathcal{J}(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(p)} = \det A_\alpha(p) \cdot \det g_{\alpha\beta}(p) \cdot \det A_\beta^{-1}(p) > 0 \qquad \forall\, p \in U_\alpha \cap U_\beta,\ \alpha, \beta \in \mathcal{A}.$$

Thus, the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ satisfies (4).

An orientation for a smooth manifold M is an orientation for the vector bundle $TM \longrightarrow M$; a manifold with a choice of orientation is called oriented. A diffeomorphism $f : M \longrightarrow N$ between oriented manifolds is called orientation-preserving (orientation-reversing) if the differential

$$\mathrm{d}_p f : T_p M \longrightarrow T_{f(p)} N$$

is an orientation-preserving (orientation-reversing) isomorphism for every $p\in M$; if M is connected, this is the case if and only if $\mathrm{d}_p f$ is orientation-preserving (orientation-reversing) for a single point $p\in M$.

If M is a smooth manifold, the sphere bundle

$$\pi\colon S\big(\Lambda^{\mathrm{top}}T^*M\big) \longrightarrow M$$

is a two-to-one covering map. By Lemma 12.1 and Corollary 12.2, if M is connected, the domain of π is connected if and only if M is not orientable. For each $p\in M$,

$$\pi^{-1}(p) \equiv \{\Omega_p, -\Omega_p\} \subset S\big(\Lambda^{\mathrm{top}}T_p^*M\big) \subset \Lambda^{\mathrm{top}}T_p^*M$$

is a pair on nonzero top forms on T_p^*M, which define opposite orientations of T_pM. Thus, $S(\Lambda^{\mathrm{top}}T^*M)$ can be thought as the set of orientations on the fibers of M; it is called the orientation double cover of M.

Smooth maps $f,g\colon M\longrightarrow N$ are called smoothly homotopic if there exists a smooth map

$$H\colon M\times[0,1] \longrightarrow N \qquad \text{s.t.} \qquad H(p,0)=f(p), \quad H(p,1)=g(p) \quad \forall\, p\in M.$$

Diffeomorphisms $f,g\colon M\longrightarrow N$ are called isotopic if there exists a smooth map H as above such that the map

$$H_t\colon M \longrightarrow N, \qquad p\longrightarrow (p,t),$$

is a diffeomorphism for every $t\in[0,1]$. We leave proofs of the following lemmas as an exercise; both can be proved using Corollary 12.2.

Lemma 12.3. *The orientation double cover of any smooth manifold is orientable.*

Lemma 12.4. *Let $f,g\colon M\longrightarrow N$ be isotopic diffeomorphisms between oriented manifolds. If f is orientation-preserving (orientation-reversing), then so is g.*

Exercises

1. Let $\pi\colon V\longrightarrow M$ be a vector bundle. Show that

(a) the scalar-multiplication map (7.1) is smooth;

(b) the space $V\times_M V$ is a smooth submanifold of $V\times V$ and the addition map (7.2) is smooth.

2. Let $\pi : V \longrightarrow M$ be a smooth vector bundle of rank k and $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ a collection of trivializations covering M. Show that a section s of π is continuous (smooth) if and only if the map

$$s_\alpha \equiv \pi_2\circ h_\alpha\circ s\colon U_\alpha \longrightarrow \mathbb{R}^k\,,$$

where $\pi_2\colon U_\alpha\times\mathbb{R}^k\longrightarrow\mathbb{R}^k$ is the projection on the second component, is continuous (smooth) for every $\alpha\in\mathcal{A}$.

3. Let $\pi\colon V\longrightarrow M$ be a submersion satisfying ($\mathbb{R}$VB1)-($\mathbb{R}$VB3) in Definition 7.1. Show that

(a) if $s_1, \ldots, s_k : U \longrightarrow V|_U$ are smooth sections over an open subset $U \subset M$ such that $\{s_i(x)\}_i$ is a basis for V_x for all $x \in U$, then the map (8.2) is a diffeomorphism;

(b) $\pi : V \longrightarrow M$ is a vector bundle of rank k if and only if for every $p \in M$ there exist a neighborhood U of p in M and smooth sections $s_1, \ldots, s_k : U \longrightarrow V|_U$ such that $\{s_i(p)\}_i$ is a basis for V_p.

4. Show that the two versions of the last condition on $\tilde{f}$ in (2) in Definition 8.2 are indeed equivalent.

5. Let M be a smooth manifold and $X, Y, Z \in \mathrm{VF}(M)$. Show that

(a) $[X, Y]$ is indeed a smooth vector field on M and

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X \qquad \forall\, f, g \in C^\infty(M);$$

(b) $[\cdot, \cdot]$ is bilinear, anti-symmetric, and

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

6. Verify all claims made in Example 7.5, thus establishing that the tangent bundle TM of a smooth manifold is indeed a vector bundle. What is its transition data?

7. Show that the tangent bundle TS^1 of S^1 is isomorphic to the trivial real line bundle over S^1.

8. Show that the tautological line bundle $\gamma_n \longrightarrow \mathbb{R}P^n$ is non-trivial for $n \geq 1$.

9. Show that the complex tautological line bundle $\gamma_n \longrightarrow \mathbb{C}P^n$ is indeed a complex line bundle as claimed in Example 7.8. What is its transition data? Why is it non-trivial for $n \geq 1$?

10. Let $q : \tilde{M} \longrightarrow M$ be a smooth covering projection. Show that

(a) the map $\mathrm{d}q : \tilde{M} \longrightarrow M$ is a covering projection and a bundle homomorphism covering q as in (8.4);

(b) there is a natural isomorphism

$$\mathrm{VF}(M) \approx VF(\tilde{M})^{\mathrm{d}q} \equiv \big\{X \in \mathrm{VF} \colon \mathrm{d}_{p_1}q(X(p_1)) = \mathrm{d}_{p_2}q(X(p_2))\ \forall\, p_1, p_2 \in M \text{ s.t. } q(p_1) = q(p_2)\big\}.$$

11. Let M be a smooth m-manifold. Show that

(TM1) the topology on TM constructed in Example 7.5 is the unique one so that $\pi : TM \longrightarrow M$ is a topological vector bundle with the canonical vector-space structure on the fibers and so that for every vector field X on TM and smooth function $f : U \longrightarrow \mathbb{R}$, where U is an open subset of $\mathbb{R}$, the function $X(f) : U \longrightarrow \mathbb{R}$ is continuous if and only if X is continuous;

(TM2) the smooth structure on TM constructed in Example 7.5 is the unique one so that $\pi : TM \longrightarrow M$ is a smooth vector bundle with the canonical vector-space structure on the fibers and so that for every vector field X on TM and smooth function $f : U \longrightarrow \mathbb{R}$, where U is an open subset of $\mathbb{R}$, the function $X(f) : U \longrightarrow \mathbb{R}$ is smooth if and only if X is smooth.

12. Suppose that $f\colon M \longrightarrow N$ is a smooth map and $\pi\colon V \longrightarrow N$ is a smooth vector bundle of rank k with transition data $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_n\mathbb{R}\}_{\alpha,\beta\in\mathcal{A}}$. Show that

(a) the space f^*V defined by (10.1) is a smooth submanifold of $M\times V$ and the projection $\pi_1\colon f^*V \longrightarrow M$ is a vector bundle of rank k with transition data

$$\big\{f^*g_{\alpha\beta} = g_{\alpha\beta}\circ f\colon f^{-1}(U_\alpha)\cap f^{-1}(U_\beta) \longrightarrow \mathrm{GL}_n\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}\,;$$

(b) if M is an embedded submanifold of N and f is the inclusion map, then the projection $\pi_2\colon f^*V \longrightarrow V$ induces an isomorphism $f^*V \longrightarrow V|_M$ of vector bundles over M.

13. Let $f\colon M \longrightarrow V$ be a smooth map and $V \longrightarrow N$ a vector bundle. Show that

(a) if $V \longrightarrow N$ is a trivial vector bundle, then so is $f^*V \longrightarrow M$;

(b) $f^*V \longrightarrow M$ may be trivial even if $V \longrightarrow N$ is not.

14. Let $f\colon M \longrightarrow N$ be a smooth map. Show that the bundle homomorphisms in diagrams (10.4) and (10.5) are indeed smooth.

15. Verify Lemma 10.2.

16. Let $f\colon M \longrightarrow N$ be a smooth map and $\varphi\colon V \longrightarrow W$ a smooth vector-bundle homomorphism over N. Show that the pullback vector-bundle homomorphism $f^*\varphi\colon f^*V \longrightarrow f^*W$ is also smooth.

17. Let $\pi\colon V \longrightarrow M$ be a smooth vector bundle of rank k and $V' \subset V$ a smooth submanifold so that $V'_p \equiv V_p \cap V'$ is a k'-dimensional linear subspace of V_p for every $p\in M$. Show that

(a) for every $p \in M = s_0(M)$ there exist an open neighborhood U of p in V' and smooth charts

$$\varphi\colon U \longrightarrow \mathbb{R}^m \times \mathbb{R}^{k'} \quad \text{and} \quad \psi\colon U\cap M \longrightarrow \mathbb{R}^m \qquad \text{s.t.} \qquad \psi\circ\pi = \pi_1\circ\varphi\,,$$

where $\pi_1\colon \mathbb{R}^m\times\mathbb{R}^{k'} \longrightarrow \mathbb{R}^m$ is the projection on the first component;

(b) $V' \subset V$ is a vector subbundle of rank k'.

18. Let $\varphi\colon V \longrightarrow W$ be a smooth surjective vector-bundle homomorphism over a smooth manifold M. Show that

$$\ker\varphi \equiv \big\{v\in V\colon\ \varphi(v)=0\big\} \longrightarrow M$$

is a subbundle of V.

19. Let $\mathcal{D} \subset TM$ a rank 1 distribution on a smooth manifold M. Show that $\Gamma(M;\mathcal{D}) \subset \mathrm{VF}(M)$ is a Lie subalgebra. *Hint:* use Exercise 5.

20. Let $\{\iota_\alpha\colon M_\alpha \longrightarrow N\}_{\alpha\in\mathcal{A}}$ be a foliation of N^n by immersions from m-manifolds. Show that

$$\mathcal{D} \equiv \bigcup_{\alpha\in\mathcal{A}}\bigcup_{p\in M_\alpha} \mathrm{Im}\,\mathrm{d}_p\iota_\alpha \subset TN$$

is a subbundle of rank m.

21. Verify all claims made in Examples 10.6 and 10.7.

22. Verify all claims made in Example 10.8.

23. Let $\pi\colon V \longrightarrow M$ be a smooth vector bundle. Show that

(a) the fibers of π form a proper foliation of V;

(b) the corresponding subbundle $\mathcal{D} \subset TV$ is isomorphic to π^*V as vector bundles over V;

(c) there is a short exact sequence of vector bundles

$$0 \longrightarrow \pi^*V \longrightarrow TV \xrightarrow{\mathrm{d}\pi} \pi^*TM \longrightarrow 0$$

24. Let $V \longrightarrow M$ be a vector bundle of rank k and $V' \subset V$ a smooth subbundle of rank k'. Show that

(a) there exists a collection $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for V covering M so that (10.9) holds and thus the corresponding transition data has the form

$$g_{\alpha\beta} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{R},$$

where the top left block is $k' \times k'$;

(b) the vector-bundle structure on V/V' described in Section 10 is the unique one so that the natural projection map $V \longrightarrow V/V'$ is a smooth vector-bundle homomorphism;

(c) if $\varphi\colon V \longrightarrow W$ is a vector-bundle homomorphism over M such that $\varphi(v)=0$ for all $v \in V'$, then the induced vector-bundle homomorphism $\bar{\varphi}\colon V/V' \longrightarrow W$ is smooth.

25. Verify Lemmas 10.9 and 10.10.

26. Obtain Corollary 10.12 from Theorem 10.11.

27. Let $f=(f_1,\ldots,f_k)\colon \mathbb{R}^m \longrightarrow \mathbb{R}^k$ be a smooth map, $q \in \mathbb{R}^k$ a regular value of f, and $X=f^{-1}(q)$. Denote by ∇f_i the gradient of f_i. Show that

$$TX = \big\{(p,v) \in X \times \mathbb{R}^m\colon\ \nabla f_i|_p \cdot v = 0 \ \forall\, i=1,2,\ldots,k\big\}$$

under the canonical identifications $TX \subset T\mathbb{R}^m|_X$ and $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$. Use this description of TX to give a trivialization of $\mathcal{N}_{\mathbb{R}^m}X$.

28. Let $V, V' \longrightarrow M$ be smooth vector bundles. Show that the two constructions of $V \oplus V'$ in Section 10 produce the same vector bundle and that this is the unique vector-bundle structure on the total space of

$$V \oplus V' = \bigsqcup_{p\in M} V_p \oplus V'_p$$

so that

(VB$\bigoplus$1) the projection maps $V \oplus V' \longrightarrow V, V'$ are smooth bundle homomorphisms over M;

(VB$\bigoplus$2) the inclusion maps $V, V' \longrightarrow V \oplus V'$ are smooth bundle homomorphisms over M.

29. Let $\pi_V\colon V \longrightarrow M$ and $\pi_W\colon W \longrightarrow N$ be smooth vector bundles and $\pi_M, \pi_N\colon M \times N \longrightarrow M, N$ the component projection maps. Show that the total of the vector bundle

$$\pi\colon \pi_M^*V \oplus \pi_N^*W \longrightarrow M \times N$$

is $V \times W$ (with the product smooth structure) and $\pi = \pi_V \times \pi_W$.

30. Verify Lemmas 10.13 and 10.14.

31. Let M and N be smooth manifolds and $\pi_M, \pi_N\colon M\times N \longrightarrow M, N$ the projection maps. Show that $\mathrm{d}\pi_M$ and $\mathrm{d}\pi_N$ viewed as maps from $T(M\times N)$ to

(a) TM and TN, respectively, induce a diffeomorphism $T(M\times N)\longrightarrow TM\times TN$ that commutes with the projections from the tangent bundles to the manifolds and is linear on the fibers of these projections;

(b) π_M^*TM and π_N^*TN, respectively, induce a vector-bundle isomorphism

$$T(M\times N) \longrightarrow \pi_M^*TM \oplus \pi_N^*TN.$$

Why are the above two statements the same?

32. Verify Lemmas 10.15 and 10.16.

33. Show that the vector-bundle structure on the total space of V^* constructed in Section 10 is the unique one so that the map (10.15) is smooth.

34. Verify Lemmas 10.17-10.19.

35. Let $V \longrightarrow M$ be a smooth vector bundle of rank k and $W \subset V$ a smooth subbundle of V of rank k'. Show that

$$\mathrm{Ann}(W) \equiv \big\{\alpha \in V_p^*\colon\ \alpha(w)=0\ \forall\, w\in W,\ p\in M\big\}$$

is a smooth subbundle of V^* of rank $k-k'$.

36. Verify Lemmas 10.20-10.23.

37. Let $\pi\colon V\longrightarrow M$ be a vector bundle. Show that there is an isomorphism

$$\Lambda^k(V^*) \longrightarrow \big(\Lambda^k V\big)^*$$

of vector bundles over M.

38. Let Ω be a volume form an m-manifold M. Show that for every $p\in M$ there exists a chart $(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m$ around p such that

$$\Omega|_U = \mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_m.$$

39. Show that every complex vector bundle $V\longrightarrow M$ admits a Hermitian metric.

40. Let $\pi\colon L\longrightarrow M$ be a real line bundle over a smooth manifold. Show that $L^{\otimes 2}\approx\tau_1^{\mathbb{R}}$ as real line bundles over M.

41. Let $V, W\longrightarrow M$ be vector bundles. Show that

(a) if V is orientable, then W is orientable if and only if $V\oplus W$ is;

(b) if V and W are non-orientable, then $V\oplus W$ may be orientable or non-orientable.

42. Let M be a connected manifold. Show that every real line bundle $L\longrightarrow M$ is orientable if and only if $\pi_1(M)$ contains no subgroup of index 2.

43. Let M and N be nonempty smooth manifolds. Show that $M\times N$ is orientable if and only if M and N are.

44. (a) Let $\varphi\colon M\longrightarrow\mathbb{R}^N$ be an immersion. Show that M is orientable if and only if the normal bundle to the immersion φ is orientable.

(b) Show that the unit sphere S^n with its natural smooth structure is orientable.

45. Verify Lemmas 12.3 and 12.4.

46. (a) Show that the antipodal map on $S^n\subset\mathbb{R}^{n+1}$ (i.e. $x\longrightarrow -x$) is orientation-preserving if n is odd and orientation-reversing if n is even.

(b) Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.

(c) Describe the orientable double cover of $\mathbb{R}P^n\times\mathbb{R}P^n$ with n even.

47. Let $\gamma_n\longrightarrow\mathbb{C}P^n$ be the tautological line bundle as in Example 7.8. If $P:\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ is a homogeneous polynomial of degree $d\geq 0$, let

$$s_P\colon\mathbb{C}P^n\longrightarrow\gamma_n^*,\qquad \big\{s_P(\ell)\big\}(\ell,v^{\otimes d})=P(v)\ \ \forall\,(\ell,v)\in\gamma_n\subset\mathbb{C}P^n\times\mathbb{C}^{n+1}.$$

Show that

(a) s_P is a well-defined holomorphic section of $\gamma_n^{*\otimes d}$;

(b) if s is a holomorphic section of $\gamma_n^{*\otimes d}$ with $d\geq 0$, then $s=s_P$ for some homogeneous polynomial $P\colon\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ of degree d;

(c) the line bundle $\gamma_n^{\otimes}\longrightarrow\mathbb{C}P^n$ admits no nonzero holomorphic section for any $d\in\mathbb{Z}^+$.

48. Let $\gamma_n\longrightarrow\mathbb{C}P^n$ be the tautological line bundle as in Example 7.8. Show that there is a short exact sequence

$$0\longrightarrow\mathbb{C}P^n\times\mathbb{C}\longrightarrow(n+1)\gamma_n^*\longrightarrow T\mathbb{C}P^n\longrightarrow 0$$

of complex (even holomorphic) vector bundles over $\mathbb{C}P^n$.

49. Suppose $k<n$ and let $\gamma_k\longrightarrow\mathbb{C}P^k$ be the tautological line bundle as in Example 7.8. Show that the map

$$\iota\colon\mathbb{C}P^k\longrightarrow\mathbb{C}P^n,\qquad [X_0,\ldots,X_k]\longrightarrow[X_0,\ldots,X_k,\underbrace{0,\ldots,0}_{n-k}],$$

is a complex embedding (i.e. a smooth embedding that induces holomorphic maps between the charts that determine the complex structures on $\mathbb{C}P^k$ and $\mathbb{C}P^n$) and that the normal bundle to this immersion, $\mathcal{N}_\iota$, is isomorphic to

$$(n-k)\gamma_k^*\equiv\underbrace{\gamma_k^*\oplus\ldots\oplus\gamma_k^*}_{n-k}$$

as a complex (even holomorphic) vector bundle over $\mathbb{C}P^k$. *Hint:* there are a number of ways of doing this, including:

(i) use Exercise 48;

(ii) construct an isomorphism between the two vector bundles;

(iii) determine transition data for $\mathcal{N}_\iota$ and $(n-k)\gamma_k^*$;

(iv) show that there exists a holomorphic diffeomorphism between $(n-k)\gamma_k^*$ and a neighborhood of $\iota(\mathbb{C}P^k)$ in $\mathbb{C}P^n$, fixing $\iota(\mathbb{C}P^k)$, and that this implies that $\mathcal{N}_\iota = (n-k)\gamma_k^*$.

50. Let $\gamma_n \longrightarrow \mathbb{C}P^n$ and $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ be the tautological line bundle as in Example 7.8 and the top exterior power of the vector bundle $T\mathbb{C}P^n$ taken over $\mathbb{C}$, respectively. Show that there is an isomorphism

$$\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \approx {\gamma_n^*}^{\otimes(n+1)} \equiv \underbrace{\gamma_n^* \otimes \ldots \otimes \gamma_n^*}_{n+1}$$

of complex (even holomorphic) line bundles over $\mathbb{C}P^n$. *Hint:* see suggestions for Exercise 49.

Chapter 3

Frobenius Theorems

13 Integral Curves

Recall from Section 8 that a vector field X on a smooth manifold M is a section of the tangent bundle $TM \longrightarrow M$. Thus, $X: M \longrightarrow TM$ is a map such that $X(p) \in T_pM$ for all $p \in M$. If $\varphi = (x_1, \ldots, x_m): U \longrightarrow M$ is a smooth chart on M, then

$$X(p) = \sum_{i=1}^{i=m} c_i(p) \frac{\partial}{\partial x_i}\bigg|_p \qquad \forall\, p \in U,$$

for some functions $c_1, \ldots, c_m: U \longrightarrow \mathbb{R}$. The vector field X is smooth (as a map between the smooth manifolds M and TM) if and only if the functions $c_1, \ldots, c_m$ corresponding to every smooth chart on M are smooth. This is the case if and only if $X(f): M \longrightarrow \mathbb{R}$ is smooth function for every $f \in C^\infty(M)$.

As defined in Section 4, a smooth curve on M is a smooth map $\gamma: (a,b) \longrightarrow M$. For $t \in (a,b)$, the tangent vector to a smooth curve γ at t is the vector

$$\gamma'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \equiv \mathrm{d}_t\gamma\big(\partial_{e_1}|_t\big) \in T_{\gamma(t)}M,$$

where $e_1 = 1 \in \mathbb{R}^1$ is the oriented unit vector.

Definition 13.1. *Let X be a smooth vector field on a smooth manifold M. An* integral curve for X *is a smooth curve*

$$\gamma: (a,b) \longrightarrow M \qquad s.t. \qquad \gamma'(t) = X(\gamma(t)) \quad \forall\, t \in (a,b). \tag{13.1}$$

For example, a smooth vector field X on $\mathbb{R}^2$ has the form

$$X(x,y) = f(x,y)\frac{\partial}{\partial x}\bigg|_{(x,y)} + g(x,y)\frac{\partial}{\partial y}\bigg|_{(x,y)}$$

for some $f, g \in C^\infty(\mathbb{R}^2)$. A smooth map $\gamma = (\gamma_1, \gamma_2): (a,b) \longrightarrow \mathbb{R}^2$ is an integral curve for such a vector field if

$$\begin{aligned} \gamma'(t) &= \frac{\partial\gamma_1(t)}{\partial t}\frac{\partial}{\partial x}\bigg|_{\gamma(t)} + \frac{\partial\gamma_2(t)}{\partial t}\frac{\partial}{\partial y}\bigg|_{\gamma(t)} \\ &= f(\gamma(t))\frac{\partial}{\partial x}\bigg|_{(\gamma(t))} + g(\gamma(t))\frac{\partial}{\partial y}\bigg|_{(\gamma(t))} \end{aligned} \qquad \Longleftrightarrow \qquad \begin{cases} \gamma_1'(t) = f\big(\gamma_1(t), \gamma_2(t)\big) \\ \gamma_2'(t) = g\big(\gamma_1(t), \gamma_2(t)\big) \end{cases}$$

This is a system of two ordinary autonomous first-order differential equations for $\gamma=(\gamma_1,\gamma_2)$ as a function of t.

Lemma 13.2. *Let X be a smooth vector field on a smooth manifold M. For every $p\in M$, there exists an integral curve $\gamma\colon(-\epsilon,\epsilon)\longrightarrow M$ for X such that $\gamma(0)=p$. If $\gamma,\tilde\gamma:(-\epsilon,\epsilon)\longrightarrow M$ are two such integral curves, then $\gamma=\tilde\gamma$.*

Let $\varphi=(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m$ be a smooth chart on M around p and $c_1,\ldots,c_m\colon U\longrightarrow\mathbb{R}$ smooth functions such that

$$X(p')=\sum_{i=1}^{i=m}c_i(p')\frac{\partial}{\partial x_i}\bigg|_{p'}\qquad\forall\,p'\in U.$$

For any smooth map $\gamma\colon(a,b)\longrightarrow U\subset M$, let

$$(\gamma_1,\ldots,\gamma_m)=\varphi\circ\gamma\colon(a,b)\longrightarrow\mathbb{R}^m\,.$$

By the chain rule (4.5) and the definition of the coordinate vector fields (4.12), the condition (13.1) on γ is then equivalent to

$$\{\varphi\circ\gamma\}'(t)=\mathrm{d}_t\varphi\big(\gamma'(t)\big)=\sum_{i=1}^{i=m}c_i\big(\gamma(t)\big)\frac{\partial}{\partial x_i}\bigg|_{\varphi(\gamma(t))}\qquad\Longleftrightarrow\qquad\gamma_i'(t)=c_i\circ\varphi^{-1}\big(\gamma_1(t),\ldots,\gamma_m(t)\big)\ \ \forall\,i.$$

Since the functions $c_i\circ\varphi^{-1}$ are smooth on $\mathbb{R}^m$, the initial-value problem

$$\begin{cases}\gamma_i'(t)=c_i\circ\varphi^{-1}\big(\gamma_1(t),\ldots,\gamma_m(t)\big) & i=1,2,\ldots,m\\ \big(\gamma_1(0),\ldots,\gamma_m(0)\big)=\varphi(p)\end{cases}\tag{13.2}$$

has a solution $(\gamma_1,\ldots,\gamma_m)\colon(-\epsilon,\epsilon)\longrightarrow\mathbb{R}^m$ for some $\epsilon>0$ by the *Existence Theorem for First-Order Differential Equations* [1, A.2]. By the *Uniqueness Theorem for First-Order Differential Equations* [1, A.1], any two solutions of this initial-value problem must agree on the intersection of the domains of their definition.

Corollary 13.3. *Let X be a smooth vector field on a smooth manifold M and $p\in M$. If $a,\tilde a\in\mathbb{R}^-$, $b,\tilde b\in\mathbb{R}^+$, and $\gamma\colon(a,b)\longrightarrow M$ and $\tilde\gamma\colon(\tilde a,\tilde b)\longrightarrow M$ are integral curves for X, then*

$$\gamma|_{(a,b)\cap(\tilde a,\tilde b)}=\tilde\gamma|_{(a,b)\cap(\tilde a,\tilde b)}\,.$$

The subset

$$A\equiv\big\{t\in(a,b)\cap(\tilde a,\tilde b)\colon\ \gamma(t)=\tilde\gamma(t)\big\}\subset(a,b)\cap(\tilde a,\tilde b)$$

is nonempty (as it contains 0) and closed (as γ and $\tilde\gamma$ are continuous). Since $(a,b)\cap(\tilde a,\tilde b)$ is connected, it is sufficient to show that S is open. If

$$t_0\in S\qquad\text{and}\qquad(t_0-\epsilon,t_0+\epsilon)\subset(a,b)\cap(\tilde a,\tilde b),$$

define smooth curves

$$\alpha,\beta\colon(-\epsilon,\epsilon)\longrightarrow M\qquad\text{by}\qquad\alpha(t)=\gamma(t+t_0),\quad\beta(t)=\tilde\gamma(t+t_0).$$

Since γ and $\tilde\gamma$ are integral curves for X,

$$\alpha'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t+t_0)=X\big(\gamma(t+t_0)\big)=X\big(\alpha(t)\big),\qquad\beta'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\tilde\gamma(t+t_0)=X\big(\tilde\gamma(t+t_0)\big)=X\big(\beta(t)\big).$$

Thus, α and β are integral curves for X. Since

$$\alpha(0) = \gamma(t_0) = \tilde{\gamma}(t_0) = \beta(0),$$

$\alpha = \beta$ by Lemma 13.2 and thus $(t_0 - \epsilon, t_0 + \epsilon) \subset (a, b) \cap (\tilde{a}, \tilde{b})$.

Corollary 13.4. *Let X be a smooth vector field on a smooth manifold M. For every $p \in M$, there exists a unique maximal integral curve $\gamma_p \colon (a_p, b_p) \longrightarrow M$ for X such that $\gamma_p(0) = p$, where $a_p \in [-\infty, 0)$ and $b_p \in (0, \infty]$. If $t \in (a_p, b_p)$, then*

$$\big(a_{\gamma_p(t)}, b_{\gamma_p(t)}\big) = (a_p - t, b_p - t), \qquad \gamma_{\gamma_p(t)}(-t) = p. \tag{13.3}$$

(1) Let $\{\gamma_\alpha \colon (a_\alpha, b_\alpha) \longrightarrow M\}_{\alpha \in \mathcal{A}}$ be the collection of all integral curves for X such that $\gamma_\alpha(0) = p$. Define

$$(a_p, b_p) = \bigcup_{\alpha \in \mathcal{A}} (a_\alpha, b_\alpha), \qquad \gamma_p \colon (a_p, b_p) \longrightarrow M, \quad \gamma_p(t) = \gamma_\alpha(t) \quad \forall\, t \in (a_\alpha, b_\alpha),\ \alpha \in \mathcal{A}.$$

By Corollary 13.3, $\gamma_\alpha(t)$ is independent of the choice of $\alpha \in \mathcal{A}$ such that $t \in (a_\alpha, b_\alpha)$. Thus, γ_p is well-defined. It is smooth, since its restriction to each open subset (a_α, b_α) is smooth and these subsets cover (a_p, b_p). It is an integral curve for X, since this is the case on the open subsets (a_α, b_α). It is immediate that $\gamma_p(0) = p$. By construction, γ_p is a maximal integral curve for X.

(2) If $t \in (a_p, b_p)$, define

$$\gamma \colon (a_p - t, b_p - t) \longrightarrow M \qquad \text{by} \qquad \gamma(\tau) = \gamma_p(\tau + t).$$

This is a smooth map such that

$$\gamma(0) = \gamma_p(t), \qquad \gamma(-t) = \gamma_p(0) = p, \qquad \gamma'(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma_p(\tau + t) = X\big(\gamma_p(\tau + t)\big) = X\big(\gamma(t)\big);$$

the second-to-last equality above holds because γ_p is an integral curve for X. Thus, γ is an integral curve for X such that $\gamma(0) = \gamma_p(t)$. In particular, by the first statement of Corollary 13.4,

$$\begin{aligned}
\big(a_{\gamma_p(t)}, b_{\gamma_p(t)}\big) \supset (a_p - t, b_p - t),\ \gamma_{\gamma_p(t)}\big|_{(a_p - t, b_p - t)} = \gamma \quad &\Longrightarrow \quad -t \in \big(a_{\gamma_p(t)}, b_{\gamma_p(t)}\big),\ \gamma_{\gamma_p(t)}(-t) = p \\
&\Longrightarrow \quad (a_p, b_p) = \big(a_{\gamma_{\gamma_p(t)}(-t)}, b_{\gamma_{\gamma_p(t)}(-t)}\big) \supset \big(a_{\gamma_p(t)} + t, b_{\gamma_p(t)} + t\big).
\end{aligned}$$

This confirms (13.3).

If X is a smooth vector field on M, for each $t \in \mathbb{R}$ let

$$\mathrm{Dom}_t(X) = \big\{p \in M \colon t \in (a_p, b_p)\big\}, \qquad \mathbb{X}_t \colon \mathrm{Dom}_t(X) \longrightarrow M, \quad \mathbb{X}_t(p) = \gamma_p(t).$$

The map $\mathbb{X}_t$ is called the time t flow of vector field X.

Example 13.5. Let X be the smooth vector field on $M = \mathbb{R}$ given by

$$X(x) = -x^2 \frac{\partial}{\partial x}\bigg|_x .$$

If $p\in\mathbb{R}$, the integral curve γ_p for X is described by

$$\gamma_p'(t) = -\gamma_p(t)^2, \quad \gamma_p(0) = p \qquad \Longleftrightarrow \qquad \gamma_p(t) = \frac{p}{1+pt}\,.$$

Thus,

$$\mathrm{Dom}_t(X) = \begin{cases} (-\infty,-1/t), & \text{if } t<0; \\ (-\infty,\infty), & \text{if } t=0; \\ (-1/t,\infty), & \text{if } t>0; \end{cases} \qquad \mathbb{X}_t\colon \mathrm{Dom}_t(X) \longrightarrow \mathbb{R}, \quad \mathbb{X}_t(p) = \frac{p}{1+tp}.$$

Example 13.6. Let $q\colon \mathbb{R}\longrightarrow S^1$, $\theta\longrightarrow \mathrm{e}^{2\pi \mathrm{i}\theta}$, be the usual covering map and X the vector field on S^1 defined by

$$X\big(\mathrm{e}^{2\pi\mathrm{i}\theta}\big) = \mathrm{d}_\theta q(e_1),$$

where $e_1=1$ is the usual oriented unit vector in $\mathbb{R}=T_\theta\mathbb{R}$. If $q(\theta_1)=q(\theta_2)$, there exists $n\in\mathbb{Z}$ such that $\theta_2=\theta_1+n$. Define

$$h_n\colon \mathbb{R}\longrightarrow\mathbb{R} \qquad \text{by} \qquad \theta\longrightarrow\theta+n.$$

Since $\mathrm{d}_{\theta_1}h(e_1)=e_1$ and $q=q\circ h_n$, by the chain rule (4.5)

$$\mathrm{d}_{\theta_2}q(e_1) = \mathrm{d}_{\theta_2}q\big(\mathrm{d}_{\theta_1}h_n(e_1)\big) = \mathrm{d}_{\theta_1}\{q\circ h_n\}(e_1) = \mathrm{d}_{\theta_1}q(e_1).$$

Thus, the vector field X is well-defined (the value of X at $\mathrm{e}^{2\pi\mathrm{i}\theta}$ depends only on $\mathrm{e}^{2\pi\mathrm{i}\theta}$, and not on θ). This vector field is smooth, since e_1 defines a smooth vector field on $\mathbb{R}$, while $q\colon\mathbb{R}\longrightarrow S^1$ and $\mathrm{d}q\colon T\mathbb{R}\longrightarrow TS^1$ are covering projections (and in particular local diffeomorphisms). If $p\in S^1$, $\tilde{p}\in q^{-1}(p)\subset\mathbb{R}$, and $\gamma\colon(a,b)\longrightarrow S^1$ is a smooth curve such that $\gamma(0)=p$, let $\tilde{\gamma}\colon(a,b)\longrightarrow\mathbb{R}$ be the continuous lift of γ over q such that $\tilde{\gamma}(0)=\tilde{p}$; since q is a local diffeomorphism, this map is smooth. The integral curve γ_p is then described by

$$\begin{aligned} \tilde{\gamma}_p'(t) = 1, \quad \tilde{\gamma}_p(0)=\tilde{p} \qquad &\Longleftrightarrow \qquad \tilde{\gamma}_p(t) = \tilde{p}+t\in\mathbb{R} \\ &\Longrightarrow \qquad \gamma_p(t) = q\big(\tilde{\gamma}_p(t)\big) = \mathrm{e}^{2\pi\mathrm{i}(\tilde{p}+t)} = \mathrm{e}^{2\pi\mathrm{i}\tilde{p}}\cdot\mathrm{e}^{2\pi\mathrm{i}t} = \mathrm{e}^{2\pi\mathrm{i}t}\cdot p\in S^1\,. \end{aligned}$$

Thus, $\gamma_p(t)$ is defined for all $t\in\mathbb{R}$, and the time t flow of X is given by

$$\mathbb{X}_t\colon \mathrm{Dom}_t(X)=S^1\longrightarrow S^1\,, \qquad p\longrightarrow \mathrm{e}^{2\pi\mathrm{i}t}p.$$

This is the rotation by the angle $2\pi t$.

Proposition 13.7. *If X is a smooth vector field on a smooth manifold M, then*

(1) $\mathrm{Dom}_0(X)=M$, $\mathbb{X}_0=\mathrm{id}_M$, *and*

$$M = \bigcup_{t>0}\mathrm{Dom}_t(X) = \bigcup_{t<0}\mathrm{Dom}_t(X)\,;$$

(2) for all $s,t\in\mathbb{R}$, $\mathbb{X}_{s+t}=\mathbb{X}_s\circ\mathbb{X}_t\colon \mathrm{Dom}(\mathbb{X}_s\circ\mathbb{X}_t)=\mathbb{X}_t^{-1}\big(\mathrm{Dom}_s(X)\big)\longrightarrow M$;

(3) for all $p\in M$, there exist an open neighborhood U of p in M and $\epsilon\in\mathbb{R}^+$ such that the map

$$\mathbb{X}\colon(-\epsilon,\epsilon)\times U\longrightarrow M, \qquad (t,p')\longrightarrow\mathbb{X}_t(p')\equiv\gamma_{p'}(t), \tag{13.4}$$

is defined and smooth;

(4) for all $t\in\mathbb{R}$, $\mathrm{Dom}(X_t)\subset M$ is an open subset;

(5) for all $t\in\mathbb{R}$, $\mathbb{X}_t\colon \mathrm{Dom}_t(X)\longrightarrow \mathrm{Dom}_{-t}(X)$ is a diffeomorphism with inverse $\mathbb{X}_{-t}$.

Proof: (1) By Lemma 13.2, for each $p\in M$ there exists an integral curve $\gamma\colon(-\epsilon,\epsilon)\longrightarrow M$ for X such that $\gamma(0)=p$. Thus, $p\in\mathrm{Dom}_{\pm\epsilon/2}(X)\subset\mathrm{Dom}_0(X)$; this implies the first and last claims in (1). The middle claim follows from the requirement that $\mathbb{X}_0(p)\equiv\gamma_p(0)=p$ for all $p\in M$.

(2) Since $\mathrm{Dom}(\mathbb{X}_s)=\mathrm{Dom}_s(X)$, $\mathrm{Dom}(\mathbb{X}_s\circ\mathbb{X}_t)=\mathbb{X}_t^{-1}(\mathrm{Dom}_s(X))$. If $p\in\mathbb{X}_t^{-1}(\mathrm{Dom}_s(X))$,

$$s\in\big(a_{\mathbb{X}_t(p)},b_{\mathbb{X}_t(p)}\big)=\big(a_{\gamma_p(t)},b_{\gamma_p(t)}\big).$$

Thus, $s+t\in(a_p,b_p)$ by (13.3) and $\mathbb{X}_t^{-1}(\mathrm{Dom}_s(X))\subset\mathrm{Dom}(\mathbb{X}_{s+t})$.[1] Define

$$\gamma\colon\big(a_{\mathbb{X}_t(p)},b_{\mathbb{X}_t(p)}\big)\longrightarrow M\qquad\text{by}\qquad\gamma(\tau)=\gamma_p(\tau+t);$$

by (13.3), $\gamma_p(\tau+t)$ is defined for all $\tau\in(a_{\mathbb{X}_t(p)},b_{\mathbb{X}_t(p)})$. The map γ is smooth and satisfies

$$\gamma(0)=\gamma_p(t)=\mathbb{X}_t(p),\qquad\gamma'(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma_p(\tau+t)=X\big(\gamma_p(\tau+t)\big)=X\big(\gamma(\tau)\big);$$

the second-to-last equality holds because γ_p is an integral curve for X. Thus, by Corollary 13.4, $\gamma=\gamma_{\mathbb{X}_t(p)}$ and so

$$\mathbb{X}_{s+t}(p)\equiv\gamma_p(s+t)=\gamma(s)=\gamma_{\mathbb{X}(t)}(s)\equiv\mathbb{X}_s\big(\mathbb{X}_t(p)\big)$$

for all $s\in(a_{\mathbb{X}_t(p)},b_{\mathbb{X}_t(p)})$.

(3) As in the proof of Lemma 13.2, the requirement for a smooth map $\gamma\colon(a,b)\longrightarrow M$ to be an integral curve for X passing through p corresponds to an initial-value problem (13.2) in a smooth chart around p. Thus, the claim follows from the smooth dependence of solutions of (13.2) on the parameters [1, A.4].

(4) Since $\mathrm{Dom}_0(X)=M$ and $\mathrm{Dom}_{-t}(X)=\mathrm{Dom}_t(-X)$, it is sufficient to prove this statement for $t\in\mathbb{R}^+$. Let $p\in\mathrm{Dom}_t(X)$ and $W\subset M$ be an open neighborhood of $\mathbb{X}_t(p)=\gamma_p(t)$ in M. Since the interval $[0,t]$ is compact, by (3) and Lebesgue Number Lemma (Lemma B.1.2), there exist $\epsilon>0$ and a neighborhood U of $\gamma_p([0,t])$ such that the map (13.4) is defined and smooth. Let $n\in\mathbb{Z}^+$ be such that $t/n<\epsilon$. We inductively define subsets $W_i\subset M$ by

$$W_n=W,\qquad W_i=\mathbb{X}_{t/n}^{-1}(W_{i+1})\cap U=\big\{\mathbb{X}_{t/n}|_U\big\}^{-1}(W_{i+1})\quad\forall\,i=0,1,\ldots,n-1.$$

By induction, $W_i\subset U$ is an open neighborhood of $\gamma_p(it/n)$, $W_i\subset\mathbb{X}_{t/n}^{-1}(\mathrm{Dom}(\mathbb{X}_{(n-1-i)t/n}))$, and thus

$$\mathbb{X}_{(n-i)t/n}=\mathbb{X}_{t/n}\circ\mathbb{X}_{(n-1-i)t/n}\colon W_i\longrightarrow U\subset M$$

by (2). It follows that $W_0\subset M$ is an open neighborhood of p in M such that $W_0\subset\mathrm{Dom}_t(X)$.

(5) By (13.3) and (2), $\mathrm{Im}\,\mathbb{X}_t=\mathrm{Dom}_{-t}(X)$ and $\mathbb{X}_{-t}$ is the inverse of $\mathbb{X}_t$. If $p\in\mathrm{Dom}_t(X)$ and W_0 is a neighborhood of p in M as in the proof of (4), $\mathbb{X}_t|_{W_0}$ is a smooth map. Thus, $\mathbb{X}_t$ is smooth on the open subset $\mathrm{Dom}_t(X)\subset M$.

[1] The domain of $\mathbb{X}_{s+t}$ might be larger than $\mathrm{Dom}(\mathbb{X}_s\circ\mathbb{X}_t)$. For example, if $s=-t$, the former is all of M.

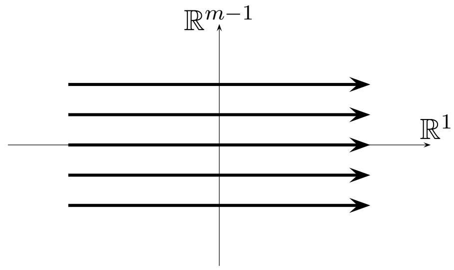


Figure 3.1: Flows of a nonvanishing vector field and integral immersions of a rank 1 distribution are horizontal slices in a coordinate chart.

Lemma 13.8. *Let X be a smooth vector field on a smooth manifold M and $p\in M$. If $X(p)\neq 0$, there exists a smooth chart*

$$\varphi\equiv(x_1,\ldots,x_m)\colon U\longrightarrow(-\delta,\delta)\times\mathbb{R}^{m-1}$$

around p on M such that

$$X(p')=\frac{\partial}{\partial x_1}\bigg|_{p'}\qquad\forall\,p'\in U. \tag{13.5}$$

By Proposition 13.7-(3), there exist an open neighborhood U of p in M and $\epsilon\in\mathbb{R}^+$ such that the map (13.4) is smooth. Let U' be a neighborhood of p in U and

$$\phi\equiv(y_1,\ldots,y_m)\colon(U',p)\longrightarrow(\mathbb{R}^m,\mathbf{0})$$

a smooth chart such that (13.5) holds with x_1 replaced by y_1 for $p'=p$; such a chart can be obtained by composing another chart with a rigid transformation of $\mathbb{R}^m$. Define

$$\psi\colon(-\epsilon,\epsilon)\times\mathbb{R}^{m-1}\longrightarrow M\qquad\text{by}\qquad\psi(x_1,x_2,\ldots,x_m)=\mathbb{X}_{x_1}\big(\phi^{-1}(0,x_2,\ldots,x_m)\big).$$

This smooth map sends $(x_1,x_2,\ldots,x_m)$ to the time x_1-flow from the point $\phi^{-1}(0,x_2,\ldots,x_m)$ on the coordinate hyperplane $\phi^{-1}(0\times\mathbb{R}^{m-1})$; see Figure 3.1. Note that

$$\begin{aligned}
\mathrm{d}_{\mathbf{0}}\psi\big(\partial_{e_1}|_{\phi(p)}\big)&=\frac{\mathrm{d}}{\mathrm{d}t}\psi(t,0,\ldots,0)\bigg|_{t=0}=\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{X}_t(p)\bigg|_{t=0}=X(p)=\frac{\partial}{\partial y_1}\bigg|_p;\\
\mathrm{d}_{\mathbf{0}}\psi\big(\partial_{e_i}|_{\phi(p)}\big)&=\frac{\mathrm{d}}{\mathrm{d}t}\psi(0,\ldots,0,t,0,\ldots,0)\bigg|_{t=0}=\frac{\mathrm{d}}{\mathrm{d}t}\phi^{-1}(0,\ldots,0,t,0,\ldots,0)\bigg|_{t=0}=\frac{\partial}{\partial y_i}\bigg|_p\quad\forall\,i\geq 2;
\end{aligned}$$

on the second line, t is inserted into the i-th slot. Thus, the differential of ψ at $\mathbf{0}$,

$$\mathrm{d}_{\mathbf{0}}\psi\colon T_{\mathbf{0}}\mathbb{R}^m\longrightarrow T_pM$$

is an isomorphism. By the *Inverse Function Theorem for Manifolds* (Corollary 4.9), there are neighborhoods U of p in M and V of $\mathbf{0}$ in $\mathbb{R}^m$ such that $\psi: V\longrightarrow U$ is a diffeomorphism. The inverse of this diffeomorphism is a smooth chart around p on M satisfying (13.5).

Corollary 13.9. *If $\mathcal{D}\subset TM$ is a rank 1 distribution on a smooth manifold M, there exists a foliation $\{\iota_\alpha\colon\mathbb{R}\longrightarrow M\}_{\alpha\in\mathcal{A}}$ by injective immersions integral to the distribution $\mathcal{D}$ on M.*

Let $h\colon \mathcal{D}|_W \longrightarrow W\times\mathbb{R}$ be a trivialization of $\mathcal{D}$ over an open subset $W\subset M$ and

$$X(p) = h^{-1}(p,1) \in \mathcal{D}_p \subset T_pM \qquad \forall\, p\in W.$$

Since h is a smooth, X is a smooth nowhere 0 vector field on the open subset $W\subset M$. Since the rank of $\mathcal{D}$ is 1, $\mathcal{D}_p=\mathbb{R}X(p)$ for all $p\in W$. Let

$$\varphi\equiv(x_1,\ldots,x_m): U \longrightarrow (-\delta,\delta)\times\mathbb{R}^{m-1}$$

be a coordinate chart on $W\subset M$ satisfying (13.5) and $\phi:\mathbb{R}\longrightarrow(-\delta,\delta)$ any diffeomorphism. For $y\in\mathbb{R}^{m-1}$, define

$$\iota_y\colon \mathbb{R}\longrightarrow M \qquad \text{by} \qquad \iota_y(t)=\varphi^{-1}\big(\phi(t),0,\ldots,0\big).$$

This is an injective immersion such that $\operatorname{Im}\iota_y$ is contained in (in fact is) the horizontal slice $\varphi^{-1}(\mathbb{R}\times y)$ and

$$\operatorname{Im}\mathrm{d}_t\iota_y = \mathbb{R}\frac{\partial}{\partial x_1}\bigg|_{\iota_y(t)} = \mathbb{R}\,X\big(\iota_y(t)\big) = \mathcal{D}_{\iota_y(t)} \qquad \forall\, t\in\mathbb{R}.$$

Thus, $\{\iota_y\}_{y\in\mathbb{R}^{m-1}}$ is a foliation of the open subset $U\subset M$ by immersions integral to $\mathcal{D}$.

The flows of a vector field X provide a way of differentiating other vector fields and differential forms in the direction of X.

Definition 13.10. *Let X be a smooth vector field on a smooth manifold M and $p\in M$.*

(1) The Lie derivative of a smooth vector field $Y\in\mathrm{VF}(M)$ on M with respect to X at p is the vector

$$(L_XY)_p = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}_{\mathbb{X}_t(p)}\mathbb{X}_{-t}\big(Y(\mathbb{X}_t(p))\big)\bigg|_{t=0} \equiv \lim_{t\longrightarrow0}\frac{\mathrm{d}_{\mathbb{X}_t(p)}\mathbb{X}_{-t}\big(Y(\mathbb{X}_t(p))\big)-Y(p)}{t} \in T_pM\,.$$

(2) The Lie derivative of a smooth k-form $\alpha\in E^k(M)$ on M with respect to X at p is the alternating k-tensor

$$(L_XY)_p = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{X}_t^*\big(\alpha\big(\mathbb{X}_t(p)\big)\big)\bigg|_{t=0} \equiv \lim_{t\longrightarrow0}\frac{\mathbb{X}_t^*\big(\alpha\big(\mathbb{X}_t(p)\big)\big)-\alpha(p)}{t} \in \Lambda^k\big(T_p^*M\big).$$

Thus, the Lie derivative L_X measures the rate of change of a smooth vector field Y at p by bringing $Y(\mathbb{X}_t(p))\in T_{\mathbb{X}_t(p)}M$ back to T_pM by the differential of the inverse flow $\mathbb{X}_{-t}$. Similarly, L_X measures the rate of change of a smooth k-form α at p by pulling $\alpha(\mathbb{X}_t(p))\in\Lambda^k(T^*_{\mathbb{X}_t(p)}M)$ back to $\Lambda^k(T_p^*M)$ by

$$\mathbb{X}_t^* = \Lambda^k\big(\mathrm{d}_p\mathbb{X}_t\big)^*\colon \Lambda^k(T^*_{\mathbb{X}_t(p)}M) \longrightarrow \Lambda^k(T_p^*M).$$

As indicated by the following proposition, $(L_XY)_p$ and $(L_X\alpha)_p$ typically depend on the germ of X at p, and not just on $X(p)$.

Proposition 13.11. *Let X be a smooth vector field on a smooth manifold M and $p\in M$.*

(1) If $f\in C^\infty(M)$, $(L_Xf)_p = X_p(f)$.

(2) If $Y\in\mathrm{VF}(M)$, $(L_XY)_p=[X,Y]$.

(3) If $\alpha\in E^k(M)$ *and* $Y_1,Y_2,\ldots,Y_k\in\mathrm{VF}(M)$,

$$\begin{aligned}\big(L_X\big(\alpha(Y_1,\ldots,Y_k)\big)\big)_p &= \{L_X\alpha\}_p\big(Y_1(p),\ldots,Y_k(p)\big)\\ &\quad+\sum_{i=1}^{i=k}\alpha_p\big(Y_1(p),\ldots,Y_{i-1}(p),(L_XY_i)_p,Y_{i+1}(p),\ldots,Y_k(p)\big).\end{aligned}$$

Corollary 13.12. *If* $X,Y\in\mathrm{VF}(M)$ *are smooth vector fields on a smooth manifold* M,

$$L_{[X,Y]}=\big[L_X,L_Y\big]\equiv L_X\circ L_Y-L_Y\circ L_X\colon \mathrm{VF}(M)\longrightarrow\mathrm{VF}(M),\ \ E^k(M)\longrightarrow E^k(M).$$

Exercises

1. Let V be the vector field on $\mathbb{R}^3$ given by

$$V(x,y,z)=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+\frac{\partial}{\partial z}.$$

Explicitly describe and sketch the flow of V.

2. Let X be a smooth vector field on a manifold M. Show that

(a) if $\gamma\colon(a,b)\longrightarrow M$ is an integral curve for X such that $\gamma'(t)=0$ for some $t\in(a,b)$, then γ is a constant map.

(b) if X is compactly supported, i.e.

$$\mathrm{supp}\,X\equiv\overline{\big\{p\in M\colon\ X_p\neq 0\big\}}$$

is a compact subset of M, then $\mathrm{Dom}_t(X)=M$ for all $t\in\mathbb{R}$.

3. (a) Let M be a smooth compact manifold and $X\in\mathrm{VF}(M)$ a nowhere-zero vector field on M, i.e. $X(p)\neq 0$ for all $p\in M$. Show that the flow $\mathbb{X}_t\colon M\longrightarrow M$ of X has no fixed points for some $t\in\mathbb{R}$.

(b) Show that S^n admits a smooth nowhere-zero vector field if and only if n is odd. *Hint:* Exercises 46 and 12 in Chapter 2 might be helpful for n even.

(c) Show that the tangent bundle of S^n is not trivial if $n\geq 1$ is even. (In fact, TS^n is trivial if and only if $n=1,3,7$ [2].)

4. Let $\gamma\colon(a,b)\longrightarrow\mathbb{R}^2$ be an integral curve for a smooth vector field X on $\mathbb{R}^2$. Show that γ is an embedding.

5. Let X be a smooth vector field on a smooth manifold M. Show that

(a) for $t\in\mathbb{R}$, $\mathrm{Dom}_{-t}(X)=\mathrm{Dom}_t(-X)$;

(b) if $s,t\in\mathbb{R}$ have the same sign, then $\mathrm{Dom}(\mathbb{X}_{s+t})=\mathrm{Dom}(\mathbb{X}_s\circ\mathbb{X}_t)$;

(c) if $\mathrm{Dom}_t(X)=M$ for some $t\in\mathbb{R}^+$, then $\mathrm{Dom}_t(X)=M$ for all $t\in\mathbb{R}^+$.

6. Suppose X and Y are smooth vector fields on a manifold M. Show that for every $p \in M$ and $f \in C^\infty(M)$,

$$\lim_{s,t \longrightarrow 0} \frac{f\big(\mathbb{Y}_{-s}(\mathbb{X}_{-t}\big(\mathbb{Y}_s(\mathbb{X}_t(p))\big))\big) - f(p)}{s\,t} = [X,Y]_p f \in \mathbb{R}.$$

Do not forget to explain why the limit exists.

7. Let X be the vector field on $\mathbb{R}^n$ given by $X = \sum_{i=1}^{i=n} x_i \frac{\partial}{\partial x_i}$.

(a) Determine the time t-flow $\mathbb{X}_t \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of X (give a formula).

(b) Use (a) to show directly from the definition of the Lie derivative L_X that the homomorphism defined by

$$R_k : E^k(\mathbb{R}^n) \longrightarrow E^k(\mathbb{R}^n), \qquad f \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} \longrightarrow \left(\int_0^1 s^{k-1} f(sx) \mathrm{d}s \right) \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

is a left inverse for L_X if $k \geq 1$.

(c) Is R_k also a right inverse for L_X for $k \geq 1$? What happens for $k = 0$?

8. Verify Corollary 13.12.

9. Let U and V be the vector fields on $\mathbb{R}^3$ given by

$$U(x,y,z) = \frac{\partial}{\partial x} \qquad \text{and} \qquad V(x,y,z) = F(x,y,z)\frac{\partial}{\partial y} + G(x,y,z)\frac{\partial}{\partial z},$$

where F and G are smooth functions on $\mathbb{R}^3$. Show that there exists a proper foliation of $\mathbb{R}^3$ by 2-dimensional embedded submanifolds such that the vector fields U and V everywhere span the tangent spaces of these submanifolds if and only if

$$F(x,y,z) = f(y,z)\, e^{h(x,y,z)} \qquad \text{and} \qquad G(x,y,z) = g(y,z)\, e^{h(x,y,z)}$$

for some $f, g \in C^\infty(\mathbb{R}^2)$ and $h \in C^\infty(\mathbb{R}^3)$ such that (f,g) does not vanish on $\mathbb{R}^2$.

10. Let α be a k-form on a smooth manifold M and $X_0, \ldots, X_k \in \mathrm{VF}(M)$. Show that

$$\mathrm{d}\alpha(X_0,\ldots,X_k) = \sum_{i=0}^{i=k} (-1)^i X_i\big(\alpha(X_0,\ldots,\widehat{X_i},\ldots,X_k)\big) + \sum_{i<j} (-1)^{i+j} \alpha\big([X_i,X_j], X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k\big).$$

Hint: first show that the values of both sides at any point $p \in M$ depend only on the values of vector fields $X_i|_p$ at p and on the restriction $\alpha|_U$ of α to any neighborhood U of p; then compute in a smooth chart.

11. Let α be a nowhere-zero closed $(m-1)$-form on an m-manifold M. Show that for every $p \in M$ there exists a chart $(x_1,\ldots,x_m) \colon U \longrightarrow \mathbb{R}^m$ around p such that

$$\alpha|_U = \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \ldots \wedge \mathrm{d}x_m.$$

Hint: Exercises 38 in Chapter 2 and 3 in Appendix A might be helpful.

12. Let ω be a smooth closed everywhere nondegenerate[2] two-form on a smooth manifold M.

(a) Show that the dimension of M is even and the map

$$TM \longrightarrow T^*M, \qquad X \longrightarrow i_X\omega,$$

is a vector-bundle isomorphism (i_X is the contraction w.r.t. X, i.e. the dual of $X\wedge$).

(b) If $H : M \longrightarrow \mathbb{R}$ is a smooth map, let $X_H \in \mathrm{VF}(M)$ be the preimage of $\mathrm{d}H$ under this isomorphism. Assume that the flow

$$\varphi\colon \mathbb{R}\times M \longrightarrow M, \qquad (t,p) \longrightarrow \varphi_t(p)$$

of X_H is defined for all (t,p). Show that for every $t\in\mathbb{R}$, the time-t flow $\varphi_t\colon M \longrightarrow M$ is a symplectomorphism, i.e. $\varphi_t^*\omega=\omega$.

13. Suppose M is a 3-manifold, α is a nowhere-zero one-form on M, and $p\in M$. Show that

(a) if there exists an embedded 2-dimensional submanifold $P\subset M$ such that $p\in P$ and $\alpha|_{TP}=0$, then $(\alpha\wedge \mathrm{d}\alpha)|_p = 0$.

(b) if there exists a neighborhood U of p in M such that $(\alpha\wedge \mathrm{d}\alpha)|_U = 0$, then there exists an embedded 2-dimensional submanifold $P\subset M$ such that $p\in P$ and $\alpha|_{TP}=0$.

14. Let $\alpha=\mathrm{d}x_1 + f\mathrm{d}x_2$ be a smooth 1-form on $\mathbb{R}^3$ (so $f\in C^\infty(\mathbb{R}^3)$). Show that for every $p\in\mathbb{R}^3$ there exists a diffeomorphism

$$\varphi=(y_1,y_2,y_3)\colon U \longrightarrow V$$

from a neighborhood U of p to an open subset V of $\mathbb{R}^3$ such that $\alpha|_U = g\mathrm{d}y_1$ for some $g\in C^\infty(U)$ if and only if f does not depend on x_3.

15. Let X be a non-vanishing vector field on $\mathbb{R}^3$, written in coordinates as

$$X(x,y,z) = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z} \qquad \text{for some} \quad f,g,h \in C^\infty(\mathbb{R}^3).$$

(a) Find a one-form α on $\mathbb{R}^3$ so that at each point of $\mathbb{R}^3$ the kernel of α is orthogonal to X, with respect to the standard inner-product on $\mathbb{R}^3$.

(b) Find a necessary and sufficient condition on X so that for every point $p\in\mathbb{R}^3$ there exists a surface $S\subset\mathbb{R}^3$ passing through p which is everywhere orthogonal to X (i.e. S is a smooth two-dimensional submanifold of $\mathbb{R}^3$ and $T_qS\subset T_q\mathbb{R}^3$ is orthogonal to $X(q)$ for all $q\in S$).

[2] This means that $\omega_p\in\Lambda^2T_p^*M$ is nondegenerate for every $p\in M$, i.e. for every $v\in T_pM-0$ there exists $v'\in T_pM$ such that $\omega_p(v,v')\neq 0$.

Appendix A

Linear Algebra

A.1

Exercises

1. Let V be a finite-dimensional vector space and $v \in V - 0$. Show that

(a) if $w \in \Lambda^k V$, $v \wedge w = 0 \in \Lambda^{k+1} V$ if and only if $w = v \wedge u$ for some $u \in \Lambda^{k-1} V$;

(b) the sequence of vector spaces

$$0 \longrightarrow \Lambda^0 V \xrightarrow{v \wedge \cdot} \Lambda^1 V \xrightarrow{v \wedge \cdot} \Lambda^2 V \xrightarrow{v \wedge \cdot} \dots$$

is exact.

2. Let V be a vector space of dimension n and $\omega \in \Lambda^2 V$ an element such that $\omega^n \neq 0 \in \Lambda^{2n} V$. Show that the homomorphism

$$\omega^k \wedge \cdot : \Lambda^{n-k} V \longrightarrow \Lambda^{n+k} V, \qquad w \longrightarrow \omega^k \wedge w,$$

is an isomorphism for all $k \in \mathbb{Z}^+$.

3. Let V be a vector space of dimension n and $\Omega \in \Lambda^n V^*$ a nonzero element. Show that the homomorphism

$$V \longrightarrow \Lambda^{n-1} V^*, \qquad v \longrightarrow i_v \Omega,$$

where i_v is the contraction map, is an isomorphism.

4. Show that every short exact sequence of vector spaces,

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{f} C \longrightarrow 0$$

induces a canonical isomorphism $\Lambda^{\mathrm{top}} A \otimes \Lambda^{\mathrm{top}} C \longrightarrow \Lambda^{\mathrm{top}} B$ (the isomorphism is determined by f and g).

5. Let $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_k\}$ be $\mathbb{C}$-bases for a vector space V over $\mathbb{C}$. Let A be the complex change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$ and B the real change-of-basis matrix from $\{e_1, \mathrm{i}e_1, \dots, e_k, \mathrm{i}e_k\}$ to $\{f_1, \mathrm{i}f_1, \dots, f_k, \mathrm{i}f_k\}$. Show that

$$\det B = (\det A)\overline{\det A}.$$

Appendix B

Topology

B.1

Lemma B.1.1. *Let M be a set and $\{\varphi_\alpha : U_\alpha \longrightarrow M_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of bijections from subsets U_α of M to topological spaces M_α such that*

$$\varphi_\alpha\circ\varphi_\beta^{-1} : \varphi_\beta\big(U_\alpha\cap U_\beta\big) \longrightarrow \varphi_\alpha\big(U_\alpha\cap U_\beta\big)$$

is a homeomorphism between open subsets of M_β and M_α, respectively, for all $\alpha,\beta\in\mathcal{A}$. If the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M, then M admits a unique topology $\mathcal{T}_M$ such that each map φ_α is a homeomorphism. If in addition

(1) *the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ separates points in M, then the topology $\mathcal{T}_M$ is Hausdorff;*

(2) *there exists a countable subset $\mathcal{A}_0\subset\mathcal{A}$ such that the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}_0}$ covers M and M_α is second-countable for all $\alpha\in\mathcal{A}_0$, then the topology $\mathcal{T}_M$ is second-countable.*

A basis for the topology $\mathcal{T}_M$ consists of the subsets $U\subset M$ such that $U\subset U_\alpha$ and $\varphi_\alpha(U)\subset M_\alpha$ is open for some $\alpha\in\mathcal{A}$.

Lemma B.1.2 (Lebesgue Number Lemma,[7, Lemma 27.5])**.** *Let (M,d) be a compact metric space. For every open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M, there exists $\delta\in\mathbb{R}$ with the property that for every subset $S\subset M$ with $\mathrm{diam}_d(S)<\epsilon$ there exists $\alpha\in\mathcal{A}$ such that $S\subset U_\alpha$.*

B.2 Fundamental Group and Covering Projections

Exercises

1. Show that every Hausdorff locally Euclidean space is regular.

2. Show that every regular second-countable space is normal.

3. Verify Lemma B.1.1.

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