MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 8

Problem 1 (15pts)

Suppose X is a topological space and $\mathcal{P}=\{S_U;\rho_{U,V}\}$ is a presheaf on X. Let

$$\bar{S}_U=\Big\{(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}: U_\alpha\subset U \text{ open},\ U=\bigcup_{\alpha\in\mathcal{A}}U_\alpha;\ f_\alpha\in S_{U_\alpha};$$
$$\forall\,\alpha,\beta\in\mathcal{A},\ p\in U_\alpha\cap U_\beta\ \exists\,W\subset U_\alpha\cap U_\beta \text{ open s.t. } p\in W,\ \rho_{W,U_\alpha}f_\alpha=\rho_{W,U_\beta}f_\beta\Big\}\big/\sim,$$

where $\quad (U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}\sim(U'_{\alpha'},f'_{\alpha'})_{\alpha'\in\mathcal{A}'}$ *if* $\ \forall\,\alpha\in\mathcal{A},\ \alpha'\in\mathcal{A}',\ p\in U_\alpha\cap U'_{\alpha'}$
$$\exists\,W\subset U_\alpha\cap U'_{\alpha'} \text{ s.t. } p\in W,\ \rho_{W,U_\alpha}f_\alpha=\rho_{W,U'_{\alpha'}}f'_{\alpha'}.$$

Whenever $U\subset V$ are open subsets of X, the homomorphisms $\rho_{U,V}$ induce homomorphisms

$$\bar\rho_{U,V}:\bar{S}_V\longrightarrow\bar{S}_U,\qquad \big[(V_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}\big]\longrightarrow\big[(V_\alpha\cap U,\rho_{V_\alpha\cap U,V_\alpha}f_\alpha)_{\alpha\in\mathcal{A}}\big],$$

so that $\bar{\mathcal{P}}\equiv\{\bar{S}_X;\bar\rho_{U,V}\}$ is a presheaf on X. Show that

(a) $\bar{\mathcal{P}}=\alpha(\beta(\mathcal{P}))$;

(b) *the presheaf homomorphism $\{\varphi_U\}:\mathcal{P}\longrightarrow\bar{\mathcal{P}}$*

$$\varphi_U: S_U\longrightarrow\bar{S}_U,\qquad f\longrightarrow\big[(U,f)\big],$$

is injective (resp. an isomorphism) if and only if $\mathcal{P}$ satisfies 5.7(C_1) (resp. is complete);

(c) *if $\mathcal{R}$ is a subsheaf of $\mathcal{S}$, then $\alpha(\mathcal{S}/\mathcal{R})\approx\overline{\alpha(\mathcal{S})/\alpha(\mathcal{R})}$.*

(a) By definition, $\alpha(\beta(\mathcal{P}))$ is the presheaf of continuous sections of the sheaf $\pi:\beta(\mathcal{P})\longrightarrow X$, i.e.

$$S_U=\Gamma\big(U;\beta(\mathcal{P})\big),\qquad S_V\longrightarrow S_U,\quad f\longrightarrow f|_U\,,$$

whenever $U\subset V$ are open subsets of X. If $f:U\longrightarrow\beta(\mathcal{P})$ is any section, $f(p)\in\beta(\mathcal{P})_p$ and so $f(p)=\rho_{p,U_p}(f_p)$ for some neighborhood U_p of p in U and $f_p\in S_{U_p}$. If in addition s is continuous,

$$f^{-1}(\mathcal{O}_{f_p})\equiv\big\{q\in U_p:\ f(q)=\rho_{q,U_p}(f_p)\big\}$$

is an open neighborhood of p in U. On the other hand, if $q\in U_p\cap U_{p'}$ and $\rho_{q,U_p}(f_p)=\rho_{q,U_{p'}}(f_{p'})$, then there exists a neighborhood W of q in $U_p\cap U_{p'}$ such that $\rho_{W,U_p}(f_p)=\rho_{W,U_{p'}}(f_{p'})$. Thus, for every $f\in\Gamma(U;\beta(\mathcal{P}))$, there exists a tuple $(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}$ such that $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of U, $f_\alpha\in S_{U_\alpha}$, for all $\alpha,\beta\in\mathcal{A}$ and $p\in U_\alpha\cap U_\beta$ there exists a neighborhood W of p in $U_\alpha\cap U_\beta$ such that $\rho_{W,U_\alpha}f_\alpha=\rho_{W,U_\beta}f_\beta$, and

$$f(p)=\rho_{p,U_\alpha}(f_\alpha)\qquad\forall\,q\in U_\alpha,\ \alpha\in\mathcal{A}.$$

Conversely, any collection $(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}$ with $f_\alpha\in S_{U_\alpha}$, covering U, and satisfying the overlap condition determines an element of $f\in\Gamma(U;\beta(\mathcal{P}))$ by the last displayed expression; the value of $f(p)$ is independent of the choice of $\alpha\in\mathcal{A}$ such that $p\in U_\alpha$ because of the overlap condition. Collections $(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}$

and $(U'_{\alpha'}, f'_{\alpha'})_{\alpha'\in\mathcal{A}'}$ with $f_\alpha \in S_{U_\alpha}$, $f'_{\alpha'} \in S_{U_{\alpha'}}$, covering U, and satisfying the overlap condition determine the same element $f\in\Gamma(U;\beta(\mathcal{P}))$ if and only if for all $\alpha\in\mathcal{A}$, $\alpha'\in\mathcal{A}'$, and $p\in U_\alpha\cap U'_{\alpha'}$ there exists a neighborhood W of p in $U_{\alpha'}\cap U'_{\alpha'}$ such that $\rho_{W,U_\alpha} f_\alpha = \rho_{W,U'_{\alpha'}} f'_{\alpha'}$, since

$$\rho_{p,U_\alpha}(f_\alpha) = \rho_{p,W}\rho_{W,U_\alpha}(f_\alpha), \qquad \rho_{p,W}\rho_{W,U'_{\alpha'}}(f'_{\alpha'}) = \rho_{p,U'_{\alpha'}}(f'_{\alpha'})$$

and so $\rho_{p,U_\alpha}(f_\alpha) = \rho_{p,U'_{\alpha'}}(f'_{\alpha'})$ is equivalent to the existence of W as above. Thus, we have constructed a bijective map

$$\bar{S}_U \longrightarrow \Gamma(U;\beta(\mathcal{P})).$$

This map is a homomorphism of K-modules, since the K-module operations on $\Gamma(U;\beta(\mathcal{P}))$ are determined by the K-operations on sections over open covers $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of U, which in turn correspond to the K-module operations in $\{S_{U_\alpha}\}_{\alpha\in\mathcal{A}}$. Whenever $U\subset V$ are open subsets of X, the diagram

$$\begin{array}{ccc} \bar{S}_V & \longrightarrow & \Gamma(V;\beta(\mathcal{P})) \\ \big\downarrow & & \big\downarrow \\ \bar{S}_U & \longrightarrow & \Gamma(U;\beta(\mathcal{P})) \end{array}$$

commutes, since both vertical arrows are restrictions on elements of open covers of U. Thus, the presheaves $\bar{\mathcal{P}}$ and $\alpha(\beta(\mathcal{P}))$ are isomorphic.

(b) If $[(U,f)]=0\in\bar{S}_U$, every $p\in U$ has a neighborhood $W_p\subset U$ such that $\rho_{W_p,U} f=0$. So,

$$\ker\varphi_U = \big\{ f\in S_U \colon \exists \text{ open cover } \{U_\alpha\}_{\alpha\in\mathcal{A}} \text{ of } U \text{ s.t. } \rho_{U_\alpha,U} f = \rho_{U_\alpha,U} 0 \quad \forall\, \alpha\in\mathcal{A} \big\}.$$

Thus, $\ker\varphi_U = 0$ for all if and only if $\mathcal{P}$ satisfies 5.7(C_1).

Suppose φ_U is an isomorphism for all open subsets U of X. If $(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}$ is a collection such that $f_\alpha\in U_\alpha$, $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of U, and

$$\rho_{U_\alpha\cap U_\beta, U_\alpha} f_\alpha = \rho_{U_\alpha\cap U_\beta, U_\beta} f_\beta \qquad \forall\ \alpha,\beta\in\mathcal{A},$$

then $[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}]\in\bar{S}_U$ ($W=U_\alpha\cap U_\beta$ can be used for the overlap condition). Thus, there exists $f\in S_U$ such that

$$\big[(U,f)\big] = \big[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}\big] \in \bar{S}_U\,.$$

This means that for every $\alpha\in\mathcal{A}$ and $p\in U_\alpha$, there exists an open neighborhood $W_{\alpha;p}$ of p in U_α such that

$$\rho_{W_{\alpha;p},U_\alpha} f_\alpha = \rho_{W_{\alpha;p},U} f = \rho_{W_{\alpha;p},U_\alpha}\rho_{U_\alpha,U} f \in S_{W_{\alpha;p}}\,.$$

Since $\{W_{\alpha;p}\}_{p\in U_\alpha}$ is an open cover of U_α and $\mathcal{P}$ satisfies 5.7(C_1), it follows that $f_\alpha = \rho_{U_\alpha,U} f \in S_{U_\alpha}$ for every $\alpha\in\mathcal{A}$. Thus, $\mathcal{P}$ satisfies (C_2).

Conversely, suppose $\mathcal{P}$ is a complete presheaf and $[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}]\in\bar{S}_U$. For every pair $\alpha,\beta\in\mathcal{A}$ and every point $p\in U_\alpha\cap U_\beta$ there exists a neighborhood $W_{\alpha,\beta;p}$ of p in $U_\alpha\cap U_\beta$ such that

$$\rho_{W_{\alpha,\beta;p},U_\alpha} f_\alpha = \rho_{W_{\alpha,\beta;p},U_\beta} f_\beta\,.$$

Since $\{W_{\alpha,\beta;p}\}_{p\in U_\alpha\cap U_\beta}$ is an open cover of $U_\alpha\cap U_\beta$,

$$\rho_{W_{\alpha,\beta;p},U_\alpha\cap U_\beta}\big(\rho_{U_\alpha\cap U_\beta,U_\alpha}f_\alpha\big) = \rho_{W_{\alpha,\beta;p},U_\alpha}f_\alpha = \rho_{W_{\alpha,\beta;p},U_\beta}f_\beta = \rho_{W_{\alpha,\beta;p},U_\alpha\cap U_\beta}\big(\rho_{U_\alpha\cap U_\beta,U_\beta}f_\beta\big),$$

and P satisfies (C_1) of Definition 5.7, it follows that $\rho_{U_\alpha\cap U_\beta,U_\alpha}f_\alpha=\rho_{U_\alpha\cap U_\beta,U_\beta}f_\beta$ for all $\alpha,\beta\in\mathcal{A}$. Since P also satisfies (C_2) of Definition 5.7, there exists $f\in S_U$ such that

$$\rho_{U_\alpha,U}f = f_\alpha \quad \forall\,\alpha\in\mathcal{A} \qquad \Longrightarrow \qquad [U,f] = \big[(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}\big],$$

i.e. the homomorphism φ_U is surjective.

(c) Let $q\colon\mathcal{S}\longrightarrow\mathcal{S}/\mathcal{R}$ and $q_U\colon\Gamma(U;\mathcal{S})\longrightarrow\Gamma(U;\mathcal{S})/\Gamma(U;\mathcal{R})$ be the quotient projection maps. Denote by $\{\bar\Gamma(U;\mathcal{S}/\mathcal{R})\}$ the completion of $\{\Gamma(U;\mathcal{S})/\Gamma(U;\mathcal{R})\}$, as in the statement of the problem. Define a homomorphism of presheaves

$$\{\varphi_U\}\colon\overline{\alpha(\mathcal{S})/\alpha(\mathcal{R})}\longrightarrow\alpha(\mathcal{S}/\mathcal{R}) \qquad\text{by}\qquad \varphi_U\colon\bar\Gamma(U;\mathcal{S}/\mathcal{R})\longrightarrow\Gamma(U;\mathcal{S}/\mathcal{R}),$$
$$\varphi_U\big(\big[(U_\alpha,q_{U_\alpha}(f_\alpha))_{\alpha\in\mathcal{A}}\big]\big)\big|_{U_\alpha} = q\circ f_\alpha\in\Gamma(U_\alpha;\mathcal{S}/\mathcal{R}).$$

If $q_{U_\alpha}(f_\alpha)=q_{U_\alpha}(f'_\alpha)$, then $f_\alpha-f'_\alpha\in\Gamma(U_\alpha;\mathcal{R})$ and thus $q\circ f_\alpha=q\circ f'_\alpha\in\Gamma(U_\alpha;\mathcal{S}/\mathcal{R})$. Since

$$\big[(U_\alpha,q_{U_\alpha}(f_\alpha))_{\alpha\in\mathcal{A}}\big]\in\bar\Gamma(U;\mathcal{S}/\mathcal{R}),$$

for every pair $\alpha,\beta\in\mathcal{A}$ and every point $p\in U_\alpha\cap U_\beta$ there exists a neighborhood $W\subset U_\alpha\cap U_\beta$ of p such that $q_W(f_\alpha|_W)=q_W(f_\beta|_W)$ and thus $q\circ f_\alpha|_W=q\circ f_\beta|_W$. Since $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of U, it follows that $\varphi_U([(U_\alpha,q_{U_\alpha}(f_\alpha))_{\alpha\in\mathcal{A}}])$ is a well-defined (continuous) section of $\mathcal{S}/\mathcal{R}$ over U, i.e. an element of $\Gamma(U;\mathcal{S}/\mathcal{R})$, as required. If

$$(U_\alpha,q_{U_\alpha}(f_\alpha))_{\alpha\in\mathcal{A}}\sim(U'_{\alpha'},q_{U'_{\alpha'}}(f'_{\alpha'}))_{\alpha'\in\mathcal{A}'},$$

for every $p\in U_\alpha\cap U'_{\alpha'}$ there exists a neighborhood W of p in $U_\alpha\cap U'_{\alpha'}$ such that $q_W(f_\alpha|_W)=q_W(f_{\alpha'}|_W)$ and thus $q\circ f_\alpha|_W=q\circ f_{\alpha'}\big|_W$. So $\varphi_U([(U_\alpha,q_{U_\alpha}(f_\alpha))_{\alpha\in\mathcal{A}}])$ depends only on $[(U_\alpha,q_{U_\alpha}(f_\alpha))_{\alpha\in\mathcal{A}}]$, and not $(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}$, i.e. the map

$$\varphi_U\colon\bar\Gamma(U;\mathcal{S}/\mathcal{R})\longrightarrow\Gamma(U;\mathcal{S}/\mathcal{R})$$

is well-defined. Since q and q_{U_α} are homomorphisms of K-modules, so is φ_U. It is immediate that φ_U commutes with the restriction maps and therefore $\{\varphi_U\}$ is a homomorphism of presheaves.

If $q\circ f_\alpha=0\in\Gamma(U;\mathcal{S}/\mathcal{R})$, then $f_\alpha\in\Gamma(U;\mathcal{R})\subset\Gamma(U;\mathcal{S})$ and $q_{U_\alpha}(f_\alpha)=0\in\bar\Gamma(U;\mathcal{S}/\mathcal{R})$. Thus, $\{\varphi_U\}$ is injective. On the other hand, suppose $g\in\Gamma(U;\mathcal{S}/\mathcal{R})$. For each $p\in X$, choose $s(p)\in\mathcal{S}_p$ such that $q(s(p))=g(p)$. Since $\pi\colon\mathcal{S}\longrightarrow X$ and $\pi'\colon\mathcal{S}/\mathcal{R}\longrightarrow X$ are local homeomorphisms, for each $p\in X$ there exist neighborhoods U_p of p in U, $U_{s(p)}$ of $s(p)$ in $\mathcal{S}$, and $U_{g(p)}$ of $g(p)$ in $\mathcal{S}/\mathcal{R}$ such that

$$\pi\colon U_{s(p)}\longrightarrow U_p \qquad\text{and}\qquad \pi'\colon U_{g(p)}\longrightarrow U_p$$

are homeomorphisms. Since $\pi=\pi'\circ q$, $q\colon U_{s(p)}\longrightarrow U_{g(p)}$ is also a homeomorphism. Let

$$f_p=\big\{\pi|_{U_{s(p)}}\big\}^{-1}\colon U_p\longrightarrow U_{s(p)}\subset\mathcal{S}.$$

Since $\pi' \circ g = \mathrm{id}_X$, it follows that

$$q \circ f_p = q \circ \{\pi|_{U_{s(p)}}\}^{-1} = q \circ \{q|_{U_{s(p)}}\}^{-1} \circ \{\pi'|_{U_{g(p)}}\}^{-1} \circ \{\pi' \circ g\}|_{U_p} = g|_{U_p}.$$

We conclude that

$$\big[(U_p, q_{U_p}(f_p))_{p\in U}\big] \in \bar{\Gamma}(U; \mathcal{S}/\mathcal{R}) \qquad \text{and} \qquad \varphi_U\big(\big[(U_p, q_{U_p}(f_p))_{p\in U}\big]\big) = g,$$

i.e. $\{\varphi_U\}$ is surjective. Note that if $p_1, p_2 \in U$, then

$$q \circ f_{p_1}\big|_{U_{p_1}\cap U_{p_2}} = g|_{U_{p_1}\cap U_{p_2}} = q \circ f_{p_2}\big|_{U_{p_1}\cap U_{p_2}} \quad \Longrightarrow \quad q_{U_{p_1}\cap U_{p_2}}\big(f_{p_1}|_{U_{p_1}\cap U_{p_2}}\big) = q_{U_{p_1}\cap U_{p_2}}\big(f_{p_2}|_{U_{p_1}\cap U_{p_2}}\big),$$

i.e. the overlap condition is indeed satisfied.

Note: It follows from (a) that $\bar{\mathcal{P}}$ is a complete pre-sheaf. We now check this directly.

If $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of U and $f_\alpha \in S_{U_\alpha}$ are elements satisfying the overlap condition in the definition of $\bar{S}_U$ above, denote by $[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}] \in \bar{S}_U$ the equivalence class of $(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}$. If $r \in K$, let

$$r \cdot \big[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}\big] = \big[(U_\alpha, r\cdot f_\alpha)_{\alpha\in\mathcal{A}}\big].$$

The tuple $(U_\alpha, r\cdot f_\alpha)_{\alpha\in\mathcal{A}}$ still satisfies the overlap condition (the same open sets W work, since ρ_{W,U_α} and ρ_{W,U_β} are homomorphisms of K-modules) and therefore $\big[(U_\alpha, r\cdot f_\alpha)_{\alpha\in\mathcal{A}}\big] \in \bar{S}_U$. If $(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}} \sim (U'_{\alpha'}, f'_{\alpha'})_{\alpha'\in\mathcal{A}'}$, then

$$(U_\alpha, r\cdot f_\alpha)_{\alpha\in\mathcal{A}} \sim (U'_{\alpha'}, r\cdot f'_{\alpha'})_{\alpha'\in\mathcal{A}'}$$

(the same open sets W work) and therefore the multiplication map $K \times \bar{S}_U \longrightarrow \bar{S}_U$ is well-defined. If

$$\big[(U_{\alpha_1}, f_{\alpha_1})_{\alpha_1\in\mathcal{A}_1}\big], \big[(U_{\alpha_2}, f_{\alpha_2})_{\alpha_2\in\mathcal{A}_2}\big] \in \bar{S}_U,$$

define

$$\big[(U_{\alpha_1}, f_{\alpha_1})_{\alpha_1\in\mathcal{A}_1}\big] + \big[(U_{\alpha_2}, f_{\alpha_2})_{\alpha_2\in\mathcal{A}_2}\big] = \big[(U_{\alpha_1}\cap U_{\alpha_2}, \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_1}} f_{\alpha_1} + \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_2}} f_{\alpha_2})_{\alpha_1\in\mathcal{A}_1,\alpha_2\in\mathcal{A}_2}\big].$$

The tuple

$$(U_{\alpha_1}\cap U_{\alpha_2}, \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_1}} f_{\alpha_1} + \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_2}} f_{\alpha_2})_{\alpha_1\in\mathcal{A}_1,\alpha_2\in\mathcal{A}_2}$$

satisfies the overlap condition for the following reason. If $\alpha_1, \beta_1 \in \mathcal{A}_1$ and $\alpha_2, \beta_2 \in \mathcal{A}_2$ and $p \in U_{\alpha_1}\cap U_{\alpha_2}\cap U_{\beta_1}\cap U_{\beta_2}$, there exists an open neighborhood W of p in this 4-fold intersection such that

$$\rho_{W,U_{\alpha_1}} f_{\alpha_1} = \rho_{W,U_{\beta_1}} f_{\beta_1}\,, \qquad \rho_{W,U_{\alpha_2}} f_{\alpha_2} = \rho_{W,U_{\beta_2}} f_{\beta_2}\,;$$

this open set W is obtained by intersecting the two W's in the overlap condition for the two tuples being summed. This open set W also works for the sum tuple:

$$\begin{aligned} \rho_{W,U_{\alpha_1}\cap U_{\alpha_2}}\big(\rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_1}} f_{\alpha_1} + \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_2}} f_{\alpha_2}\big) &= \rho_{W,U_{\alpha_1}} f_{\alpha_1} + \rho_{W,U_{\alpha_2}} f_{\alpha_2} \\ &= \rho_{W,U_{\beta_1}} f_{\beta_1} + \rho_{W,U_{\beta_2}} f_{\beta_2} \\ &= \rho_{W,U_{\beta_1}\cap U_{\beta_2}}\big(\rho_{U_{\beta_1}\cap U_{\beta_2},U_{\beta_1}} f_{\beta_1} + \rho_{U_{\beta_1}\cap U_{\beta_2},U_{\beta_2}} f_{\beta_2}\big). \end{aligned} \tag{1}$$

Thus,

$$\big[(U_{\alpha_1}\cap U_{\alpha_2}, \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_1}} f_{\alpha_1} + \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_2}} f_{\alpha_2})_{\alpha_1\in\mathcal{A}_1,\alpha_2\in\mathcal{A}_2}\big] \in \bar{S}_U.$$

If

$$(U_{\alpha_1}, f_{\alpha_1})_{\alpha_1\in\mathcal{A}_1} \sim (U'_{\alpha'_1}, f'_{\alpha'_1})_{\alpha'_1\in\mathcal{A}'_1} \qquad \text{and} \qquad (U_{\alpha_2}, f_{\alpha_2})_{\alpha_2\in\mathcal{A}_2} \sim (U'_{\alpha'_2}, f'_{\alpha'_2})_{\alpha'_2\in\mathcal{A}'_2},$$

then

$$\begin{aligned}(U_{\alpha_1}\cap U_{\alpha_2}, \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_1}} f_{\alpha_1} + \rho_{U_{\alpha_1}\cap U_{\alpha_2},U_{\alpha_2}} f_{\alpha_2})_{\alpha_1\in\mathcal{A}_1,\alpha_2\in\mathcal{A}_2}\\ \sim (U'_{\alpha'_1}\cap U'_{\alpha'_2}, \rho_{U'_{\alpha'_1}\cap U'_{\alpha'_2},U'_{\alpha'_1}} f'_{\alpha'_1} + \rho_{U'_{\alpha'_1}\cap U'_{\alpha'_2},U'_{\alpha'_2}} f'_{\alpha'_2})_{\alpha'_1\in\mathcal{A}'_1,\alpha'_2\in\mathcal{A}'_2}\end{aligned}$$

as we can use the intersection of the open sets W corresponding to (α_1,α'_1) and (α'_2,α'_2) by a computation similar to (1) above. Thus, $\bar{S}_U$ is a K-module.

If $U\subset V$ and $[(V_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}] \in \bar{S}_V$, then

$$\big[(V_\alpha\cap U, \rho_{V_\alpha\cap U,V_\alpha} f_\alpha)_{\alpha\in\mathcal{A}}\big] \in \bar{S}_U;$$

the overlap condition still holds, with W used for the first tuple replaced by $W\cap U$. Furthermore,

$$\begin{aligned}\big[(V_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}\big] = \big[(V'_{\alpha'}, f'_{\alpha'})_{\alpha'\in\mathcal{A}'}\big] \in \bar{S}_V\\ \Longrightarrow \quad \big[(V_\alpha\cap U, \rho_{V_\alpha\cap U,V_\alpha} f_\alpha)_{\alpha\in\mathcal{A}}\big] = \big[(V'_\alpha\cap U, \rho_{V'_{\alpha'}\cap U,V'_{\alpha'}} f'_{\alpha'})_{\alpha'\in\mathcal{A}'}\big] \in \bar{S}_U\,,\end{aligned}$$

as each W used in the first equivalence condition can be replaced with $W\cap U$. Thus, the map $\bar{\rho}_{U,V}$ is well-defined. It must be a homomorphism of K-modules, because $\rho_{U,V}$ is. Furthermore, if $U\subset V\subset W$,

$$\bar{\rho}_{U,W} = \bar{\rho}_{U,V}\circ\bar{\rho}_{V,W}$$

since $\rho_{U',W'} = \rho_{U',V'}\circ\rho_{V',W'}$ whenever $U'\subset V'\subset W'$ are open subsets of X. Thus, $\bar{\mathcal{P}} = \{\bar{S}, \bar{\rho}_{U,V}\}$ is indeed a presheaf on X.

Suppose U is an open subset of X and $\{V_\gamma\}_{\gamma\in\Gamma}$ is an open cover of U. Suppose in addition that

$$\big[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}\big]\in\bar{S}_U, \quad \big[(U'_{\alpha'}, f'_{\alpha'})_{\alpha'\in\mathcal{A}'}\big]\in\bar{S}_U, \quad \bar{\rho}_{V_\gamma,U}\big[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}\big] = \bar{\rho}_{V_\gamma,U}\big[(U'_{\alpha'}, f'_{\alpha'})_{\alpha'\in\mathcal{A}'}\big] \ \ \forall\, \gamma\in\Gamma.$$

By definition,

$$\begin{aligned}\bar{\rho}_{V_\gamma,U}\big[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}\big] &= \big[(U_\alpha\cap V_\gamma, \rho_{U_\alpha\cap V_\gamma,U_\alpha} f_\alpha)_{\alpha\in\mathcal{A}}\big] \in \bar{S}_{V_\gamma},\\ \bar{\rho}_{V_\gamma,U}\big[(U'_{\alpha'}, f'_{\alpha'})_{\alpha'\in\mathcal{A}'}\big] &= \big[(U'_{\alpha'}\cap V_\gamma, \rho_{U'_{\alpha'}\cap V_\gamma,U'_{\alpha'}} f'_{\alpha'})_{\alpha'\in\mathcal{A}'}\big] \in \bar{S}_{V_\gamma}.\end{aligned}$$

By definition of the equality of the two, for every $p\in U_\alpha\cap U'_{\alpha'}\cap V_\gamma$ there exists a neighborhood $W\subset U_\alpha\cap U'_{\alpha'}\cap V_\gamma$ of p such that

$$\begin{aligned}\rho_{W,U_\alpha} f_\alpha &= \rho_{W,U_\alpha\cap V_\gamma}\big(\rho_{U_\alpha\cap V_\gamma,U_\alpha} f_\alpha\big)\\ &= \rho_{W,U'_{\alpha'}\cap V_\gamma}\big(\rho_{U'_{\alpha'}\cap V_\gamma,U'_{\alpha'}} f_{\alpha'}\big) = \rho_{W,U_{\alpha'}} f'_{\alpha'}.\end{aligned}$$

By definition of equivalence, this means that

$$(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}} \sim (U_\alpha, f'_{\alpha'})_{\alpha'\in\mathcal{A}'} \qquad \text{i.e.} \qquad \big[(U_\alpha, f_\alpha)_{\alpha\in\mathcal{A}}\big] = \big[(U'_{\alpha'}, f'_{\alpha'})_{\alpha'\in\mathcal{A}'}\big] \in \bar{S}_U.$$

Thus, $\bar{\mathcal{P}}$ satisfies (C_1) of Definition 5.7.

Suppose U is an open subset of X, $\{V_\gamma\}_{\gamma\in\Gamma}$ is an open cover of U, $[(U_{\gamma,\alpha},f_{\gamma,\alpha})_{\alpha\in\mathcal{A}_\gamma}]\in\bar{S}_{V_\gamma}$, and

$$\bar{\rho}_{V_{\gamma_1}\cap V_{\gamma_2},V_{\gamma_1}}\big[(U_{\gamma_1,\alpha},f_{\gamma_1,\alpha})_{\alpha\in\mathcal{A}_{\gamma_1}}\big]=\bar{\rho}_{V_{\gamma_1}\cap V_{\gamma_2},V_{\gamma_2}}\big[(U_{\gamma_2,\alpha},f_{\gamma_2,\alpha})_{\alpha\in\mathcal{A}_{\gamma_2}}\big]\quad\forall\,\gamma_1,\gamma_2\in\Gamma.$$

By definition, this equality implies that for all $\gamma_1,\gamma_2\in\Gamma$, $\alpha_1\in\mathcal{A}_{\gamma_1}$, $\alpha_2\in\mathcal{A}_{\gamma_2}$, and $p\in U_{\gamma_1,\alpha_1}\cap U_{\gamma_2,\alpha_2}$, there exists a neighborhood W of p in $U_{\gamma_1,\alpha_1}\cap U_{\gamma_2,\alpha_2}$ such that

$$\begin{aligned}\rho_{W,U_{\gamma_1,\alpha_1}}f_{\gamma_1,\alpha_1}&=\rho_{W,U_{\gamma_1,\alpha_1}\cap V_{\gamma_2}}\big(\rho_{U_{\gamma_1,\alpha_1}\cap V_{\gamma_2},U_{\gamma_1,\alpha_1}}f_{\gamma_1,\alpha_1}\big)\\&=\rho_{W,U_{\gamma_2,\alpha_2}\cap V_{\gamma_1}}\big(\rho_{U_{\gamma_2,\alpha_2}\cap V_{\gamma_1},U_{\gamma_2,\alpha_2}}f_{\gamma_2,\alpha_2}\big)=\rho_{W,U_{\gamma_2,\alpha_2}}f_{\gamma_2,\alpha_2}.\end{aligned}$$

Thus, the collection $(U_{\gamma,\alpha},f_{\gamma,\alpha})_{\alpha\in\mathcal{A}_\gamma,\gamma\in\Gamma}$ satisfies the overlap condition in the definition of $\bar{S}_U$ and

$$\big[(U_{\gamma,\alpha},f_{\gamma,\alpha})_{\alpha\in\mathcal{A}_\gamma,\gamma\in\Gamma}\big]\in\bar{S}_U\quad\text{s.t.}\quad\bar{\rho}_{V_\gamma,U}\big(\big[(U_{\gamma,\alpha},f_{\gamma,\alpha})_{\alpha\in\mathcal{A}_\gamma,\gamma\in\Gamma}\big]\big)=[(U_{\gamma,\alpha},f_{\gamma,\alpha})_{\alpha\in\mathcal{A}_\gamma}]\quad\forall\,\gamma\in\Gamma.$$

Thus, $\bar{\mathcal{P}}$ satisfies (C_2) of Definition 5.7 and therefore is a complete presheaf.

Problem 2: Chapter 5, #17 (5pts)

Give an example of a fine sheaf which contains a subsheaf which is not fine.

The sheaf $\mathcal{S}$ of germs of continuous functions over a topological space X is a fine sheaf by 5.10 (the argument in the continuous case is the same as in the smooth case). It contains the sheaf $\underline{\mathbb{R}}\equiv X\times\mathbb{R}_{\text{discreet}}$ of germs of locally constant functions. By 5.31, $\check{H}^p(X;\underline{\mathbb{R}})\approx H^p_{\text{sing}}(X;\mathbb{R})$; thus, by 5.33 $\underline{\mathbb{R}}$ is not a fine sheaf as long as $H^p_{\text{sing}}(X;\mathbb{R})\neq 0$ for some $p\neq 0$ (e.g. $X=S^1$ by deRham or Hurewicz's theorem). A similar example is obtained by considering $\underline{\mathbb{R}}$ as the subsheaf of germs of locally constant smooth functions contained in the sheaf of germs of smooth 0-forms $\mathcal{E}^0$. In addition, if $\mathcal{S}$ is any sheaf, the sheaf $\mathcal{S}_0$ of germs of discontinuous sections of $\mathcal{S}$ is a fine sheaf (see 5.22) and contains $\mathcal{S}$ as the subsheaf of germs of continuous sections of $\mathcal{S}$; this provides an example whenever $\mathcal{S}$ is not a fine sheaf.

In fact, the sheaf $\underline{K}\equiv X\times K_{\text{discreet}}$, where K is a ring with unity 1, is not fine as long as X contains two nonempty connected subsets V_1 and V_2 such that $V_1\cap V_2\neq\emptyset$, $V_1\not\subset V_2$, and $V_2\not\subset V_1$. If so, let U_1 be the complement of a point x_2 in V_2-V_1 and U_2 be the complement of a point x_1 in V_1-V_2. Then, $\{U_1,U_2\}$ is a locally finite open cover of X (as long as X is T1). If $L\colon\underline{K}\longrightarrow\underline{K}$ is any sheaf homomorphism, the sets

$$\begin{aligned}A&\equiv\big\{x\in X:\ L_x=0\big\}=\pi_1\big(L^{-1}(X\times0)\cap X\times1\big),\\B&\equiv\big\{x\in X:\ L_x\neq0\big\}=\pi_1\big(L^{-1}(X\times(K-0))\cap X\times1\big)\end{aligned}$$

are open and disjoint in X, since $X\times0$, $X\times1$ and $X\times(K-0)$ are open in $\underline{K}$, L a continuous map, and $\pi_1\colon\underline{K}\longrightarrow X$ is a local homeomorphism which is injective on $X\times1$. Thus, $(V_1\cup V_2)\cap A$ and $(V_1\cup V_2)\cap B$ form an open partition of $V_1\cup V_2$. Since $V_1\cup V_2$ is connected, every sheaf homomorphism $L\colon\underline{K}\longrightarrow\underline{K}$ is either identically 0 on $V_1\cup V_2$ or nowhere 0 on $V_1\cup V_2$. It follows that there exists no partition of identity $\{L_1,L_2\}$ on $\underline{K}$ subordinate to $\{U_1,U_2\}$: L_1 would have to vanish at $p_2\in V_2-U_1$, because the support of L_1 must be contained in U_1, and would have to equal the identity at $p_1\in V_1-U_2$, because the support of L_2 must be contained in U_2 and $L_1+L_2=\text{id}$.

Problem 3 (10pts)

Let K be any ring containing 1. For each $i \in \mathbb{Z}^+$, let $V_i = K$; this is a K-module. Whenever $i \leq j$, define

$$\rho_{ji}\colon V_i \longrightarrow V_j \qquad \text{by} \qquad \rho_{ji}(v) = 2^{j-i}v;$$

this is a homomorphism of K-modules. Since $\rho_{ki} = \rho_{kj}\rho_{ji}$ whenever $i \leq j \leq k$, we have a directed system and get a direct-limit K-module

$$V_\infty = \varinjlim_{\mathbb{Z}^+} V_i = \lim_{i \longrightarrow \infty} V_i\,.$$

(a) *Suppose $2=0\in K$ (e.g. $K=\mathbb{Z}_2$). Show that $V_\infty=\{0\}$.*

(b) *Suppose 2 is a unit in K (e.g. $K=\mathbb{R}$). Show that $V_\infty \approx K$ as K-modules.*

(c) *Suppose 2 is not a unit in K, but $2 \neq 0 \in K$, and K is an integral domain (e.g. $K=\mathbb{Z}$). Show that the K-module V_∞ is not finitely generated.*

An element of V_∞ is an equivalence class $[i,v]$, where $i \in \mathbb{Z}^+$, $v \in V_i$, and $[i,v]=[j,w]$ if there exists $k \geq i,j$ such that $2^{k-i}v = 2^{k-j}w \in K$; in particular, $[i,v]=[j,2^{j-i}v]$ whenever $i \leq j$.

(a) Since $2=0$, $[i,v]=[i+1,2v]=[i+1,0]$ for all $[i,v]\in V_\infty$; so $V_\infty=\{0\}$.

(b) Define a homomorphism

$$h\colon K \longrightarrow V_\infty \qquad \text{by} \qquad v \longrightarrow [1,v].$$

If $h(v)=0$ for some $v \in K$, then $2^{j+1-1}v = 0 \in V_{j+1}$ for some $j \geq 1$. Since 2 is a unit in K (has an inverse), it follows that $v=0\in V_{j+1}=K$; thus, h is injective. On the other hand, for every $j \in \mathbb{Z}^+$ and $w \in V_j = K$,

$$w = 2^{j-1}\cdot(2^{-1})^{j-1}w \qquad \Longrightarrow \qquad [j,w] = \big[1,(2^{-1})^{j-1}w\big] = h\big((2^{-1})^{j-1}w\big);$$

thus, h is also surjective.

(c) Suppose V_∞ is spanned by $[i_1,v_1],\ldots,[i_k,v_k]$ for some $i_1,\ldots,i_k \in \mathbb{Z}^+$ and $v_1,\ldots,v_k \in K$. Let $i = \max\{i_1,\ldots,i_k\}$. Since $[i_l,v_l] = [i,2^{i-i_l}v_k\cdot 1]$, V_∞ is spanned by the single element $[i,1]$. In particular, $[i+1,1]=k[i,1]$ for some $k\in K$ and so

$$2^{(j+1)-(i+1)}\cdot 1 = 2^{(j+1)-i}\cdot k \in V_{j+1} = K$$

for some $j \geq i$. Thus, $2^{j-i}(2k-1) = 0 \in K$. Since K is an integral domain, it follows that $2k=1$, contrary to the assumption that 2 is not a unit in K.