MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 7

Problem 1 (15pts)

Let X be a path-connected topological space and $(\mathcal{S}_*(X), \partial)$ the singular chain complex of continuous *simplices into X with* integer *coefficients. Denote by $H_1(X;\mathbb{Z})$ the corresponding first homology group.*
(a) *Show that there exists a well-defined surjective homomorphism*

$$h\colon \pi_1(X,x_0) \longrightarrow H_1(X;\mathbb{Z}).$$

(b) *Show that the kernel of this homomorphism is the commutator subgroup of $\pi_1(X,x_0)$ so that h induces an isomorphism*

$$\Phi\colon \pi_1(X,x_0)/\big[\pi_1(X,x_0),\pi_1(X,x_0)\big] \longrightarrow H_1(X;\mathbb{Z}).$$

This is the first part of the Hurewicz Theorem.

The motivation for this result is that $\pi_1(X,x_0)$ is generated by loops based at $x_0\in X$, i.e. continuous maps $\alpha\colon I\longrightarrow X$ such that $\alpha(0)=\alpha(1)=x_0$, while $H_1(X;\mathbb{Z})$ is generated by formal linear combinations of 1-simplicies, i.e. continuous maps

$$f\colon \Delta^1 = I \longrightarrow X.$$

In particular, a loop (as well as any path) in X is a 1-simplex. However, the equivalence relations on paths and 1-simplicies used to define $\pi_1(X,x_0)$ and $H_1(X;\mathbb{Z})$ and the groups structures are quite different. So we will need to show that equivalent paths are equivalent as 1-simplicies and a product of two paths corresponds to the sum of the two 1-simplicies.

We will denote the path-homotopy equivalence class of a path α (loop or not) by $[\alpha]$ and the image of a 1-simplex in $S_1(X)/\partial S_2(X)$ by $\{\alpha\}$. It will be essential to distinguish between a point $x_0\in X$ and the k-simplex taking the entire standard k-simplex Δ^k to x_0. Denote the latter by f_{k,x_0}.

Lemma 0: If $\alpha\colon I\longrightarrow X$ is a loop, $\partial\alpha=0$.
Lemma 1: If $x_0\in X$, $f_{1,x_0}\in\partial\mathcal{S}_2(X)$.
Lemma 2: If $\alpha,\beta\colon I\longrightarrow X$ are path-homotopic, then $\alpha-\beta\in\partial\mathcal{S}_2(X)$.
Lemma 3: If $\alpha,\beta\colon I\longrightarrow X$ are paths such that $\alpha(1)=\beta(0)$, then $\alpha+\beta-\alpha*\beta\in\partial\mathcal{S}_2(X)$.
Lemma 4: If $\alpha : I\longrightarrow X$ and $\bar\alpha\colon I\longrightarrow X$ is its inverse, then $\alpha+\bar\alpha\in\partial\mathcal{S}_2(X)$.
Lemma 5: If $F\colon\Delta^2\longrightarrow X$ is a 2-simplex, then

$$\big[(F\circ\iota_0^2)*\overline{(F\circ\iota_1^2)}*(F\circ\iota_2^2)\big] = [\mathrm{id}] \in \pi_1\big(X, F(1,0)\big).$$

First, recall the maps ι_j^1 and ι_j^2 used to define the boundaries of 1- and 2-simplicies:

$$\iota_j^1\colon \Delta^0 \longrightarrow \Delta^1, \qquad \iota_0^1(0)=1, \quad \iota_1^1(0)=0;$$
$$\iota_j^2\colon \Delta^1 \longrightarrow \Delta^2, \qquad \iota_0^2(s)=(1-s,s), \quad \iota_1^2(s)=(0,s), \quad \iota_2^2(s)=(s,0) \quad \forall\ s\in I;$$

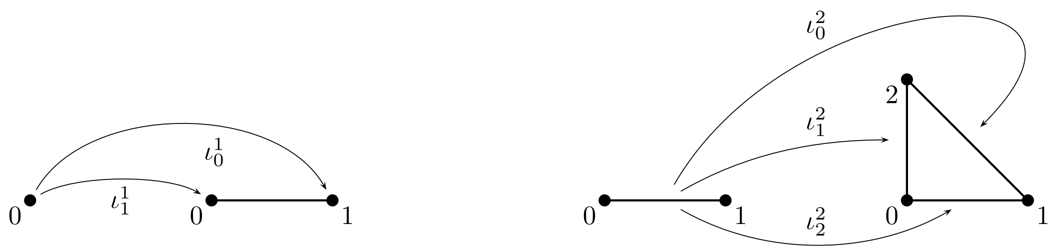

Figure 1: The boundary maps $\iota_j^1\colon \Delta^0 \longrightarrow \Delta^1$ and $\iota_j^2\colon \Delta^1 \longrightarrow \Delta^2$.

see Figure 1. These maps respect the orders of the vertices. By the above, if α is a loop based at x_0,

$$\partial\alpha = \alpha\circ\iota_0^1 - \alpha\circ\iota_1^1 = f_{0,\alpha(1)} - f_{0,\alpha(0)} = f_{0,x_0} - f_{0,x_0} = 0.$$

For Lemma 1, note that

$$\partial f_{2,x_0} = f_{2,x_0}\circ\iota_0^2 - f_{2,x_0}\circ\iota_1^2 + f_{2,x_0}\circ\iota_2^2 = f_{1,x_0} - f_{1,x_0} + f_{1,x_0} = f_{1,x_0},$$

since $f_{2,x_0}\circ\iota_j^2$ maps all of I to x_0. For Lemma 2, choose a path-homotopy from α to β, i.e. a continuous map

$$F\colon I\times I \longrightarrow X \qquad \text{s.t.} \qquad F(s,0)=\alpha(s),\quad F(s,1)=\beta(s),\quad F(0,t)=F(1,t) \quad \forall\ s,t\in[0,1].$$

There is a quotient map

$$q\colon I\times I \longrightarrow \Delta^2 \qquad \text{s.t.} \qquad q(s,0)=(s,0),\quad q(s,1)=q(0,s),\quad q(0,t)=(0,0),\quad q(1,t)=(1-t,t),$$

i.e. q contracts the left edge of $I\times I$ and maps the other three edges linearly onto the edges of Δ^2. Since F is constant along the fibers of q, F induces a continuous map

$$\begin{aligned} \bar F\colon \Delta^2 \longrightarrow X \quad \text{s.t.} \quad F=\bar F\circ q \quad &\Longrightarrow \quad \bar F(s,0)=\alpha(s),\ \ \bar F(0,t)=\beta(t),\ \ \bar F(s,1-s)=x_1 \quad \forall\ s,t\in I \\ &\Longrightarrow \quad \partial\bar F = \bar F\circ\iota_0^2 - \bar F\circ\iota_1^2 + \bar F\circ\iota_2^2 = f_{1,x_1} - \beta + \alpha; \end{aligned}$$

see Figure 2. Along with Lemma 1, this implies Lemma 2.

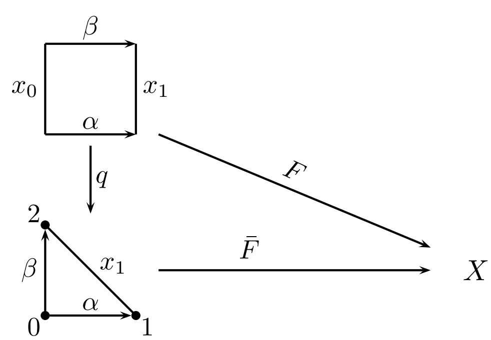

Figure 2: A path homotopy gives rise to a boundary between the corresponding 1-simplices.

For Lemma 3, define

$$F\colon \Delta^2 \longrightarrow X \qquad \text{by} \qquad F(x,y) = \begin{cases} \alpha(x+2y), & \text{if } x+2y\le 1; \\ \beta(x+2y-1), & \text{if } x+2y\ge 1. \end{cases}$$

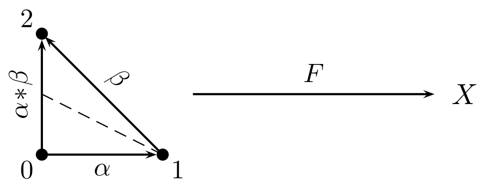

Figure 3: A boundary between the 1-simplex corresponding to a composition of paths and the sum of the 1-simplices corresponding to the paths.

see Figure 3. This map is well-defined and continuous, since it is continuous on the two closed sets and agrees on the overlap, where it equals $\alpha(1)=\beta(0)$. Furthermore,

$$F\big(\iota_0^2(s)\big) = F(1-s,s) = \beta(s), \qquad F\big(\iota_2^2(s)\big) = F(s,0) = \alpha(s);$$
$$F\big(\iota_1^2(s)\big) = F(0,s) = \begin{cases} \alpha(2s), & \text{if } 2s\le 1; \\ \beta(2s-1), & \text{if } 2s\ge 1; \end{cases}$$
$$\Longrightarrow \qquad \partial F = F\circ\iota_0^2 - F\circ\iota_1^2 + F\circ\iota_2^2 = \beta - \alpha*\beta + \alpha.$$

For Lemma 4, note that

$$\alpha+\bar{\alpha} = \big(\alpha+\bar{\alpha}-\alpha*\bar{\alpha}\big) + \big(\alpha*\bar{\alpha}-f_{1,\alpha(0)}\big) + f_{1,\alpha(0)}.$$

Since $\alpha*\bar{\alpha}$ is path-homotopic to the constant path $f_{1,\alpha(0)}$, each of the three expressions above belongs to $\partial\mathcal{S}_2(X)$ by Lemmas 1-3. This implies Lemma 4.

For Lemma 5, choose a continuous map $q\colon I\times I\longrightarrow\Delta^2$ such that

$$q(s,0) = \begin{cases} (1-2s,2s), & \text{if } s\in[0,1/2]; \\ (0,3-4s), & \text{if } s\in[1/2,3/4]; \\ (4s-3,0), & \text{if } s\in[3/4,1]; \end{cases} \qquad q(s,1)=q(0,t)=q(1,t)=(1,0) \quad \forall\ s,t\in I.$$

Then, $F\circ q$ is a path-homotopy from $(F\circ\iota_0^2)*\big(\overline{(F\circ\iota_1^2)}*(F\circ\iota_2^2)\big)$ to the constant loop $f_{1,F(1,0)}$; see Figure 4.

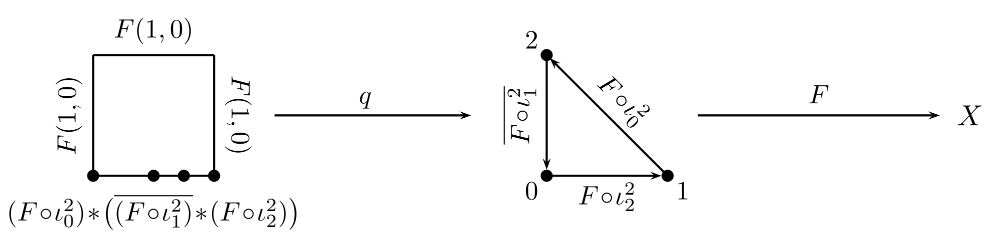

Figure 4: Boundary of a 2-simplex is loop homotopic to the constant loop.

(a) We now define the homomorphism

$$h\colon \pi_1(X,x_0)\longrightarrow H_1(X;\mathbb{Z}) \qquad \text{by} \qquad h\big([\alpha]\big) = \{\alpha\}\in H_1(X;\mathbb{Z}).$$

By Lemma 0, $\partial\alpha=0$ and thus $\{\alpha\}\in H_1(X;\mathbb{Z})$. By Lemma 2, the map h is well-defined, i.e.

$$[\alpha]=[\beta] \qquad \Longrightarrow \qquad \{\alpha\}=\{\beta\}.$$

By Lemma 3, h is indeed a homomorphism:

$$h\big([\alpha]*[\beta]\big) = h\big([\alpha*\beta]\big) = \{\alpha*\beta\} = \{\alpha\}+\{\beta\} = h\big([\alpha]\big) + h\big([\beta]\big).$$

To show that h is surjective, for each $x\in X$ choose a path $\gamma_x\colon (I,0,1)\longrightarrow(X,x_0,x)$ from x_0 to x. If

$$c = \sum_{i=1}^{N} a_i\sigma_i \in \mathcal{S}_1(X),$$

let
$$\alpha_c = \big(\gamma_{\sigma_1(0)}*\sigma_1*\bar{\gamma}_{\sigma_1(1)}\big)^{a_1} * \ldots * \big(\gamma_{\sigma_N(0)}*\sigma_N*\bar{\gamma}_{\sigma_N(1)}\big)^{a_N}.$$

This is a product of loops at x_0. It is essential that $a_i\in\mathbb{Z}$, i.e. we are dealing with integer homology. The loop α_c is not uniquely determined by c, even if the paths γ_x are fixed, as it depends on the ordering of the σ_i's. This is irrelevant, however, at this point. Since h is a homomorphism,

$$\begin{aligned} h\big([\alpha_c]\big) &= \sum_{i=1}^{N} a_i h\big([\gamma_{\sigma_i(0)}*\sigma_i*\bar{\gamma}_{\sigma_i(1)}]\big) = \sum_{i=1}^{N} a_i\big\{\gamma_{\sigma_i(0)}*\sigma_i*\bar{\gamma}_{\sigma_i(1)}\big\} \\ &= \sum_{i=1}^{N} a_i\big(\{\gamma_{\sigma_i(0)}\}+\{\sigma_i\}-\{\gamma_{\sigma_i(1)}\}\big) = \{c\} + \sum_{i=1}^{N} a_i\big(\{\gamma_{\sigma_i(0)}\} - \{\gamma_{\sigma_i(1)}\}\big). \end{aligned}$$

The third equality above follows from Lemma 3 (not from h being a homomorphism). If $c\in\ker\partial$,

$$\begin{aligned} &\sum_{i=1}^{N} a_i\big(f_{1,\sigma_i(1)}-f_{1,\sigma_i(0)}\big) = \partial c = 0 \qquad\Longrightarrow\qquad \sum_{i=1}^{N} a_i\big(\{\gamma_{\sigma_i(0)}\} - \{\gamma_{\sigma_i(1)}\}\big) = 0 \\ &\Longrightarrow\qquad h\big([\alpha_c]\big) = \{c\} \in H_1(X;\mathbb{Z}) \qquad \forall\ c \in \ker\partial. \end{aligned}$$

This shows that h is surjective.

(b) Since the group $H_1(X;\mathbb{Z})$ is abelian, h must vanish on the commutator subgroup of $\pi_1(X;x_0)$. Since this subgroup is normal, h induces a group homomorphism

$$\Phi\colon \mathrm{Abel}\big(\pi_1(X,x_0)\big)\equiv\pi_1(X,x_0)/\big[\pi_1(X,x_0),\pi_1(X,x_0)\big] \longrightarrow H_1(X;\mathbb{Z}).$$

We will show that this map is an isomorphism by constructing an inverse Ψ for Φ.

If α is a loop based at x_0, denote its image (and the image of $[\alpha]$) in $\mathrm{Abel}(\pi_1(X,x_0))$ by $\langle\alpha\rangle$. For each 1-simplex $\sigma\in\mathcal{S}_1(X)$, let

$$g(\sigma) = \langle\alpha_\sigma\rangle \in \mathrm{Abel}\big(\pi_1(X,x_0)\big).$$

Since $\mathcal{S}_1(X)$ is a free abelian group with a basis consisting of 1-simplicies σ *and* $\mathrm{Abel}(\pi_1(X,x_0))$ is abelian, g extends to a homomorphism

$$g\colon \mathcal{S}_1(X) \longrightarrow \mathrm{Abel}\big(\pi_1(X,x_0)\big).$$

If $F\colon\Delta^2\longrightarrow X$ is a 2-simplex,

$$\begin{aligned} g(\partial F) &= g(F\circ\iota_0^2) - g(F\circ\iota_1^2) + g(F\circ\iota_2^2) \\ &= \big\langle\gamma_{F(\iota_0^2(0))}*(F\circ\iota_0^2)*\bar{\gamma}_{F(\iota_0^2(1))}\big\rangle - \big\langle\gamma_{F(\iota_1^2(0))}*(F\circ\iota_1^2)*\bar{\gamma}_{F(\iota_1^2(1))}\big\rangle + \big\langle\gamma_{F(\iota_2^2(0))}*(F\circ\iota_2^2)*\bar{\gamma}_{F(\iota_2^2(1))}\big\rangle \\ &= \big\langle\big(\gamma_{F(1,0)}*(F\circ\iota_0^2)*\bar{\gamma}_{F(0,1)}\big) * \big(\gamma_{F(0,0)}*(F\circ\iota_1^2)*\bar{\gamma}_{F(0,1)}\big)^{-1} * \big(\gamma_{F(0,0)}*(F\circ\iota_2^2)*\bar{\gamma}_{F(1,0)}\big)\big\rangle \\ &= \big\langle\gamma_{F(1,0)} * \big((F\circ\iota_0^2)*\overline{(F\circ\iota_1^2)}*(F\circ\iota_2^2)\big) * \bar{\gamma}_{F(1,0)}\big\rangle. \end{aligned}$$

By Lemma 5, $(F\circ\iota_0^2)*\overline{(F\circ\iota_1^2)}*(F\circ\iota_2^2)$ is path-homotopic to the constant loop at $F(1,0)$ and thus

$$\begin{aligned} &\big[\gamma_{F(1,0)} * \big((F\circ\iota_0^2)*\overline{(F\circ\iota_1^2)}*(F\circ\iota_2^2)\big) * \bar\gamma_{F(1,0)}\big] = [\mathrm{id}] \in \pi_1(X,x_0) \\ \Longrightarrow \qquad & g(\partial F) = \big\langle \gamma_{F(1,0)} * \big((F\circ\iota_0^2)*\overline{(F\circ\iota_1^2)}*(F\circ\iota_2^2)\big) * \bar\gamma_{F(1,0)}\big\rangle = 0 \in \mathrm{Abel}\big(\pi_1(X,x_0)\big). \end{aligned}$$

It follows that g vanishes on the subgroup $\partial\mathcal{S}_2(X)$ of $\mathcal{S}_1(X)$ and therefore induces a homomorphism

$$\Psi\colon \mathcal{S}_1(X)/\partial\mathcal{S}_2(X) \longrightarrow \mathrm{Abel}\big(\pi_1(X,x_0)\big).$$

If α is a loop at x_0, γ_{x_0} is a loop at x_0, and thus

$$\begin{aligned} &\Psi\big(\Phi(\langle\alpha\rangle)\big) = \Psi\big(\{\alpha\}\big) = \big\{\gamma_{\alpha(0)}*\alpha*\bar\gamma_{\alpha(1)}\big\} = \big\{\gamma_{x_0}*\alpha*\bar\gamma_{x_0}\big\} = \{\gamma_{x_0}\} + \{\alpha\} - \{\gamma_{x_0}\} = \{\alpha\} \\ \Longrightarrow \qquad & \Psi\circ\Phi = \mathrm{Id}\colon \mathrm{Abel}\big(\pi_1(X,x_0)\big) \longrightarrow \mathrm{Abel}\big(\pi_1(X,x_0)\big). \end{aligned}$$

This implies that Φ is injective. On the other hand, it is surjective by part (a).

Problem 2 (10pts)

(a) *Prove* Mayer-Vietoris for Cohomology: *If M is a smooth manifold, $U,V\subset M$ open subsets, and $M=U\cup V$, then there exists an exact sequence*

$$\begin{aligned} 0 &\longrightarrow H^0_{\mathrm{de\,R}}(M) \xrightarrow{f_0} H^0_{\mathrm{de\,R}}(U)\oplus H^0_{\mathrm{de\,R}}(V) \xrightarrow{g_0} H^0_{\mathrm{de\,R}}(U\cap V) \xrightarrow{\delta_0} \\ &\xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(M) \xrightarrow{f_1} H^1_{\mathrm{de\,R}}(U)\oplus H^1_{\mathrm{de\,R}}(V) \xrightarrow{g_1} H^1_{\mathrm{de\,R}}(U\cap V) \xrightarrow{\delta_1} \\ &\xrightarrow{\delta_1} \dots \\ &\vdots \end{aligned}$$

where $\qquad f_i(\alpha) = \big(\alpha|_U, \alpha|_V\big) \quad$ and $\quad g_i(\beta,\gamma) = \beta|_{U\cap V} - \gamma|_{U\cap V}$.

(b) *Suppose M is a compact connected orientable n-dimensional submanifold of $\mathbb{R}^{n+1}$. Show that $\mathbb{R}^{n+1}-M$ has exactly two connected components. How is the compactness of M used?*

(a) We construct an exact sequence of cochain complexes and then apply Proposition 5.17 (*ses of cochain complexes gives les in cohomology*). Define

$$\begin{aligned} &\underline{0} \longrightarrow \big(E^*(M),\mathrm{d}_M\big) \xrightarrow{f} \big(E^*(U)\oplus E^*(V), \mathrm{d}_U\oplus\mathrm{d}_V\big) \xrightarrow{g} \big(E^*(U\cap V), \mathrm{d}_{U\cap V}\big) \longrightarrow \underline{0} \\ &\text{by} \qquad f(\alpha) = \big(\alpha|_U, \alpha|_V\big) \quad\text{and}\quad g(\beta,\gamma) = \beta|_{U\cap V} - \gamma|_{U\cap V}. \end{aligned}$$

The homomorphisms f and g preserve the grading of the complexes (take p-forms to p-forms) and commute with the differentials by Proposition 2.23b (restriction to a submanifold is the same as the pullback by the inclusion map). Thus, f and g are indeed homomorphisms of cochain complexes. The homomorphisms f is injective since $M=U\cup V$ and it is immediate that $g\circ f=0$, i.e. $\mathrm{Im} f\subset\ker g$. By the Pasting Lemma for smooth functions, $\mathrm{Im} f\supset\ker g$. Thus, the sequence above is exact at the first two positions. To see that it is exact at the third position, i.e. g is surjective, let $\{\varphi_U,\varphi_V\}$ be a partition of unity subordinate to the open cover $\{U,V\}$ of M, i.e.

$$\varphi_U,\varphi_V\colon M\longrightarrow[0,1], \quad \mathrm{supp}\,\varphi_U\subset U, \quad \mathrm{supp}\,\varphi_V\subset V, \quad \varphi_U+\varphi_V\equiv 1.$$

If $\omega\in E^*(U\cap V)$, define $\varphi_V\omega\in E^*(U)$ and $\varphi_U\omega\in E^*(V)$ by

$$\{\varphi_V\omega\}\big|_p=\begin{cases}\varphi_V(p)\{\omega|_p\}, & \text{if } p\in U\cap V;\\ 0, & \text{if } p\in U-\operatorname{supp}\varphi_V;\end{cases}\qquad \{\varphi_U\omega\}\big|_p=\begin{cases}\varphi_U(p)\{\omega|_p\}, & \text{if } p\in U\cap V;\\ 0, & \text{if } p\in V-\operatorname{supp}\varphi_U.\end{cases}$$

Since $\operatorname{supp}\varphi_V\subset V$ is a closed subset of M, U is the union of the open subsets $U\cap V$ and $U-\operatorname{supp}\varphi_V$. Since the definition of $\varphi_V\omega$ is smooth on $U\cap V$ and $U-\operatorname{supp}\varphi_V$ and agrees on the overlap, $\varphi_V\omega$ is a well-defined smooth form on U, i.e. an element of $E^*(U)$. Similarly, $\varphi_U\omega\in E^*(V)$. By definition,

$$g\big(\varphi_V\omega,-\varphi_U\omega\big)=\{\varphi_V\omega\}\big|_{U\cap V}-\big(-\{\varphi_U\omega\}\big|_{U\cap V}\big)=\varphi_V|_{U\cap V}\omega+\varphi_U|_{U\cap V}\omega=\omega.$$

Thus, g is surjective. The Mayer-Vietoris sequence in cohomology is the long exact sequence corresponding to the above short exact sequence of chain complexes via Proposition 5.17.

Note: According to the above and the proof of Proposition 5.17, the MV boundary homomorphism δ is obtained as follows. Choose $\varphi\in C^\infty(M)$ such that $\operatorname{supp}\varphi\subset V$ and $\operatorname{supp}\{1-\varphi\}\subset U$. Then,

$$\mathrm{d}\varphi\in E^1(M)\qquad \text{s.t.}\qquad \operatorname{supp}\mathrm{d}\varphi\subset U\cap V.$$

Thus, if $\omega\in E^k(U\cap V)$, then $\mathrm{d}\varphi\wedge\omega$ is a well-defined k-form on M (it is 0 outside of $\operatorname{supp}\mathrm{d}\varphi\subset U\cap V$). If in addition $\mathrm{d}\omega=0$, then $\mathrm{d}(\mathrm{d}\varphi\wedge\omega)=0$ and so $\mathrm{d}\varphi\wedge\omega$ determines an element of $H^{p+1}_{\mathrm{de\,R}}(M)$. Furthermore, for every $\eta\in E^{k-1}(U\cap V)$, $\mathrm{d}\varphi\wedge\eta$ is a well-defined k-form on M and

$$\mathrm{d}\big(\mathrm{d}\varphi\wedge\eta\big)=\mathrm{d}\varphi\wedge\mathrm{d}\eta\in E^{k+1}(M).$$

Thus, the homomorphism

$$\delta_p\colon H^p(U\cap V)\longrightarrow H^{p+1}(M),\qquad [\omega]\longrightarrow[\mathrm{d}\varphi\wedge\omega],$$

is well-defined (the image of $[\omega]$ is independent of the choice of representative ω, since any two such choices differ by an image of d, which is sent to zero by h). This is the boundary homomorphism δ_p of Proposition 5.17 in the given case, with $\varphi_V=\varphi$ and $\varphi_U=1-\varphi$. Furthermore, this homomorphism is independent of the choice of φ by Proposition 5.17, but this can also be seen directly. If $\varphi'\in C^\infty(M)$ is another function such that $\operatorname{supp}\varphi'\subset V$ and $\operatorname{supp}\{1-\varphi'\}\subset U$, then $\operatorname{supp}\{\varphi-\varphi'\}\subset U\cap V$ and thus $(\varphi-\varphi')\omega$ is a well-defined k-form on M for every k-form ω on $U\cap V$. If in addition, ω is closed,

$$\mathrm{d}\big((\varphi-\varphi')\omega\big)=\big(\mathrm{d}\varphi-\mathrm{d}\varphi'\big)\wedge\omega=\mathrm{d}\varphi\wedge\omega-\mathrm{d}\varphi'\wedge\omega\qquad\Longrightarrow\qquad[\mathrm{d}\varphi\wedge\omega]=[\mathrm{d}\varphi'\wedge\omega]\in H^{p+1}(M).$$

In contrast, $\mathrm{d}\varphi\wedge\omega$ need not be an exact form on M; it looks like $\mathrm{d}(\varphi\omega)$ if $\mathrm{d}\omega=0$, but $\varphi\omega$ is not a well-defined k-form on M because $\operatorname{supp}\varphi$ is contained in V, not in $U\cap V$, and ω is defined only on $U\cap V$. On the other hand, if $\omega\in E^k(V)$, $\varphi\omega$ is a well-defined k-form on M, and so $[\mathrm{d}\varphi\wedge\omega]=0$ in $H^{p+1}_{\mathrm{de\,R}}(M)$; this corresponds to $\delta_k\circ g_k=0$.

(b) Since M is a compact subspace of the Hausdorff space $\mathbb{R}^{n+1}$, $\mathbb{R}^{n+1}-M$ is an open subspace of $\mathbb{R}^{n+1}$ and thus a smooth manifold. Thus, the number of connected components is the dimension of $H^0_{\mathrm{de\,R}}(\mathbb{R}^{n+1}-M)$ as a real vector space. We will apply Mayer-Vietoris with $U=\mathbb{R}^{n+1}-M$ and V a nice neighborhood of M in $\mathbb{R}^{n+1}$, so that $\mathbb{R}^{n+1}=U\cup V$. The goal is not to determine $H^*_{\mathrm{de\,R}}(\mathbb{R}^{n+1})$, but $H^0_{\mathrm{de\,R}}(U)$.

Let $\mathcal{N}\longrightarrow M$ be the normal bundle of M in $\mathbb{R}^{n+1}$. Since M and $\mathbb{R}^{n+1}$ are orientable, $\mathcal{N}$ is orientable by Problem 4 on PS6. Since the codimension of M in $\mathbb{R}^{n+1}$ is one, $\mathcal{N}$ is a line bundle. Since it is orientable, $\mathcal{N}$ is trivial, i.e. isomorphic to $M\times\mathbb{R}$, by Lemma 12.1 in *Lecture Notes*. In particular, $(\mathcal{N},M)$ is diffeomorphic to $(M\times\mathbb{R},M\times 0)$, via a diffeomorphism restricting to the identity on M. In general, we can choose a neighborhood V of M in $\mathbb{R}^{n+1}$ so that (V,M) is diffeomorphic to $(\mathcal{N},M)$, via a diffeomorphism restricting to the identity on M. Thus, in this case, we can choose a neighborhood V of M in $\mathbb{R}^{n+1}$ such that $(\mathcal{N},M)$ is diffeomorphism to $(M\times\mathbb{R},M\times 0)$, via a diffeomorphism restricting to the identity on M. This implies that

$$U\cap V=(\mathbb{R}^{n+1}-M)\cap V=V-M\approx M\times\mathbb{R}^*, \qquad \text{where} \quad \mathbb{R}^*=\mathbb{R}-\{0\}.$$

The first four terms of MV for $M=U\cup V$ are

$$0\longrightarrow H^0_{\mathrm{de\,R}}(\mathbb{R}^{n+1})\longrightarrow H^0_{\mathrm{de\,R}}(U)\oplus H^0_{\mathrm{de\,R}}(V)\longrightarrow H^0_{\mathrm{de\,R}}(U\cap V)\longrightarrow H^1_{\mathrm{de\,R}}(\mathbb{R}^{n+1}).$$

Since $\mathbb{R}^{n+1}$ and M are connected,

$$H^0_{\mathrm{de\,R}}(\mathbb{R}^{n+1})\approx\mathbb{R}, \quad H^0_{\mathrm{de\,R}}(V)\approx H^0_{\mathrm{de\,R}}(M\times\mathbb{R})\approx\mathbb{R}, \quad H^0_{\mathrm{de\,R}}(U\cap V)\approx H^0_{\mathrm{de\,R}}(M\times\mathbb{R}^*)\approx\mathbb{R}^2.$$

By the Poincare Lemma, $H^1_{\mathrm{de\,R}}(\mathbb{R}^{n+1})=0$. Thus, the above sequence reduces to

$$0\longrightarrow\mathbb{R}\longrightarrow H^0_{\mathrm{de\,R}}(U)\oplus\mathbb{R}\longrightarrow\mathbb{R}^2\longrightarrow 0.$$

Since this sequence is exact, it follows that $H^0_{\mathrm{de\,R}}(U)\approx\mathbb{R}^2$, i.e. $\mathbb{R}^{n+1}-M\approx U$ has exactly two connected components.

Problem 3 (10pts)

(a) *Show that the inclusion map* $S^n\longrightarrow\mathbb{R}^{n+1}-0$ *induces an isomorphism in cohomology.*
(b) *Show that for all* $n\ge 0$ *and* $p\in\mathbb{Z}$,

$$H^p_{\mathrm{de\,R}}(S^n)\approx\begin{cases}\mathbb{R}^2, & \text{if } p=n=0;\\ \mathbb{R}, & \text{if } p=0,n,\ n\neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

(c) *Show that* S^n *is not a product of two positive-dimensional manifolds.*

(a) Let $i\colon S^n\longrightarrow\mathbb{R}^{n+1}-0$ be the inclusion and

$$r\colon\mathbb{R}^{n+1}-0\longrightarrow S^n, \qquad r(z)=\frac{z}{|z|},$$

the standard retraction. Then, $r\circ i=\mathrm{id}_{S^n}$ and $i\circ r$ is *smoothly* homotopic to $\mathrm{id}_{\mathbb{R}^{n+1}-0}$ via the map

$$F(x,t)=(1-t)\frac{z}{|z|}+t\,z.$$

Thus, by Chapter 5, #19,

$$i^*\circ r^*=\mathrm{id}^*_{S^n}=\mathrm{id}\colon H^*_{\mathrm{de\,R}}(S^n)\longrightarrow H^*_{\mathrm{de\,R}}(S^n) \qquad \text{and}$$
$$r^*\circ i^*=\mathrm{id}^*_{\mathbb{R}^{n+1}-0}=\mathrm{id}\colon H^*_{\mathrm{de\,R}}(\mathbb{R}^{n+1}-0)\longrightarrow H^*_{\mathrm{de\,R}}(\mathbb{R}^{n+1}-0).$$

This means that

$$i^* \colon H^*_{\mathrm{de\,R}}(\mathbb{R}^{n+1}-0) \longrightarrow H^*_{\mathrm{de\,R}}(S^n)$$

is an isomorphism.

(b) If $p<0$ or $p>n$, $H^p_{\mathrm{de\,R}}(S^n)=0$ by definition because $E^p(S^n)=0$ in these cases. The space S^0 consists of two points and thus $H^0_{\mathrm{de\,R}}(S^0)\approx\mathbb{R}^2$. The $n,p=1$ case is done in 4.18 (it can also be verified from MV).

Suppose $n\geq 1$ and the statement holds for n. Let U and V be the complements of the south and north poles in S^{n+1}, respectively. Since these open subsets of S^{n+1} are diffeomorphic to $\mathbb{R}^{n+1}$,

$$H^p_{\mathrm{de\,R}}(U) \approx H^p_{\mathrm{de\,R}}(V) \approx \begin{cases} \mathbb{R}, & \text{if } p=0, \\ 0, & \text{if } p\neq 0. \end{cases}$$

by the Poincare Lemma. Furthermore, $U\cap V$ is diffeomorphic to $\mathbb{R}^{n+1}-0$. By part (a) and the induction assumption,

$$H^p_{\mathrm{de\,R}}(U\cap V) \approx H^p_{\mathrm{de\,R}}(S^n) \approx \begin{cases} \mathbb{R}, & \text{if } p=0,n; \\ 0, & \text{if } p\neq 0,n. \end{cases}$$

By MV, applied to $S^{n+1}=U\cup V$, the sequence

$$H^{p-1}_{\mathrm{de\,R}}(U)\oplus H^{p-1}_{\mathrm{de\,R}}(V) \longrightarrow H^{p-1}_{\mathrm{de\,R}}(U\cap V) \longrightarrow H^p_{\mathrm{de\,R}}(S^{n+1}) \longrightarrow H^p_{\mathrm{de\,R}}(U)\oplus H^p_{\mathrm{de\,R}}(V)$$

is exact for all $p\geq 1$. Thus, if $2\leq p\leq n$, $H^p_{\mathrm{de\,R}}(S^{n+1})=0$, since the two groups surrounding $H^p_{\mathrm{de\,R}}(S^{n+1})$ vanish. In the $p=n+1\geq 2$ case, the above sequence becomes

$$0 \longrightarrow \mathbb{R} \longrightarrow H^p_{\mathrm{de\,R}}(S^{n+1}) \longrightarrow 0.$$

Thus, $H^{n+1}_{\mathrm{de\,R}}(S^{n+1})\approx\mathbb{R}$. In the remaining $p=1$ case, we consider the first 5 terms of the long sequence:

$$0\longrightarrow H^0_{\mathrm{de\,R}}(S^{n+1})\longrightarrow H^0_{\mathrm{de\,R}}(U)\oplus H^0_{\mathrm{de\,R}}(V)\longrightarrow H^0_{\mathrm{de\,R}}(U\cap V)\xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(S^{n+1})\longrightarrow H^1_{\mathrm{de\,R}}(U)\oplus H^1_{\mathrm{de\,R}}(V).$$

Since $n\geq 1$, S^{n+1}, U, V, and $U\cap V$ are connected and this sequence reduces to

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}\oplus\mathbb{R} \longrightarrow \mathbb{R} \xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(S^{n+1}) \longrightarrow 0.$$

Since this sequence is exact, δ_0 must be the zero homomorphism and thus $H^1_{\mathrm{de\,R}}(S^{n+1})=0$. This completes verification of the inductive step.

Caution: In order to conclude that δ_0 is the zero homomorphism, it is essential that $\mathbb{R}$ is a field, rather than a ring. The same conclusion about δ_0 holds if we replace $\mathbb{R}$ by any field. However, if we replace $\mathbb{R}$ by the ring $\mathbb{Z}$, we could have

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f_0} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g_0} \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z}_2 \longrightarrow 0, \qquad f_0(a)=(a,0), \quad g_0(b,c)=(0,2c), \quad \delta_0(d)=d+2\mathbb{Z}.$$

This is an exact sequence of $\mathbb{Z}$-modules (i.e. abelian groups). In general, if $\mathbb{R}$ is a ring, the last group must be all torsion.

Remark: The fact that $H^1_{\mathrm{de\,R}}(S^n)=0$ for $n\geq 2$ can be obtained immediately, without any induction, from Hurewicz Theorem and de Rham Theorem (to be proved):

$$\pi_1(S^n)=0 \quad\Longrightarrow\quad H_1(S^n;\mathbb{Z})=\mathrm{Abel}\big(\pi_1(S^n)\big)=0 \quad\Longrightarrow\quad H_1(S^n;\mathbb{R})\approx H_1(S^n;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}=0$$
$$\Longrightarrow\quad H^1_{\mathrm{de\,R}}(S^n)\approx \big(H_1(S^n;\mathbb{R})\big)^*\approx 0.$$

(c) Suppose $S^n=M^p\times N^q$ for some $p,q>0$. Since S^n is compact and orientable, so are M and N (see Problem 5 on the 06 midterm). Let $\alpha\in E^p(M)$ and $\beta\in E^q(N)$ be nowhere-zero top forms. By Problem 5 on the 06 midterm,

$$\pi_1^*\alpha\wedge\pi_2^*\beta\in E^n(S^n)$$

is a nowhere-zero top form. Therefore,

$$\int_M \pi_1^*\alpha\wedge\pi_2^*\beta \qquad\Longrightarrow\qquad \big[\pi_1^*\alpha\wedge\pi_2^*\beta\big]\neq 0\in H^n_{\mathrm{de\,R}}(S^n)$$

by Stokes' Theorem. On the other hand,

$$\big[\pi_1^*\alpha\wedge\pi_2^*\beta\big]=\big[\pi_1^*\alpha\big]\wedge\big[\pi_2^*\beta\big]=\pi_1^*[\alpha]\wedge\pi_2^*[\beta].$$

Since $0<p,q<n$, by part (b)

$$H^p_{\mathrm{de\,R}}(S^n)=0,\;\; H^q_{\mathrm{de\,R}}(S^n)=0 \quad\Longrightarrow\quad \pi_1^*[\alpha]=0,\;\; \pi_2^*[\beta]=0 \quad\Longrightarrow\quad \big[\pi_1^*\alpha\wedge\pi_2^*\beta\big]=\pi_1^*[\alpha]\wedge\pi_2^*[\beta]=0.$$

This is a contradiction.

Problem 4 (20pts)

(a) *Use Mayer-Vietoris (not Kunneth formula) to compute $H^*_{\mathrm{de\,R}}(T^2)$, where T^2 is the two-torus, $S^1\times S^1$. Find a basis for $H^*_{\mathrm{de\,R}}(T^2)$; justify your answer.*
(b) *Let Σ_g be a compact connected orientable surface of genus g (donut with g holes). Let $B\subset\Sigma_g$ be a small closed ball or a single point. Relate $H^*_{\mathrm{de\,R}}(\Sigma_g-B)$ to $H^*_{\mathrm{de\,R}}(\Sigma_g)$.*
(c) *Show that*

$$H^p_{\mathrm{de\,R}}(\Sigma_g)=\begin{cases}\mathbb{R}, & \text{if } p=0,2;\\ \mathbb{R}^{2g}, & \text{if } p=1;\\ 0, & \text{otherwise.}\end{cases}$$

(a) View T^2 as a donut lying flat on a table. Let U and V be the complements of the top and bottom circles in T^2, respectively. Formally,

$$U=S^1\times\big(S^1-\{1\}\big)\approx S^1\times\mathbb{R},\quad V=S^1\times\big(S^1-\{-1\}\big)\approx S^1\times\mathbb{R} \qquad\Longrightarrow\qquad U\cap V\approx S^1\times\mathbb{R}^*.$$

By the invariance of the de Rham cohomology under smooth homotopies

$$H^p_{\mathrm{de\,R}}(U)\approx H^p_{\mathrm{de\,R}}(V)\approx H^p_{\mathrm{de\,R}}(S^1)\approx\begin{cases}\mathbb{R}, & \text{if } p=0,1;\\ 0, & \text{if } p\neq 0,1;\end{cases}$$

$$H^p_{\mathrm{de\,R}}(U\cap V)\approx H^p_{\mathrm{de\,R}}\big(S^1\sqcup S^1\big)\approx H^p_{\mathrm{de\,R}}(S^1)\oplus H^p_{\mathrm{de\,R}}(S^1)\approx\begin{cases}\mathbb{R}^2, & \text{if } p=0,1;\\ 0, & \text{if } p\neq 0,1.\end{cases}$$

Since T^2 is connected, $\boxed{H^0_{\mathrm{de\,R}}(T^2)\approx\mathbb{R}}$ By MV,

$$\begin{aligned}&0\longrightarrow H^0_{\mathrm{de\,R}}(T^2)\longrightarrow H^0_{\mathrm{de\,R}}(U)\oplus H^0_{\mathrm{de\,R}}(V)\longrightarrow H^0_{\mathrm{de\,R}}(U\cap V)\\&\xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(T^2)\longrightarrow H^1_{\mathrm{de\,R}}(U)\oplus H^1_{\mathrm{de\,R}}(V)\xrightarrow{g_1} H^1_{\mathrm{de\,R}}(U\cap V)\longrightarrow H^2_{\mathrm{de\,R}}(T^2)\longrightarrow H^2_{\mathrm{de\,R}}(U)\oplus H^2_{\mathrm{de\,R}}(V).\end{aligned}$$

The remaining groups vanish for dimensional reasons. Plugging in for the known groups, we obtain

$$\begin{aligned}&0\longrightarrow\mathbb{R}\longrightarrow\mathbb{R}\oplus\mathbb{R}\longrightarrow\mathbb{R}^2\\&\xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(T^2)\longrightarrow\mathbb{R}\oplus\mathbb{R}\xrightarrow{g_1}\mathbb{R}^2\longrightarrow H^2_{\mathrm{de\,R}}(T^2)\longrightarrow 0.\end{aligned}$$

By the exactness of the sequence, the image of δ_0 must be $\mathbb{R}$. Since $H^1_{\mathrm{de\,R}}(S^1)$ is nonzero and the inclusion map $S^1\times\mathbb{R}^-\longrightarrow S^1\times\mathbb{R}$ induces an isomorphism in cohomology (being a smooth homotopy equivalence), the inclusion map

$$U\cap V\approx S^1\times(\mathbb{R}^-\sqcup\mathbb{R}^+)\longrightarrow U\approx S^1\times\mathbb{R}$$

induces a nontrivial homomorphism on the first cohomology. Thus, the homomorphism g_1 in the above sequence is nontrivial. Its cokernel is $H^2_{\mathrm{de\,R}}(T^2)$. Since T^2 is compact and oriented, $H^2_{\mathrm{de\,R}}(T^2)$ is nonzero and $\operatorname{Im} g_1\subsetneq\mathbb{R}^2$. Thus, $\operatorname{Im} g_1$ is a one-dimensional subspace of $\mathbb{R}^2$ and $\boxed{H^2_{\mathrm{de\,R}}(T^2)\approx\mathbb{R}}$ (this can also be obtained by studying g_1 in more detail). The above exact sequence then induces an exact sequence

$$0\longrightarrow\operatorname{Im}\delta_0\approx\mathbb{R}\longrightarrow H^1_{\mathrm{de\,R}}(T^2)\longrightarrow\mathbb{R}\oplus\mathbb{R}\xrightarrow{g_1}\operatorname{Im} g_1\approx\mathbb{R}^1\longrightarrow 0.$$

From this exact sequence we conclude that $\boxed{H^1_{\mathrm{de\,R}}(T^2)\approx\mathbb{R}^2}$

The one-dimensional vector space $H^0_{\mathrm{de\,R}}(T^2)$ consists of the constant functions on T^2. Thus the constant function 1 forms a basis for $H^0_{\mathrm{de\,R}}(T^2)$. If $\mathrm{d}\theta$ is the standard volume form on S^1, as in 4.18, or any other nowhere-zero one-form on S^1, then $\pi_1^*\mathrm{d}\theta\wedge\pi_2^*\mathrm{d}\theta$ is a nowhere-zero top form on T^2. Therefore,

$$[\pi_1^*\mathrm{d}\theta]\wedge[\pi_2^*\mathrm{d}\theta]=[\pi_1^*\mathrm{d}\theta\wedge\pi_2^*\mathrm{d}\theta]\neq 0\in H^2_{\mathrm{de\,R}}(T^2)$$

and $\{[\pi_1^*\mathrm{d}\theta],[\pi_2^*\mathrm{d}\theta]\}$ must be a linearly independent set of vectors in $H^1_{\mathrm{de\,R}}(T^2)$. Since $H^1_{\mathrm{de\,R}}(T^2)$ is two-dimensional, this is a basis for $H^1_{\mathrm{de\,R}}(T^2)$. Finally, since $H^2_{\mathrm{de\,R}}(T^2)$ is one-dimensional and $[\pi_1^*\mathrm{d}\theta]\wedge[\pi_2^*\mathrm{d}\theta]$ is nonzero, it forms a basis for $H^2_{\mathrm{de\,R}}(T^2)$.

Remark: Note that we have determined $H^*_{\mathrm{de\,R}}(T^2)$ as a graded *ring*. By the above, we have an isomorphism of graded rings

$$H^*_{\mathrm{de\,R}}(T^2)=\Lambda^*H^1_{\mathrm{de\,R}}(T^2)=\Lambda^*\mathbb{R}\big\{[\pi_1^*\mathrm{d}\theta],[\pi_2^*\mathrm{d}\theta]\big\}\approx\Lambda^*\mathbb{R}^2,$$

where $\mathbb{R}\{[\pi_1^*d\theta],[\pi_2^*d\theta]\}$ is the vector space with basis $\{[\pi_1^*d\theta],[\pi_2^*d\theta]\}$. The first equality above holds for all tori.

(b) Since Σ_g is connected, so is Σ_g-B and therefore

$$H^0_{\mathrm{de\,R}}(\Sigma_g-B)\approx H^0_{\mathrm{de\,R}}(\Sigma_g)\approx\mathbb{R}.$$

Let V be a small open ball in Σ_g containing B. Then, $(\Sigma_g-B)\cap V$ is either an open disk with a point removed or an open annulus, so that

$$(\Sigma_g-B)\cap V \approx S^1\times(-1,1) \qquad\Longrightarrow\qquad H^p_{\mathrm{de\,R}}(\Sigma_g-B)\approx H^p_{\mathrm{de\,R}}(S^1)\approx\begin{cases}\mathbb{R}, & \text{if } p=0,1,\\ 0, & \text{if } p\neq 0,1,\end{cases}$$

by the invariance of the de Rham cohomology under smooth homotopy equivalences. Since Σ_g is the union of the open subsets Σ_g-B and V, by MV

$$\begin{aligned}0\longrightarrow H^0_{\mathrm{de\,R}}(\Sigma_g)&\longrightarrow H^0_{\mathrm{de\,R}}(\Sigma_g-B)\oplus H^0_{\mathrm{de\,R}}(V)\longrightarrow H^0_{\mathrm{de\,R}}\big((\Sigma_g-B)\cap V\big)\\ \xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(\Sigma_g)&\longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_g-B)\oplus H^1_{\mathrm{de\,R}}(V)\xrightarrow{g_1} H^1_{\mathrm{de\,R}}\big((\Sigma_g-B)\cap V\big)\\ \xrightarrow{\delta_1} H^2_{\mathrm{de\,R}}(\Sigma_g)&\longrightarrow H^2_{\mathrm{de\,R}}(\Sigma_g-B)\oplus H^2_{\mathrm{de\,R}}(V)\longrightarrow H^2_{\mathrm{de\,R}}\big((\Sigma_g-B)\cap V\big).\end{aligned}$$

Plugging in for the known groups, we obtain

$$\begin{aligned}0\longrightarrow \mathbb{R}&\longrightarrow \mathbb{R}\oplus\mathbb{R}\longrightarrow\mathbb{R}\\ \xrightarrow{\delta_0} H^1_{\mathrm{de\,R}}(\Sigma_g)&\longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_g-B)\oplus 0\xrightarrow{g_1}\mathbb{R}\\ \xrightarrow{\delta_1} H^2_{\mathrm{de\,R}}(\Sigma_g)&\longrightarrow H^2_{\mathrm{de\,R}}(\Sigma_g-B)\oplus 0\longrightarrow 0.\end{aligned}$$

By exactness, δ_0 must be zero and therefore we have an exact sequence

$$0\longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_g)\longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_g-B)\xrightarrow{g_1} H^1_{\mathrm{de\,R}}\big((\Sigma_g-B)\cap V\big)\approx\mathbb{R}\xrightarrow{\delta_1} H^2_{\mathrm{de\,R}}(\Sigma_g)\longrightarrow H^2_{\mathrm{de\,R}}(\Sigma_g-B)\longrightarrow 0.$$

In the next paragraph we show that the homomorphism δ_1 is nonzero. By exactness, g_1 must then be trivial and we obtain two exact sequences

$$0\longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_g)\longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_g-B)\xrightarrow{g_1}0, \qquad 0\longrightarrow\mathbb{R}\xrightarrow{\delta_1}H^2_{\mathrm{de\,R}}(\Sigma_g)\longrightarrow H^2_{\mathrm{de\,R}}(\Sigma_g-B)\longrightarrow 0.$$

From this, we conclude that

$$H^p_{\mathrm{de\,R}}(\Sigma_g-B)\approx\begin{cases}\mathbb{R}, & \text{if } p=0;\\ H^1_{\mathrm{de\,R}}(\Sigma_g), & \text{if } p=1;\\ H^2_{\mathrm{de\,R}}(\Sigma_g)/\mathbb{R}, & \text{if } p=2.\end{cases}$$

Furthermore, the isomorphism between $H^1_{\mathrm{de\,R}}(\Sigma_g)$ and $H^1_{\mathrm{de\,R}}(\Sigma_g-B)$ is induced by the inclusion $\Sigma_g-B\longrightarrow\Sigma_g$.

In order to see that the homomorphism

$$\delta_1\colon H^1_{\mathrm{de\,R}}\big((\Sigma_g-B)\cap V\big)\longrightarrow H^2_{\mathrm{de\,R}}(\Sigma_g)$$

is nonzero, we use Problem 2a and the definition of δ given in the *Note* there. Choose $\varphi\in C^\infty(\Sigma_g)$ such that $\mathrm{supp}\,\varphi\subset V$ and $\mathrm{supp}\{1-\varphi\}\subset\Sigma_g-B$. Let

$$\gamma=\pi_1^*\mathrm{d}\theta\in E^1\big(\mathbb{R}\times(1/2,1)\big)\approx E^1\big((\Sigma_g-B)\cap V\big).$$

We will show that

$$\delta\big([\gamma]\big)\equiv\big[\mathrm{d}\varphi\wedge\gamma\big]\equiv\big[\mathrm{d}\varphi\wedge\pi_1^*\mathrm{d}\theta\big]\neq 0\in H^2_{\mathrm{de\,R}}(\Sigma_g)$$

by showing that the integral of $\mathrm{d}\varphi\wedge\gamma$ over the *orientable* manifold Σ_g is not zero. Since the compact set $\mathrm{supp}(1-\varphi)\cap\mathrm{supp}\,\varphi$ is contained in the open annulus $V-B$, there exist $1/2<r<R<1$ such that

$$\mathrm{supp}(1-\varphi)\cap\mathrm{supp}\,\varphi\subset A\equiv S^1\times[r,R]\subset V-B \qquad\Longrightarrow\qquad \varphi|_{S^1\times r}=1,\quad \varphi|_{S^1\times R}=0.$$

Since $\mathrm{d}\varphi\wedge\pi_1^*\mathrm{d}\theta$ vanishes outside of A,

$$\begin{aligned}\int_M \mathrm{d}\varphi\wedge\pi_1^*\mathrm{d}\theta &= \int_A \mathrm{d}\varphi\wedge\pi_1^*\mathrm{d}\theta = \int_A \mathrm{d}\big(\varphi\pi_1^*\mathrm{d}\theta\big) = \int_{\partial A}\varphi\pi_1^*\mathrm{d}\theta \\ &= \pm\Big(\int_{S^1\times R}\varphi\pi_1^*\mathrm{d}\theta - \int_{S^1\times r}\varphi\pi_1^*\mathrm{d}\theta\Big) = \pm\int_{S^1\times r}\pi_1^*\mathrm{d}\theta = \pm2\pi\neq0.\end{aligned}$$

The third equality above follows from Stokes' Theorem.

Remark: If M is a connected *non*-compact n-dimensional manifold, $H^n_{\mathrm{de\,R}}(M)=0$; see Spivak p369 for a proof. This fact would simplify the solution, but first needs to be established.

(c) The cases $g=0,1$ were proved in Problem 3b and part (a) above. Suppose $g\geq1$ and the statement holds for g. Since Σ_{g+1} is connected, $H^0_{\mathrm{de\,R}}(\Sigma_{g+1})\approx\mathbb{R}$. Note that

$$\Sigma_{g+1}=\Sigma_g\#\Sigma_1=\Sigma_g\#T^2,$$

i.e. Σ_{g+1} can be obtained from Σ_g and T^2 by removing small open disks from the two surfaces and joining the two boundary circles together. We thus can write

$$\Sigma_{g+1}=(\Sigma_g-B_1)\cup(T^2-B_2),$$

where B_1 and B_2 are slightly smaller closed balls. The overlap of U and V in Σ_{g+1} is a small band around the circle joining the two surfaces. Thus,

$$(\Sigma_g-B_1)\cap(T^2-B_2)\approx S^1\times(-1,1)\quad\Longrightarrow\quad H^p_{\mathrm{de\,R}}\big((\Sigma_g-B_1)\cap(T^2-B_2)\big)\approx H^p_{\mathrm{de\,R}}(S^1)\approx\begin{cases}\mathbb{R}, & \text{if } p=0,1,\\ 0, & \text{if } p\neq0,1,\end{cases}$$

by the invariance of the de Rham cohomology under smooth homotopy equivalences. By the induction assumption and part (b),

$$H^p_{\mathrm{de\,R}}(\Sigma-B_1)\approx\begin{cases}\mathbb{R}, & \text{if } p=0;\\ \mathbb{R}^{2g}, & \text{if } p=1;\\ 0, & \text{otherwise;}\end{cases}\qquad\text{and}\qquad H^p_{\mathrm{de\,R}}(T^2-B_2)\approx\begin{cases}\mathbb{R}, & \text{if } p=0;\\ \mathbb{R}^{2}, & \text{if } p=1;\\ 0, & \text{otherwise.}\end{cases}$$

Since Σ_{g+1} is the union of open subsets Σ_g-B_1 and T^2-B_2, by MV

$$\begin{aligned}0\longrightarrow H^0_{\mathrm{de\,R}}(\Sigma_{g+1})&\longrightarrow H^0_{\mathrm{de\,R}}(\Sigma_g-B_1)\oplus H^0_{\mathrm{de\,R}}(T^2-B_2)\longrightarrow H^0_{\mathrm{de\,R}}\big((\Sigma_g-B_1)\cap(T^2-B_2)\big)\\ \stackrel{\delta_0}{\longrightarrow} H^1_{\mathrm{de\,R}}(\Sigma_{g+1})&\longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_g-B_1)\oplus H^1_{\mathrm{de\,R}}(T^2-B_2)\longrightarrow H^1_{\mathrm{de\,R}}\big((\Sigma_g-B_1)\cap(T^2-B_2)\big)\\ \longrightarrow H^2_{\mathrm{de\,R}}(\Sigma_{g+1})&\longrightarrow H^2_{\mathrm{de\,R}}(\Sigma_g-B_1)\oplus H^2_{\mathrm{de\,R}}(T^2-B_2).\end{aligned}$$

Plugging in for the known groups, we obtain

$$\begin{aligned}&0\longrightarrow\mathbb{R}\longrightarrow\mathbb{R}\oplus\mathbb{R}\longrightarrow\mathbb{R}\\ &\stackrel{\delta_0}{\longrightarrow} H^1_{\mathrm{de\,R}}(\Sigma_{g+1})\longrightarrow\mathbb{R}^{2g}\oplus\mathbb{R}^2\longrightarrow\mathbb{R}\stackrel{\delta_1}{\longrightarrow} H^2_{\mathrm{de\,R}}(\Sigma_{g+1})\longrightarrow0.\end{aligned}$$

By exactness, δ_0 must be zero and therefore we have an exact sequence

$$0 \longrightarrow H^1_{\mathrm{de\,R}}(\Sigma_{g+1}) \longrightarrow \mathbb{R}^{2g+2} \xrightarrow{g_1} \mathbb{R} \xrightarrow{\delta_1} H^2_{\mathrm{de\,R}}(\Sigma_{g+1}) \longrightarrow 0.$$

Since Σ_{g+1} is compact and orientable, $H^2_{\mathrm{de\,R}}(\Sigma_{g+1})$ is nonzero. Therefore, the homomorphism δ_1 is nonzero and thus an isomorphism, while the homomorphism g_1 is zero. It follows that

$$H^1_{\mathrm{de\,R}}(\Sigma_{g+1}) \approx \mathbb{R}^{2g+2} \qquad \text{and} \qquad H^2_{\mathrm{de\,R}}(\Sigma_{g+1}) \approx \mathbb{R}.$$

This completes verification of the inductive step.

Remark: The de Rham cohomology of Σ_g can be determined without Mayer-Vietoris. Since Σ_g is connected, $H^0_{\mathrm{de\,R}}(\Sigma_g) \approx \mathbb{R}$. Since Σ_g is a 2-dimensional compact orientable manifold, by the Poincare Duality (to be proved)

$$H^2_{\mathrm{de\,R}}(\Sigma_g) \approx \big(H^{2-2}_{\mathrm{de\,R}}(\Sigma_g)\big)^* \approx \mathbb{R}.$$

Finally, by Hurewicz Theorem (Problem 1) and de Rham Theorem (to be proved):

$$\pi_1(\Sigma_g) = \big\langle a_1, b_1, \ldots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \big\rangle \quad \Longrightarrow \quad H_1(\Sigma; \mathbb{Z}) = \mathrm{Abel}\big(\pi_1(\Sigma_g)\big) \approx \mathbb{Z}^{2g}$$
$$\Longrightarrow \quad H_1(\Sigma_g; \mathbb{R}) \approx H_1(\Sigma_g; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \approx \mathbb{R}^{2g} \qquad \Longrightarrow \qquad H^1_{\mathrm{de\,R}}(\Sigma_g) \approx \big(H_1(\Sigma_g; \mathbb{R})\big)^* \approx \mathbb{R}^{2g}.$$

Problem 5 (10pts)

(a) *Suppose $q\colon \tilde{M} \longrightarrow M$ is a regular covering projection with a finite group of deck transformations G (so that $M = \tilde{M}/G$). Show that*

$$q^*\colon H^*_{\mathrm{de\,R}}(M) \longrightarrow H^*_{\mathrm{de\,R}}(\tilde{M})^G \equiv \big\{\alpha \in H^*_{\mathrm{de\,R}}(\tilde{M})\colon g^*\alpha = \alpha \ \forall\, g \in G\big\}$$

is an isomorphism. Does this statement continue to hold if G is not assumed to be finite?
(b) *Determine $H^*_{\mathrm{de\,R}}(K)$, where K is the Klein bottle. Find a basis for $H^*_{\mathrm{de\,R}}(K)$; justify your answer.*

(a) If $g \in G$, $q = q \circ g$ and

$$q^*[\beta] = \{q \circ g\}^*[\beta] = g^* q^* \big[\beta\big] \quad \forall\, [\beta] \in H^*_{\mathrm{de\,R}}(M).$$

Thus, the image of q^* is contained in $H^*_{\mathrm{de\,R}}(\tilde{M})^G$. We next show that the image of q^* is all of $H^*_{\mathrm{de\,R}}(\tilde{M})^G$. If $\alpha \in E^*(\tilde{M})$ is such that $[\alpha] \in H^*_{\mathrm{de\,R}}(\tilde{M})^G$, let

$$\tilde{\alpha} = \frac{1}{|G|} \sum_{g \in G} g^* \alpha \in E^*(\tilde{M})^G.$$

Since $\mathrm{d} g^* = g^* \mathrm{d}$, $\mathrm{d}\tilde{\alpha} = 0$ and

$$[\tilde{\alpha}] = \frac{1}{|G|} \sum_{g \in G} [g^* \alpha] = \frac{1}{|G|} \sum_{g \in G} g^* [\alpha] = \frac{1}{|G|} \sum_{g \in G} [\alpha] = [\alpha] \in H^*_{\mathrm{de\,R}}(\tilde{M}).$$

On the other hand, since $\tilde{\alpha}\in E^*(\tilde{M})^G$, $\tilde{\alpha}=q^*\beta$ for some $\beta\in E^*(M)$ by Problem 6b on PS6. Since $\mathrm{d}\tilde{\alpha}=0$ and q is a local diffeomorphism (and thus $q^*: E^*(M)\longrightarrow E^*(\tilde{M})$ is injective), $\mathrm{d}\beta=0$. Thus, $[\beta]\in H^*(M)$ and

$$[\alpha]=[\tilde{\alpha}]=q^*[\beta]\in H^*(\tilde{M}).$$

Thus, the map

$$q^*: H^*_{\mathrm{de\,R}}(M)\longrightarrow H^*_{\mathrm{de\,R}}(\tilde{M})^G$$

is surjective. Finally, we show that q^* is injective. Suppose $\beta\in E^*(M)$ and $q^*\beta=\mathrm{d}\alpha$ for some $\alpha\in E^*(\tilde{M})$. With $\tilde{\alpha}$ defined as above,

$$\mathrm{d}\tilde{\alpha}=\frac{1}{|G|}\sum_{g\in G}\mathrm{d}\,g^*\alpha=\frac{1}{|G|}\sum_{g\in G}g^*\mathrm{d}\alpha=\frac{1}{|G|}\sum_{g\in G}g^*q^*\beta=\frac{1}{|G|}\sum_{g\in G}q^*\beta=q^*\beta.$$

Since $\tilde{\alpha}\in E^*(\tilde{M})^G$, $\tilde{\alpha}=q^*\gamma$ for some $\gamma\in E^*(M)$ and

$$q^*\mathrm{d}\gamma=\mathrm{d}\,q^*\gamma=\mathrm{d}\tilde{\alpha}=q^*\beta.$$

Since q is a local diffeomorphism, q^* is injective and thus

$$\beta=\mathrm{d}\gamma\qquad\Longrightarrow\qquad[\beta]=[0]\in H^*_{\mathrm{de\,R}}(M),$$

i.e. q^* is injective on cohomology.

The statement may not hold if G is infinite. For example, if $q:\mathbb{R}\longrightarrow S^1$ is the standard covering map, the map

$$q^*: H^1_{\mathrm{de\,R}}(S^1)\approx\mathbb{R}\longrightarrow H^1_{\mathrm{de\,R}}(\mathbb{R})=0$$

cannot be injective.

(b) Since K is connected, $H^0_{\mathrm{de\,R}}(K)\approx\mathbb{R}$. By Exercise 3 on p454 of Munkres, there is a $2:1$ covering map $q: T^2\longrightarrow K$. The corresponding group of covering transformations is isomorphic to $\mathbb{Z}_2$. Let g be the non-trivial diffeomorphism. From Exercise 3, it can be written as

$$g\big(e^{i\theta_1},e^{i\theta_2}\big)=\big(-e^{i\theta_1},e^{-i\theta_2}\big)\equiv\big(g_1(e^{i\theta_1}),g_2(e^{i\theta_2})\big).$$

With $\mathrm{d}\theta$ as in Problem 4a,

$$\begin{aligned}g^*\pi_1^*\mathrm{d}\theta&=\{\pi_1\circ g\}^*\mathrm{d}\theta=\{g_1\circ\pi_1\}^*\mathrm{d}\theta=\pi_1^*g_1^*\mathrm{d}\theta=\pi_1^*\mathrm{d}\theta;\\ g^*\pi_2^*\mathrm{d}\theta&=\{\pi_2\circ g\}^*\mathrm{d}\theta=\{g_2\circ\pi_2\}^*\mathrm{d}\theta=\pi_2^*g_2^*\mathrm{d}\theta=\pi_2^*(-\mathrm{d}\theta)=-\pi_2^*\mathrm{d}\theta;\\ g^*\big(\pi_1^*\mathrm{d}\theta\wedge\pi_2^*\mathrm{d}\theta\big)&=g^*\pi_1^*\mathrm{d}\theta\wedge g^*\pi_2^*\mathrm{d}\theta=\pi_1^*\mathrm{d}\theta\wedge(-\pi_2^*\mathrm{d}\theta)=-\pi_1^*\mathrm{d}\theta\wedge\pi_2^*\mathrm{d}\theta.\end{aligned}$$

By Problem 4a, $\{[\pi_1^*\mathrm{d}\theta],[\pi_2^*\mathrm{d}\theta]\}$ and $\{[\pi_1^*d\theta]\wedge[\pi_2^*\mathrm{d}\theta]\}$ are bases for $H^1_{\mathrm{de\,R}}(T^2)$ and $H^2_{\mathrm{de\,R}}(T^2)$, respectively. Thus, by part (a),

$$H^1_{\mathrm{de\,R}}(K)\approx H^1_{\mathrm{de\,R}}(T^2)^G=\mathbb{R}\big\{[\pi_1^*\mathrm{d}\theta]\big\}\approx\mathbb{R},\qquad H^2_{\mathrm{de\,R}}(K)\approx H^2_{\mathrm{de\,R}}(T^2)^G=0.$$

Since the isomorphisms are induced by q^*, a basis for $H^1_{\mathrm{de\,R}}(K)$ consists of the equivalence class of the one-form α on K such that $q^*\alpha=\pi_1^*\mathrm{d}\theta$. A basis for $H^0_{\mathrm{de\,R}}(K)$ is formed by the constant function 1.

Problem 6: Chapter 5, #4 (5pts)

A smooth function f on a manifold M determines a section $\mathbf{f}$ of the sheaf of germs of smooth functions, $\mathfrak{C}^\infty(M)$. The set $f^{-1}(0)$ is closed, while $\mathbf{f}^{-1}(0)$ is open. How do you reconcile these two facts? Consider examples.

The section $\mathbf{f}$ of the sheaf $\mathfrak{C}^\infty(M)$ vanishes at some $p\in M$ if its germ at p is the same as the germ of the 0-function at p. This means that for every $p\in\mathbf{f}^{-1}(0)$, there exists an open neighborhood U_p of p in M such that $f|_{U_p}\equiv 0$, so that

$$\mathbf{f}^{-1}(0) \equiv \bigcup_{p\in M} U_p$$

is open in M. In other words, vanishing of $\mathbf{f}$ at a point p means vanishing of f on a neighborhood of p; the latter is an open condition on p.

As an example, suppose $f\colon \mathbb{R}\longrightarrow\mathbb{R}$, $f(x)=x$. Then, $f(0)=0$, but $\mathbf{f}(0)\neq 0$ because f does not vanish on a neighborhood of 0. As another example, suppose $f\colon\mathbb{R}\longrightarrow\mathbb{R}$ is a smooth function such that

$$f(x) = 0 \quad \forall\, x \le 0, \qquad f(x) > 0 \quad \forall\, x > 0.$$

Then, $\mathbf{f}^{-1}(0)=\mathbb{R}^-$, while $f^{-1}(0)=\mathbb{R}^-\cup\{0\}$.

Problem 7 (5pts)

Let $K=\mathbb{Z}$ and let $\pi\colon \mathcal{S}_0\longrightarrow\mathbb{R}$ be the corresponding skyscraper sheaf, with the only non-trivial stack over $0\in\mathbb{R}$; see Subsection 5.11. What is $\mathcal{S}_0$ as a topological space?

This is a line with countably many origins, indexed by $\mathbb{Z}$. Explicitly, $S_0 = \mathbb{R}^* \sqcup 0\times\mathbb{Z}$ as sets. The projection map is given by

$$\pi\colon S_0 \longrightarrow \mathbb{R}, \qquad \pi(x) = \begin{cases} x, & \text{if } x\in\mathbb{R}^*;\\ 0, & \text{if } x\in 0\times\mathbb{Z}.\end{cases}$$

Each fiber of this projection map is a $\mathbb{Z}$-module, either 0 or $\mathbb{Z}$. A basis for the topology on $\mathcal{S}_0$ consists of the intervals (a,b) with $ab\ge 0$, and the sets

$$(a,b)_m \equiv (a,0)\sqcup\{0\times m\}\sqcup(0,b),$$

with $ab<0$ (i.e. a and b have different signs) and $m\in\mathbb{Z}$. This topology is forced on $\mathcal{S}_0$ by the requirement that each point $x\in\mathbb{R}^*$ and $(0,m)\in 0\times\mathbb{Z}$ have a neighborhood U such that $\pi\colon U\longrightarrow\pi(U)$ is a homeomorphism. With the given topology,

$$\pi\colon(-\infty,\infty)_m\longrightarrow\mathbb{R}$$

is a homeomorphism for all $m\in\mathbb{Z}$. For $k\in\mathbb{Z}$, the multiplication map by k induces a homeomorphism

$$(-\infty,\infty)_m\longrightarrow(-\infty,\infty)_{km},$$

while the addition map restricts to a homeomorphism

$$\big\{(m_1,m_2)\big\}\cup\big\{(x,x)\colon x\in\mathbb{R}^*\big\}\longrightarrow(-\infty,\infty)_{m_1+m_2}.$$

Thus, the $\mathbb{Z}$-module operations are continuous.