MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 6
Due before 5pm on Tuesday, 3/22, in Math 3-111

Note 1: This problem set has two pages. Please review Sections 10-12 of *Lecture Notes* before starting on Problems (4) and (5).

Note 2: Since this problem set covers 1.5 weeks, it is **longer than usual**. However, nearly all of the problems (especially parts of them) have *very* short solutions, but you may not be able to come up with them right away. You are very much discouraged from postponing this problem set until just before it is due or even the weekend before it is due; doing so would leave you with no time to study for the midterm.

(1) Let X be the vector field on $\mathbb{R}^n$ given by $X = \sum_{i=1}^{i=n} x_i \frac{\partial}{\partial x_i}$.

- (a) Determine the time t-flow $X_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of X (give a formula).
- (b) Use (a) to show directly from the definition of the Lie derivative L_X that the homomorphism defined by

$$R_k : E^k(\mathbb{R}^n) \longrightarrow E^k(\mathbb{R}^n), \qquad f \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} \longrightarrow \left(\int_0^1 s^{k-1} f(sx) \mathrm{d}s \right) \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

  is a left inverse for L_X if $k \geq 1$ (this is used in the proof of the Poincare Lemma).
- (c) Is R_k also a right inverse for L_X for $k \geq 1$? What happens for $k = 0$?

(2) Chapter 4, #19 (p160)

(3) Show that a one-form α on S^1 is exact if and only if

$$\int_{[0,1]} f^* \alpha = 0$$

for every smooth function $f : [0,1] \longrightarrow S^1$ such that $f(0) = f(1)$.
Note: In light of 4.14, this confirms de Rham's theorem 4.17 for $M = S^1$ and $p = 1$.

(4) (a) Suppose $\varphi : M \longrightarrow \mathbb{R}^N$ is an immersion. Show that M is orientable if and only if the normal bundle to the immersion φ is orientable.
(b) Chapter 4, #1 (p157)

(5) Let M be a smooth manifold.

- (a) Show that every real vector bundle $V \longrightarrow M$ admits a Riemannian metric and every complex vector bundle admits a hermitian metric.
- (b) Show that if M is connected and there exists a non-orientable vector bundle $V \longrightarrow M$, then M admits a connected double-cover (2:1 covering map).
- (c) Show that if the order of $\pi_1(M)$ is finite and odd, then M is orientable.

(6) (a) Show that the antipodal map on $S^n \subset \mathbb{R}^{n+1}$ (i.e. $x \longrightarrow -x$) is orientation-preserving if n is odd and orientation-reversing if n is even.
(b) Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.
(c) Describe the orientable double cover of $\mathbb{R}P^n \times \mathbb{R}P^n$ with n even.

(7) (a) Show that every diffeomorphism $f\colon S^n \longrightarrow S^n$ that has no fixed points is smoothly homotopic to the antipodal map (x is a *fixed point* of f if $f(x)=x$).
(b) Show that if $\pi\colon S^n \longrightarrow M$ is a covering projection onto a smooth manifold M and $|\pi_1(M)| \neq 2$, then M is orientable.

(8) (a) Show that if X is a smooth nowhere-vanishing[1] vector field on a compact manifold M, then the flow $X_t\colon M \longrightarrow M$ of X has no fixed points for some $t \in \mathbb{R}$.
(b) Show that S^n admits a nowhere vanishing vector field if and only if n is odd.
(c) Show that the tangent bundle of S^n is not trivial if $n \geq 1$ is even.
Note: In fact, TS^n is trivial if and only if $n=0,1,3,7$.

(9) Suppose M is a compact oriented 3-manifold with boundary and $\partial M = T^2 = S^1 \times S^1$. Let

$$\pi_1, \pi_2\colon T^2 \longrightarrow S^1$$

be the two projection maps. Show that it is impossible to extend both (as opposed to at least one of) $\alpha_1 \equiv \pi_1^* \mathrm{d}\theta$ and $\alpha_2 \equiv \pi_2^* \mathrm{d}\theta$ to closed forms on M ($\mathrm{d}\theta$ is as in Problem (3)).

[1] $X_p \neq 0$ for all $p \in M$