MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 5

Problem 1 (5pts)

Let V be a vector space of dimension n and $\Omega\in\Lambda^n V^*$ a nonzero element. Show that the homomorphism

$$V \longrightarrow \Lambda^{n-1}V^*, \qquad v \longrightarrow i_v\Omega,$$

where i_v is the contraction map, is an isomorphism.

Let $\{v_i\}$ be a basis for V and $\{v_i^*\}$ the dual basis for V^*, i.e. $v_i^*(v_j)=\delta_{ij}$. Then, for some $C\in\mathbb{R}-0$

$$\Omega = C\, v_1^*\wedge\ldots\wedge v_n^* \qquad \Longrightarrow \qquad i_{v_k}\Omega = (-1)^{k-1}C\, v_1^*\wedge\ldots\wedge v_{k-1}^*\wedge v_{k+1}^*\wedge\ldots\wedge v_n^*.$$

Thus, the above homomorphism is surjective (since every basis element is in the image) and therefore an isomorphism (since the dimensions are the same).

Problem 2 (10pts)

Suppose M is a smooth n-manifold.
(a) *Let Ω be a nowhere-zero n-form on M. Show that for every $p\in M$ there exists a smooth chart $(x_1,\ldots,x_n)\colon U\longrightarrow\mathbb{R}^n$ near p such that*

$$\Omega|_U = \mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n.$$

(b) *Let α be a closed nowhere-zero $(n{-}1)$-form on M. Show that for every $p\in M$ there exists a smooth chart $(x_1,\ldots,x_n)\colon U\longrightarrow\mathbb{R}^n$ near p such that*

$$\alpha|_U = \mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\ldots\wedge\mathrm{d}x_n.$$

(a) Let $\varphi=(y_1,\ldots,y_n)\colon V\longrightarrow\mathbb{R}^n$ be a smooth chart near p. Since $\Lambda^n T_p^*M$ is one-dimensional, there exists $f\in C^\infty(V)$ such that

$$\alpha|_V = f\,\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n.$$

Let $F\in C^\infty(V)$ be a function such that $\frac{\partial}{\partial y_1}F=f$, e.g.

$$F\big(\varphi^{-1}(y_1,\ldots,y_n)\big) = \int_0^{y_1} f\big(\varphi^{-1}(t,y_2,\ldots,y_n)\big)\,\mathrm{d}t.$$

Define smooth functions

$$(x_1,\ldots,x_n)\colon V\longrightarrow\mathbb{R}^n \qquad \text{by} \qquad x_i=\begin{cases} F, & \text{if } i=1;\\ y_i, & \text{if } i\ge 2;\end{cases}$$

$$\Longrightarrow \qquad \mathrm{d}x_i=\begin{cases}\sum_{j=1}^{j=n}\big(\frac{\partial}{\partial y_j}F\big)\mathrm{d}y_j, & \text{if } i=1\\ \mathrm{d}y_i, & \text{if } i\ge 2\end{cases} = \begin{cases} f\,\mathrm{d}y_1+\sum_{j=2}^{j=n}\big(\frac{\partial}{\partial y_j}F\big)\mathrm{d}y_j, & \text{if } i=1;\\ \mathrm{d}y_i, & \text{if } i\ge 2;\end{cases}$$

$$\Longrightarrow \qquad \mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\ldots\wedge\mathrm{d}x_n = f\,\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n = \alpha|_V,$$

as needed. It remains to check that $(x_1,\ldots,x_n)$ restricts to a smooth chart near p. Since

$$\mathrm{d}_p x_1 \wedge \ldots \wedge \mathrm{d}_p x_n = \Omega_p \in \Lambda^{\mathrm{top}} T_p^* M$$

and $\Omega_p \neq 0$, $\{\mathrm{d}_p x_1, \ldots, \mathrm{d}_p x_n\}$ is basis for $T_p^* M$. Thus, by Corollary 1.30b, there exists $U \subset V$ such that $(x_1,\ldots,x_n)\colon U \longrightarrow \mathbb{R}^n$ is a smooth chart.

(b) Let $(y_1,\ldots,y_n)\colon V \longrightarrow \mathbb{R}^n$ be a smooth chart near p. By Problem 2,

$$\alpha|_V = i_X\big(\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n\big)$$

for a unique vector field X on M. This vector field is smooth because the $(n-1)$-form α is smooth. Since $\alpha_p \neq 0$, $X_p \neq 0$. Thus, by Proposition 1.53, there exists a smooth chart around p

$$\psi = (z_1,\ldots,z_n)\colon W \longrightarrow \mathbb{R}^n \qquad \text{s.t.} \qquad W \subset V, \quad \frac{\partial}{\partial z_1} = X|_W.$$

Then, for some $f \in C^\infty(W)$

$$\begin{aligned} &\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n = f\, \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n \\ \Longrightarrow \qquad &\alpha|_W = i_X\big(\mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n\big) = i_{\partial/\partial z_1}\big(f\, \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n\big) = f\, \mathrm{d}z_2 \wedge \ldots \wedge \mathrm{d}z_n \\ &\mathrm{d}\alpha|_W = \mathrm{d}\big(f\, \mathrm{d}z_2 \wedge \ldots \wedge \mathrm{d}z_n\big) = \left(\frac{\partial}{\partial z_1} f\right) \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \ldots \wedge \mathrm{d}z_n. \end{aligned}$$

Since $\mathrm{d}\alpha = 0$, $\frac{\partial}{\partial z_1} f = 0$. Let $F \in C^\infty(W)$ be given by

$$F\big(\psi^{-1}(z_1, z_2, \ldots, z_n)\big) = \int_0^{z_2} f\big(\psi^{-1}(z_1, t, z_3, \ldots, z_n)\big)\, \mathrm{d}t \qquad \Longrightarrow \qquad \frac{\partial}{\partial z_2} F = f.$$

Define smooth functions

$$(x_1,\ldots,x_n)\colon W \longrightarrow \mathbb{R}^n \qquad \text{by} \qquad x_i = \begin{cases} F, & \text{if } i=2; \\ z_i, & \text{if } i \neq 2; \end{cases}$$

$$\Longrightarrow \quad \mathrm{d}x_i = \begin{cases} \sum_{j=1}^{j=n} \big(\frac{\partial}{\partial z_j} F\big) \mathrm{d}z_j, & \text{if } i=2 \\ \mathrm{d}z_i, & \text{if } i \neq 2 \end{cases} = \begin{cases} \big(\frac{\partial}{\partial z_1} F\big) \mathrm{d}z_1 + f \mathrm{d}z_2 + \sum_{j=3}^{j=n} \big(\frac{\partial}{\partial z_j} F\big) \mathrm{d}z_j, & \text{if } i=2; \\ \mathrm{d}z_i, & \text{if } i \neq 2. \end{cases}$$

Since $\frac{\partial}{\partial z_1} f = 0$,

$$\begin{aligned} \left(\frac{\partial}{\partial z_1} F\right)\big(\psi^{-1}(z_1, z_2, \ldots, z_n)\big) &= \frac{\partial}{\partial z_1} \int_0^{z_2} f\big(\psi^{-1}(z_1, t, z_3, \ldots, z_n)\big)\, \mathrm{d}t \\ &= \int_0^{z_2} \left(\frac{\partial}{\partial z_1} f\big(\psi^{-1}(z_1, t, z_3, \ldots, z_n)\big)\right) \mathrm{d}t = \int_0^{z_2} 0\, \mathrm{d}t = 0 \\ \Longrightarrow \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \ldots \wedge \mathrm{d}x_n &= f\, \mathrm{d}z_2 \wedge \ldots \wedge \mathrm{d}z_n = \alpha|_W, \end{aligned}$$

as needed. It remains to check that $(x_1,\ldots,x_n)$ restricts to a smooth chart near p. Note that

$$\mathrm{d}_p x_1 \wedge \mathrm{d}_p x_2 \wedge \ldots \wedge \mathrm{d}_p x_n = \mathrm{d}_p z_1 \wedge \big(f\, \mathrm{d}_p z_2 \wedge \ldots \wedge \mathrm{d}_p z_n\big) = f\, \mathrm{d}_p z_1 \wedge \ldots \wedge \mathrm{d}_p z_n \in \Lambda^{\mathrm{top}} T_p^* M.$$

Since $\alpha_p \neq 0$ and thus $f(p) \neq 0$, $\{\mathrm{d}_p x_1, \ldots, \mathrm{d}_p x_n\}$ is basis for $T_p^* M$. Therefore, by Corollary 1.30b, there exists $U \subset V$ such that $(x_1,\ldots,x_n)\colon U \longrightarrow \mathbb{R}^n$ is a smooth chart.

Problem 3 (5pts)

Let M be a smooth manifold and $X, Y \in \Gamma(M;TM)$ smooth vector fields on M. Show that the Lie derivative satisfies

$$L_{[X,Y]} = [L_X, L_Y] \equiv L_X \circ L_Y - L_Y \circ L_X$$

as homomorphisms on $\Gamma(M;TM)$ and $E^k(M)$.

If $f \in C^\infty(M) = E^0(M)$, by 2.25a and the definition of $[X,Y]$

$$L_{[X,Y]}f = [X,Y]f = X(Yf) - Y(Xf) = L_X(L_Y f) - L_Y(L_X f) = [L_X, L_Y]f.$$

If $Z \in \Gamma(M;TM)$, by 2.25b and 1.45cd

$$L_{[X,Y]}Z = \big[[X,Y],Z\big] = -\big[Z,[X,Y]\big] = \big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] = L_X(L_Y Z) - L_Y(L_X Z) = [L_X, L_Y]Z.$$

If $\alpha \in E^k$ and $Z_1, \ldots, Z_k \in \Gamma(M;TM)$, by 2.25e and the two identifies above

$$\begin{aligned}\big\{L_{[X,Y]}\alpha\big\}(Z_1,\ldots,Z_k) &= L_{[X,Y]}\big(\alpha(Z_1,\ldots,Z_k)\big) - \sum_{i=1}^{i=k} \alpha\big(Z_1,\ldots,Z_{i-1}, L_{[X,Y]}Z, Z_{i+1},\ldots,Z_k\big) \\ &= [L_X, L_Y]\big(\alpha(Z_1,\ldots,Z_k)\big) - \sum_{i=1}^{i=k} \alpha\big(Z_1,\ldots,Z_{i-1},[L_X,L_Y]Z, Z_{i+1},\ldots,Z_k\big).\end{aligned}$$

Using 2.25e again gives

$$\begin{aligned}L_X\big(L_Y\big(\alpha(Z_1,\ldots,Z_k)\big)\big) &= L_X\bigg(\{L_Y\alpha\}(Z_1,\ldots,Z_k) + \sum_{i=1}^{i=k} \alpha(Z_1,\ldots,Z_{i-1},L_Y Z, Z_{i+1},\ldots,Z_k)\bigg) \\ &= \big\{L_X(L_Y\alpha)\big\}(Z_1,\ldots,Z_k) + \sum_{i=1}^{i=k} \{L_Y\alpha\}\big(Z_1,\ldots,Z_{i-1},L_X Z, Z_{i+1},\ldots,Z_k\big) \\ &\quad + \sum_{i=1}^{i=k} \Big(\{L_X\alpha\}\big(Z_1,\ldots,Z_{i-1},L_Y Z,Z_{i+1},\ldots,Z_k\big)\Big) + \alpha(Z_1,\ldots,Z_{i-1},L_X(L_Y Z),Z_{i+1},\ldots,Z_k) \\ &\quad + \sum_{i\neq j} \alpha\big(Z_1,\ldots,Z_{i-1},L_Y Z,Z_{i+1},\ldots,L_X Z_{j-1},L_X Z_j,L_X Z_{j+1},\ldots,Z_k\big).\end{aligned}$$

Interchanging X and Y above and taking the difference of the two expressions gives

$$[L_X,L_Y]\big(\alpha(Z_1,\ldots,Z_k)\big) = \big\{[L_X,L_Y]\alpha\big\}(Z_1,\ldots,Z_k) + \sum_{i=1}^{i=k} \alpha(Z_1,\ldots,Z_{i-1},[L_X,L_Y]Z,Z_{i+1},\ldots,Z_k).$$

Combining this with the first expression above involving α gives

$$\big\{L_{[X,Y]}\alpha\big\}(Z_1,\ldots,Z_k) = \big\{[L_X,L_Y]\alpha\big\}(Z_1,\ldots,Z_k).$$

Since this holds for all smooth vector fields $Z_1,\ldots,Z_k$ on M, it follows that $L_{[X,Y]}\alpha = [L_X,L_Y]\alpha$.

Problem 4 (10pts)

Let α be a k-form on a smooth manifold M and $X_0, X_1, \ldots, X_k$ smooth vector fields on M. Show directly from the definitions that

$$\begin{aligned}\mathrm{d}\alpha(X_0, X_1, \ldots, X_k) = &\sum_{i=0}^{i=k}(-1)^i X_i\big(\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\big) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha\big([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\big).\end{aligned}$$

Since $\mathrm{d}\alpha$ is a $(k+1)$-form, the value of LHS of this identity at any $p \in M$ depends only on the values of $X_0, X_1, \ldots, X_k$ at p. We next show that RHS of this identity is also linear over $C^\infty(M)$ in each of the inputs. If $\mathrm{RHS}_\alpha^{(1)}$ and $\mathrm{RHS}_\alpha^{(2)}$ denote the two terms on RHS and $f \in C^\infty(M)$,

$$\begin{aligned}\mathrm{RHS}_\alpha^{(1)}(fX_0, X_1, \ldots, X_k) &= (-1)^0(fX_0)\alpha(X_1, \ldots, X_k) + \sum_{i=1}^{i=k}(-1)^i X_i\big(\alpha(fX_0, X_1, \ldots, \widehat{X_i}, \ldots, X_k)\big) \\ &= fX_0\big(\alpha(X_1, \ldots, X_k)\big) + \sum_{i=1}^{i=k}(-1)^i X_i\big(f\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\big) \\ &= \sum_{i=1}^{i=k}(-1)^i X_i(f)\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k) + f\sum_{i=0}^{i=k}(-1)^i X_i\big(\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\big); \\ \mathrm{RHS}_\alpha^{(2)}(fX_0, X_1, \ldots, X_k) &= \sum_{i=1}^{i=k}(-1)^i\alpha\big([fX_0, X_i], X_1, \ldots, \widehat{X_i}, \ldots, X_k\big) \\ &\qquad + \sum_{1 \le i<j}(-1)^{i+j}\alpha\big([X_i, X_j], fX_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\big) \\ &= \sum_{i=1}^{i=k}(-1)^i\alpha\big(f[X_0, X_i] - X_i(f)X_0, X_1, \ldots, \widehat{X_i}, \ldots, X_k\big) \\ &\qquad + f\sum_{1 \le i<j}(-1)^{i+j}\alpha\big([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\big) \\ &= -\sum_{i=1}^{i=k}(-1)^i X_i(f)\alpha\big(X_0, \ldots, \widehat{X_i}, \ldots, X_k\big) + f\sum_{i<j}(-1)^{i+j}\alpha\big([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\big).\end{aligned}$$

Thus, summing the two terms on RHS together, we obtain

$$\mathrm{RHS}_\alpha(fX_0, X_1, \ldots, X_k) = f\mathrm{RHS}_\alpha(X_0, X_1, \ldots, X_k).$$

Since RHS of the identity is alternating, it follows that

$$\mathrm{RHS}_\alpha(f_0X_0, \ldots, f_kX_k) = f_0 \ldots f_k\mathrm{RHS}_\alpha(X_0, \ldots, X_k)$$

for all $f_0, \ldots, f_k \in C^\infty(M)$. So, the value of RHS_α at a point $p \in M$ depends only on $X_0|_p, \ldots, X_k|_p$. Since both sides are alternating in the inputs, it is sufficient to check the identity for

$$\alpha = f\mathrm{d}x_I \equiv f\mathrm{d}x_{i_1} \wedge \ldots \wedge dx_{i_k}, \quad i_1 < i_2 < \ldots < i_k, \qquad X_l = \frac{\partial}{\partial x_{j_l}}, \; j_0 < j_1 < \ldots < j_k.$$

In this case,

$$[X_i, X_j] = 0, \qquad \mathrm{d}\alpha = \sum_{i=1}^{i=m} \frac{\partial f}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_I \,.$$

RHS reduces to

$$\begin{aligned}\sum_{l=0}^{l=k} (-1)^l X_l\big(\alpha(X_0, \ldots, \widehat{X_l}, \ldots, X_k)\big) &= \sum_{l=0}^{l=k} (-1)^l \frac{\partial}{\partial x_{j_l}} \bigg(f \mathrm{d}x_I \Big(\frac{\partial}{\partial x_{j_0}}, \ldots, \widehat{\frac{\partial}{\partial x_{j_l}}}, \ldots, \frac{\partial}{\partial x_{j_k}} \Big) \bigg) \\ &= \sum_{l=0}^{l=k} (-1)^l \bigg(\frac{\partial f}{\partial x_{j_l}} \bigg) \delta_{I,(j_0,\ldots,\widehat{j_l},\ldots,j_k)} \,.\end{aligned}$$

LHS becomes

$$\begin{aligned}\mathrm{d}\alpha\bigg(\frac{\partial}{\partial x_{j_0}}, \ldots, \frac{\partial}{\partial x_{j_k}} \bigg) &= \sum_{i=1}^{i=m} \sum_{l=0}^{l=k} (-1)^l \frac{\partial f}{\partial x_i} \mathrm{d}x_i \bigg(\frac{\partial}{\partial x_{j_l}} \bigg) \mathrm{d}x_I \bigg(\frac{\partial}{\partial x_{j_0}}, \ldots, \widehat{\frac{\partial}{\partial x_{j_l}}}, \ldots, \frac{\partial}{\partial x_{j_k}} \bigg) \\ &= \sum_{l=0}^{l=p} (-1)^l \bigg(\frac{\partial f}{\partial x_{j_l}} \bigg) \delta_{I,(j_0,\ldots,\widehat{j_l},\ldots,j_k)} \,.\end{aligned}$$

So the identity holds in this case.

Problem 5 (5pts)

Let $V \longrightarrow M$ be a smooth vector bundle of rank k and $W \subset V$ a smooth subbundle of V of rank k'. Show that

$$\mathrm{Ann}(W) \equiv \big\{ \alpha \in V_p^* \colon \alpha(w) = 0 \,\forall\, w \in W,\ p \in M \big\}$$

is a smooth subbundle of V^* of rank $k - k'$.

For each $p \in M$, $\mathrm{Ann}(W)_p \equiv \mathrm{Ann}(W_p)$ is a linear subspace of V_p^* of dimension $k - k'$; so we only need to show that $\mathrm{Ann}(W) \subset V^*$ is an embedded submanifold. Let $r : V^* \longrightarrow W^*$ be the bundle homomorphism induced by the restriction map on each fiber:

$$r(\alpha) = \alpha|_{W_p} \in W_p^* = \mathrm{Hom}(W_p, \mathbb{R}) \qquad \forall\ \alpha \in V_p^* \equiv \mathrm{Hom}(V_p, \mathbb{R}),\ p \in M.$$

The restriction of r to each fiber V_p^* is clearly linear. The map r is also smooth and its differential is surjective at every point (see below). Thus, by the Implicit Function Theorem,

$$\mathrm{Ann}(W) \equiv r^{-1}\big(s_0(M)\big) \subset V^* \,,$$

where $s_0(M) \subset W^*$ is the zero section, is a smooth embedded submanifold, as required (for this, it would be suffice that

$$T_{s_0(p)} W^* = \mathrm{Im}\, \mathrm{d}_\alpha r + T_{s_0(p)}\big(s_0(M)\big) \qquad \forall\, \alpha \in \mathrm{Ann}(W)_p \,,$$

and in turn this condition holds if $\mathrm{d}_\alpha r$ is onto for all $\alpha \in \mathrm{Ann}(W) \subset V^*$).

If $h_V = (\pi_V, h_1, \ldots, h_k) \colon V|_U \longrightarrow U \times \mathbb{R}^k$ is a trivialization of V such that

$$h_W \equiv \big(\pi_V, h_1, \ldots, h_{k'}\big) \colon W|_U \longrightarrow U \times \mathbb{R}^{k'} = U \times \mathbb{R}^{k'} \times 0 \subset U \times \mathbb{R}^k,$$

then

$$h_V^*: V^*|_U \longrightarrow U\times\mathbb{R}^k, \qquad \alpha \longrightarrow \big(p, \alpha(h^{-1}(p,e_1)),\ldots,\alpha(h^{-1}(p,e_k))\big) \quad \forall\ \alpha\in V_p^*,\ p\in U,$$
$$h_W^*: W^*|_U \longrightarrow U\times\mathbb{R}^{k'}, \qquad \alpha \longrightarrow \big(p, \alpha(h^{-1}(p,e_1)),\ldots,\alpha(h^{-1}(p,e_{k'}))\big) \quad \forall\ \alpha\in W_p^*,\ p\in U,$$

are trivializations for V^* and W^*, and

$$h_W^*\circ r\circ (h_V^*)^{-1}: U\times\mathbb{R}^k \longrightarrow U\times\mathbb{R}^{k'} = U\times\mathbb{R}^{k'}\times 0 \subset U\times\mathbb{R}^k$$

is the projection map. Thus, r is smooth and is a submersion. In fact,

$$h_{\mathrm{Ann}(W)}: \mathrm{Ann}(W)|_U \longrightarrow U\times\mathbb{R}^{k-k'} = U\times 0\times\mathbb{R}^{k'} \subset U\times\mathbb{R}^k,$$
$$\alpha \longrightarrow \big(p, \alpha(h^{-1}(p,e_{k'+1})),\ldots,\alpha(h^{-1}(p,e_k))\big) \quad \forall\ \alpha\in\mathrm{Ann}(W)_p,\ p\in U,$$

is a trivialization for the subbundle $\mathrm{Ann}(W)\subset V$. However, W^* is not a subbundle of V^* in a canonical way (it is the orthogonal complement of $\mathrm{Ann}(W)$, but this depends on the choice of the metric on the fibers).

Problem 6 (10pts)

Suppose M is a 3-manifold, α is a nowhere-zero one-form on M, and $p\in M$. Show that

(a) *if there exists an embedded 2-dimensional submanifold $P\subset M$ such that $p\in P$ and $\alpha|_{TP}=0$, then $(\alpha\wedge \mathrm{d}\alpha)|_p = 0$;*

(b) *if there exists a neighborhood U of p in M such that $(\alpha\wedge\mathrm{d}\alpha)|_U = 0$, then there exists an embedded 2-dimensional submanifold $P\subset M$ such that $p\in P$ and $\alpha|_{TP}=0$.*

Note: If the top form $\alpha\wedge\mathrm{d}\alpha$ on M is nowhere-zero, α is called a contact form*. In this case, it has no integrable submanifolds at all.*

(a) Suppose $P\subset M$ is an embedded two-dimensional submanifold such that $p\in P$ and

$$i^*\alpha = \alpha|_{TP} = 0,$$

where $i: P\longrightarrow M$ is the inclusion map. Then,

$$(\mathrm{d}\alpha)_p|_{T_pP} = (i^*\mathrm{d}\alpha)_p = (\mathrm{d}i^*\alpha)_p = \mathrm{d}0 = 0.$$

Since α_p and $\mathrm{d}\alpha|_p$ vanish on the codimension-one subspace T_pP of T_pM, it follows that their wedge product vanishes on T_pM, i.e. $(\alpha\wedge\mathrm{d}\alpha)_p = 0$.

(b) We first note if V is any vector space of dimension n, $\alpha\in V$, $\alpha\neq 0$, $\gamma\in\Lambda^{n-1}V$, and $\alpha\wedge\gamma = 0$, then $\gamma = \alpha\wedge\beta$ for some $\beta\in\Lambda^{n-2}V$. This can be seen by an argument similar to the solution of Problem 3.[1] In turn, this statement implies that if M is a smooth manifold, $\alpha\in E^1(M)$, $\alpha\neq 0$, $\gamma\in E^{n-1}(M)$, and $\alpha\wedge\gamma = 0$, then $\gamma = \alpha\wedge\beta$ for some $\beta\in E^{n-2}(M)$ (one needs to make sure that β can be chosen to be smooth).

[1]The statement is actually true for any form $\gamma\in\Lambda^k V$; see Chapter 2, #15, p80.

Since $\alpha_q \neq 0$ for all $q \in M$,

$$\mathbb{R}\alpha \equiv \big\{c\alpha_q \in T_q^*M : c \in \mathbb{R},\ q \in M\big\}$$

is a subbundle of T^*M of rank 1. Any section $\tilde{\alpha}$ of this subbundle is of the form $\tilde{\alpha} = f\alpha$ for some $f \in C^\infty(M)$; for such $\tilde{\alpha}$,

$$\mathrm{d}\tilde{\alpha} = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha.$$

If U is a neighborhood of p in M such that $(\alpha \wedge \mathrm{d}\alpha)|_U = 0$, $\mathrm{d}\alpha|_U = \alpha|_U \wedge \beta$ for some $\beta \in E^1(U)$ by the previous paragraph and thus

$$\mathrm{d}\tilde{\alpha} \in \Gamma\big(U; \mathbb{R}\alpha \wedge T^*U\big) \subset \Gamma\big(U; \Lambda^2 T^*U\big) = E^2(U) \qquad \forall \tilde{\alpha} \in \Gamma(U; \mathbb{R}\alpha) \subset E_1(U).$$

So, by the differential-form version of Frobenius Theorem (Warner's 2.32, stated in terms of vector bundles in class), for every $p \in U$ there exists a 2-dimensional embedded submanifold $P \subset U \subset M$ such that $p \in P$ and $\alpha|_{TP} = 0$.

Problem 7 (10pts)

A two-form ω on a smooth manifold M is called symplectic *if ω is closed (i.e. $\mathrm{d}\omega = 0$) and everywhere nondegenerate[2]. Suppose ω is a symplectic form on M.*
(a) *Show that the dimension of M is even and the map*

$$TM \longrightarrow T^*M, \qquad X \longrightarrow i_X\omega,$$

is a vector-bundle isomorphism ($i_X\omega$ is the contraction w.r.t. X, i.e. the dual of $X\wedge$).
(b) *If $H : M \longrightarrow \mathbb{R}$ is a smooth map, let $X_H \in \Gamma(M; TM)$ be the preimage of $\mathrm{d}H$ under this isomorphism. Assume that X_H is a complete vector field, so that the flow*

$$\varphi\colon \mathbb{R} \times M \longrightarrow M, \qquad (t, p) \longrightarrow \varphi_t(p),$$

is globally defined. Show that for every $t \in \mathbb{R}$, the time-t flow $\varphi_t : M \longrightarrow M$ is a symplectomorphism, i.e. $\varphi_t^*\omega = \omega$.
Note: In such a situation, H is called a Hamiltonian *and φ_t a* Hamiltonian symplectomorphism*.*

(a) If $p \in M$, ω_p is a nondegenerate bilinear anti-symmetric form on T_pM. Thus, it is a standard fact in linear algebra that the dimension of T_pM is even. In fact, one can choose a basis $\{v_1, \ldots, v_n\}$ for T_pM so that the matrix for ω_p with respect to this basis is

$$\begin{pmatrix} J & 0 & \ldots & 0 \\ 0 & J & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & J \end{pmatrix} \qquad \text{where} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since ω is smooth, the map

$$TM \longrightarrow T^*M, \qquad X \longrightarrow i_X\omega, \tag{1}$$

[2]This means that $\omega_p \in \Lambda^2 T_p^*M$ is nondegenerate for every $p \in M$, i.e. for every $v \in T_pM$ such that $v \neq 0$ there exists $w \in T_pM$ such that $\omega_p(v, w) \neq 0$.

is smooth. If $X \in T_pM$, then $i_X\omega_p \in T_p^*M$, i.e. eq1 is a bundle map (commutes with the projections to the base). If $X_1, X_2, Y \in T_pM$ and $a, b \in \mathbb{R}$, then

$$\begin{aligned}\big\{i_{aX_1+bX_2}\omega\big\}(Y) &\equiv \omega\big(aX_1+bX_2, Y\big) = a\,\omega(X_1,Y) + b\,\omega(X_2,Y) = a\{i_{X_1}\omega\}(Y) + b\{i_{X_2}\omega\}(Y) \\ \Longrightarrow \qquad & i_{aX_1+bX_2}\omega = a\{i_{X_1}\omega\} + b\{i_{X_2}\omega\} \in T_p^*M \quad \forall\, X_1, X_2 \in T_pM,\ a, b \in \mathbb{R}.\end{aligned}$$

Thus, eq1 is a bundle homomorphism (i.e. linear on every fiber). Finally, since ω_p is nondegenerate, if $X \in T_pM - \{0\}$, then there exists $Y \in T_pM$ such that

$$\{i_X\omega\}(Y) = \omega(X,Y) \neq 0 \qquad \Longrightarrow \qquad i_X\omega \neq 0 \in T_p^*M.$$

Thus, the bundle homomorphism eq1 is injective and therefore a bundle isomorphism (since the two bundles have the same rank).

(b) We need to show that $\varphi_t^*\omega = \omega$ for all t, i.e. for all $t \in \mathbb{R}$ and $p \in M$

$$\lim_{s\longrightarrow 0} \frac{\{\varphi_{t+s}^*\omega\}_p - \{\varphi_t^*\omega\}_p}{s} = \frac{\mathrm{d}}{\mathrm{d}s}\big(\{\varphi_{t+s}^*\omega\}_p\big)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\big(\{\varphi_s^*\omega\}_p\big)\Big|_{s=t} = 0.$$

Since $\varphi_{t+s} = \varphi_t \circ \varphi_s$ by (h) of Theorem 1.48,

$$\begin{aligned}\lim_{s\longrightarrow 0} \frac{\{\varphi_{t+s}^*\omega\}_p - \{\varphi_t^*\omega\}_p}{s} &= \lim_{s\longrightarrow 0} \frac{\big\{\varphi_s^*\{\varphi_t^*\omega\}\big\}_p - \{\varphi_t^*\omega\}_p}{s} \\ &= \lim_{s\longrightarrow 0} \frac{\varphi_s^*\{\varphi_t^*\omega\}_{\varphi_s(p)} - \{\varphi_t^*\omega\}_p}{s} = \big(L_{X_H}(\varphi_t^*\omega)\big)_p.\end{aligned}$$

Since $\mathrm{d}\omega = 0$, by (d) of Proposition 2.25 and (b) of Proposition 2.23

$$L_{X_H}(\varphi_t^*\omega) = \big\{i_{X_H}\circ \mathrm{d} + \mathrm{d}\circ i_{X_H}\big\}(\varphi_t^*\omega) = i_{X_H}\varphi_t^*\mathrm{d}\omega + \mathrm{d}\circ i_{X_H}\varphi_t^*\omega = 0 + \mathrm{d}\big(\varphi_t^*\big\{i_{\mathrm{d}\varphi_t X_H}\omega\big\}\big).$$

Since φ_t is the flow for the vector field X_H,

$$\begin{aligned}\mathrm{d}_p\varphi_t X_H = \mathrm{d}_p\varphi_t\bigg(\frac{\mathrm{d}}{\mathrm{d}s}\varphi_s(p)\Big|_{s=0}\bigg) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big(\varphi_t\circ\varphi_s(p)\Big)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\varphi_{t+s}(p)\Big|_{s=0} = X_H\big(\varphi_t(p)\big) \\ \Longrightarrow \qquad & i_{\mathrm{d}\varphi_t X_H}\omega = i_{X_H}\omega = \mathrm{d}H,\end{aligned}$$

by definition of X_H. We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\{\varphi_s^*\omega\}_p\big)\Big|_{s=t} = \mathrm{d}\big(\varphi_t^*\mathrm{d}H\big) = \varphi_t^*\mathrm{d}^2H = 0,$$

i.e. $\varphi_t^*\omega = \omega$.