MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 2

Problem 1: Chapter 1, #10 (5pts)

Suppose M is a compact nonempty manifold of dimension n and $f: M \longrightarrow \mathbb{R}^n$ is a smooth map. Show that f is not an immersion (i.e. $df|_m$ is not injective for some $m \in M$.

Solution 1 (direct): We first notice that if $h: M \longrightarrow \mathbb{R}$ is a smooth map and reaches its maximum at some $m \in M$ (which need not exist in general), then $dh|_m = 0$. If $(\mathcal{U}, \varphi)$ is a coordinate chart near m, then

$$h \circ \varphi^{-1}: \varphi(\mathcal{U}) \longrightarrow \mathbb{R}$$

is a smooth map that reaches its maximum at $\varphi(m)$. Thus,

$$\frac{\partial(h \circ \varphi^{-1})}{\partial x_i} = 0 \quad \forall\, i = 1, \ldots, n \qquad \Longrightarrow \qquad dh|_m = \sum_{i=1}^{n} \left(\frac{\partial(h \circ \varphi^{-1})}{\partial x_i} \right) dx_i = 0.$$

Suppose next that $f: M^n \longrightarrow \mathbb{R}^n$ is smooth and $r_1: \mathbb{R}^n \longrightarrow \mathbb{R}$ is the projection onto the first component. Since r_1 is a smooth map, so is

$$r_1 \circ f: M \longrightarrow \mathbb{R}.$$

Since M is compact, $r_1(f(M))$ is a closed bounded subset of $\mathbb{R}$ and thus $r_1 \circ f$ reaches its maximum at some point $m \in M$. By the above,

$$dr_1|_{f(m)} \circ df|_m = d(r_1 \circ f)|_m = 0.$$

Since the linear map

$$dr_1|_{f(m)}: T_{f(m)}\mathbb{R}^n = \mathbb{R}^n \longrightarrow T_{r_1(f(m))}\mathbb{R} = \mathbb{R}$$

is surjective (being projection onto the first component), the linear map

$$df|_m: T_m M \longrightarrow T_{f(m)}\mathbb{R}^n$$

is not surjective. Since the dimension of $T_m M$ is n, it follows that $df|_m$ is not injective either.

Solution 2 (via Inverse FT): Suppose $f: M^n \longrightarrow \mathbb{R}^n$ is an immersion. Since for every $m \in M$, the linear map

$$df|_m: T_m M \longrightarrow T_{f(m)}\mathbb{R}^n$$

is injective, $df|_m$ is an isomorphism. Thus, by the Inverse Function Theorem, for every $m \in M$ there exist open neighborhoods $\mathcal{U}_m$ of m in M and V_m of $f(m)$ in $\mathbb{R}^n$ such that

$$f|_{\mathcal{U}_m}: U_m \longrightarrow V_m$$

is a diffeomorphism. In particular,

$$f(M) = \bigcup_{m \in M} V_m \subset \mathbb{R}^n$$

is an open subset of $\mathbb{R}^n$. On the other hand, if M is compact, then so is $f(M)$. Since $\mathbb{R}^n$ is Hausdorff, $f(M)$ is then a closed subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is connected and $f(M)$ is open and closed, $f(M)$ is either empty or the entire space $\mathbb{R}^n$. The former is impossible if M is not empty; the latter is impossible because $f(M)$ is compact, while $\mathbb{R}^n$ is not.

Problem 2: Chapter 1, #7 (10pts)

Suppose N is a smooth manifold, A is a subset of N, and $\iota\colon A \longrightarrow N$ is the inclusion map.

(a) *Let $\mathcal{T}$ be a topology on A. Show that there exists at most one differentiable structure $\mathcal{F}$ on $(A,\mathcal{T})$ such that $\iota\colon (A,\mathcal{F}) \longrightarrow N$ is a submanifold of N (i.e. ι is smooth and $d\iota|_a$ is injective for all $a \in A$).*

(b) *Let $\mathcal{T}$ be the subspace topology on A (induced from the topology of N). Suppose $(A,\mathcal{T})$ admits a smooth structure $\mathcal{F}$ such that $\iota\colon (A,\mathcal{F}) \longrightarrow N$ is a submanifold of N. Show that there exists no other manifold structure $(\mathcal{T}',\mathcal{F}')$ such that $\iota\colon (A,\mathcal{F}') \longrightarrow N$ is a submanifold of N.*

(a) Suppose $\mathcal{F}$ and $\mathcal{F}'$ are smooth structures on $(A,\mathcal{T})$ such that the maps

$$\iota\colon (A,\mathcal{F}) \longrightarrow N \qquad \text{and} \qquad \iota\colon (A,\mathcal{F}') \longrightarrow N$$

are immersions. The map $\mathrm{id}\colon (A,\mathcal{F}') \longrightarrow (A,\mathcal{F})$ is a homeomorphism (and thus continuous) and $\iota = \iota \circ \mathrm{id}$:

$$\begin{tikzcd}
(A,\mathcal{F}) \arrow[dr, "\iota"] & \\
& N \\
(A,\mathcal{F}') \arrow[uu, "\mathrm{id}"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Since $\iota\colon (A,\mathcal{F}) \longrightarrow N$ is a submanifold and $\iota\colon (A,\mathcal{F}') \longrightarrow N$ is smooth, by Theorem 1.32 the map

$$\mathrm{id}\colon (A,\mathcal{F}') \longrightarrow (A,\mathcal{F})$$

is smooth. Similarly, the map $\mathrm{id}\colon (A,\mathcal{F}) \longrightarrow (A,\mathcal{F}')$ is smooth. Thus, the map

$$\mathrm{id}\colon (A,\mathcal{F}') \longrightarrow (A,\mathcal{F})$$

is a diffeomorphism. Since $\mathcal{F}$ and $\mathcal{F}'$ are maximal with respect to the smooth-overlap condition, it follows that $\mathcal{F} = \mathcal{F}'$.

(b) Suppose $(\mathcal{T}',\mathcal{F}')$ is a manifold structure on A such that the map

$$\iota\colon (A,\mathcal{F}') \longrightarrow N$$

is a submanifold of N. The map $\iota\colon (A,\mathcal{T}) \longrightarrow N$ is a topological embedding and $\iota\colon (A,\mathcal{T}') \longrightarrow N$ is continuous:

$$\begin{tikzcd}
(A,\mathcal{T},\mathcal{F}) \arrow[dr, "\iota"] & \\
& N \\
(A,\mathcal{T}',\mathcal{F}') \arrow[uu, "\mathrm{id}"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Thus, the map $\mathrm{id}: (A,\mathcal{T}') \longrightarrow (A,\mathcal{T})$ is continuous. Since $\iota: (A,\mathcal{T}) \longrightarrow N$ is a submanifold, by Theorem 1.32 the map

$$\mathrm{id}\colon (A,\mathcal{T}',\mathcal{F}') \longrightarrow (A,\mathcal{T},\mathcal{F})$$

is then smooth. Since the map

$$\iota = \iota \circ \mathrm{id}\colon (A,\mathcal{T}',\mathcal{F}') \longrightarrow (A,\mathcal{T},\mathcal{F}) \longrightarrow N$$

is an immersion, so is the map

$$\mathrm{id}\colon (A,\mathcal{T}',\mathcal{F}') \longrightarrow (A,\mathcal{T},\mathcal{F}).$$

Since it is bijective, by Problem 4 on PS1 id is a diffeomorphism. We conclude that $\mathcal{T}' = \mathcal{T}$ and $\mathcal{F}' = \mathcal{F}$.

Problem 3 (15pts)

(a) *For what values of $t \in \mathbb{R}$, is the subspace*

$$\big\{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = t\big\}$$

a smooth embedded submanifold of $\mathbb{R}^{n+1}$?
(b) *For such values of t, determine the diffeomorphism type of this submanifold (i.e. show that it is diffeomorphic to something rather standard).*

(a) Let $f\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be the smooth map given by

$$f(\mathbf{x}) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2 \qquad \text{if} \quad \mathbf{x} = (x_1,\ldots,x_n).$$

Then, $S_t = f^{-1}(t)$. The differential of f,

$$\mathrm{d}_{\mathbf{x}} f\colon T_{\mathbf{x}}\mathbb{R}^{n+1} \longrightarrow T_{f(\mathbf{x})}\mathbb{R} = \mathbb{R},$$

is given by

$$\mathrm{d}_{\mathbf{x}} f = \left(\frac{\partial f}{\partial x_1}\right)\mathrm{d}_{\mathbf{x}} x_1 + \ldots + \left(\frac{\partial f}{\partial x_{n+1}}\right)\mathrm{d}_{\mathbf{x}} x_{n+1} = 2x_1 \mathrm{d}_{\mathbf{x}} x_1 + \ldots + 2x_n \mathrm{d}_{\mathbf{x}} x_n - 2x_{n+1}\mathrm{d}_{\mathbf{x}} x_{n+1}.$$

Since the target space of $\mathrm{d}_{\mathbf{x}} f$ is a one-dimensional vector space, $\mathrm{d}_{\mathbf{x}} f$ is surjective if and only if $\mathrm{d}_{\mathbf{x}} f$ is nonzero. Since $\{\mathrm{d}_{\mathbf{x}} x_i\}$ is a basis for $T^*_{\mathbf{x}}\mathbb{R}^{n+1}$, it follows that $\mathrm{d}_{\mathbf{x}} f$ is surjective if and only $\mathbf{x} \neq \mathbf{0}$. If $t \neq 0$, then $\mathbf{0} \notin S_t$. Thus, $\mathrm{d}_{\mathbf{x}} f$ is surjective for all $\mathbf{x} \in S_t$ and S_t is an embedded submanifold of $\mathbb{R}^{n+1}$ of dimension

$$\dim S_t = \dim \mathbb{R}^{n+1} - \dim \mathbb{R} = n$$

by the Implicit Function Theorem if $t \neq 0$.

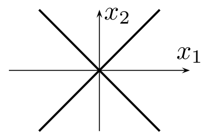
The differential of f vanishes at $\mathbf{0}\in S_0$ and the Implicit FT does not determine whether

$$S_0 = \big\{(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}\colon x_1^2+\ldots+x_n^2=x_{n+1}^2\big\}$$

is an embedded submanifold or not. The only potentially singular (non-smooth) point of S_0 is $\mathbf{0}$. To see what S_0 looks like, consider the case $n=1$:

$$S_0 = \big\{(x_1,x_2)\in\mathbb{R}^2\colon x_1^2=x_2^2\big\} = \big\{(x_1,x_2)\in\mathbb{R}^2\colon |x_1|=|x_2|\big\}.$$

Thus, if $n=1$, S_0 is the union of the lines $x_1=\pm x_2$ through the origin:



In general, the cross-section of S_0 by the hyperplane $x_{n+1}=s$ is an $(n-1)$-sphere of radius $|s|$ or a single point if $s=0$. Thus, S_0 is a union of 2 cones with the vertex at the the origin. This implies that $\mathbf{0}$ is not a smooth point of S_0. In fact, it is not even a manifold point in the topological sense, i.e. there exists no open neighborhood U of $\mathbf{0}$ in $\mathbb{R}^{n+1}$ such that $S_0\cap U$ is homeomorphic to an open subset of $\mathbb{R}^k$ for some k. In summary, S_t is a smooth embedded submanifold of $\mathbb{R}^{n+1}$ if and only if $t\neq 0$.

Remark: Here is how to see formally that if U is a neighborhood of $\mathbf{0}$ in $\mathbb{R}^{n+1}$ and V is an open subset of $\mathbb{R}^k$, then $S_0\cap U$ and V are not homeomorphic. It is enough to assume that U and V are both connected. By the Implicit Function Theorem, $S_0-\mathbf{0}$ is a smooth embedded submanifold of $\mathbb{R}^{n+1}$ of dimension n. Thus, we can also assume that $k=n$. If $n>1$, then the complement of any point in V is connected. However, $(S_0-\mathbf{0})\cap U$ is not connected:

$$\begin{gathered} S_0-\mathbf{0} = (S_0-\mathbf{0})\cap U_+ \cup (S_0-\mathbf{0})\cap U_- \qquad \text{where} \\ U_+ = \big\{(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}\colon x_{n+1}>0\big\} \qquad \text{and} \qquad U_- = \big\{(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}\colon x_{n+1}<0\big\}. \end{gathered}$$

Thus, $S_0\cap U$ and V are not homeomorphic. If $n=1$, V must be an open interval and the complement of a point in V has exactly two components. On the other hand, $(S_0-\mathbf{0})\cap U$ has (at least) four components:

$$\begin{gathered} S_0-\mathbf{0} = (S_0-\mathbf{0})\cap U_{++} \cup (S_0-\mathbf{0})\cap U_{+-} \cup (S_0-\mathbf{0})\cap U_{-+} \cup (S_0-\mathbf{0})\cap U_{--}, \\ \text{where} \qquad U_{\pm\pm} = \big\{(x_1,x_2)\in\mathbb{R}^2\colon \pm x_1>0, \pm x_2>0\big\}. \end{gathered}$$

Thus, $S_0\cap U$ and V are again not homeomorphic.

(b) Suppose $t>0$. Then, the set of solutions of the equation

$$x_1^2+\ldots+x_n^2 = t+x_{n+1}^2, \qquad (x_1,\ldots,x_n)\in\mathbb{R}^n,$$

with x_{n+1} fixed is an $(n-1)$-sphere (this is not the case for every x_{n+1} if $t\le 0$). Thus, we expect that S_t is diffeomorphic to $S^{n-1}\times\mathbb{R}$, with the second component given by x_{n+1}. Define

$$\psi\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n\times\mathbb{R} \qquad \text{by} \qquad \psi(x_1,\ldots,x_{n+1}) = \bigg(\frac{(x_1,\ldots,x_n)}{\sqrt{t+x_{n+1}^2}}, x_{n+1}\bigg).$$

Since $t>0$, this map is smooth. In fact, it is a diffeomorphism:

$$\psi^{-1}(y_1,\ldots,y_{n+1}) = \big(\sqrt{t+y_{n+1}^2}\,(y_1,\ldots,y_n), y_{n+1}\big).$$

Since S_t is a submanifold of $\mathbb{R}^{n+1}$, $\psi|_{S_t}$ is also smooth. Furthermore,

$$\psi(S_t) \subset S^{n-1}\times\mathbb{R}.$$

Since $S^{n-1}\times\mathbb{R}$ is an *embedded* submanifold of $\mathbb{R}^n\times\mathbb{R}$,

$$\psi|_{S_t}\colon S_t \longrightarrow S^{n-1}\times\mathbb{R}$$

is smooth by Theorem 1.32. Since ψ is a diffeomorphism,

$$\psi|_{S_t}\colon S_t \longrightarrow S^{n-1}\times\mathbb{R}$$

is an injective immersion. Since $\psi^{-1}(S^{n-1}\times\mathbb{R})\subset S_t$ (i.e. $f(\psi^{-1}(\mathbf{y}))=t$ for all $\mathbf{y}\in S^{n-1}\times\mathbb{R}$), this map is surjective as well. Thus, by Exercise 6 on p51 (from PS1),

$$\psi|_{S_t}\colon S_t \longrightarrow S^{n-1}\times\mathbb{R}$$

is a diffeomorphism.

Suppose $t<0$. Then, the set of solutions of the equation

$$x_{n+1}^2 = -t + x_1^2+\ldots+x_n^2, \qquad x_{n+1}\in\mathbb{R},$$

with $x_1,\ldots,x_n$ fixed is two distinct points, i.e. S^0 (this is not the case for every $(x_1,\ldots,x_n)$ if $t\geq 0$). Thus, we expect that S_t is diffeomorphic to $\mathbb{R}^n\times S^0$ ($\mathbb{R}^n\sqcup\mathbb{R}^n$), with the first component given by $(x_1,\ldots,x_n)$. Define

$$\psi\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n\times\mathbb{R} \qquad \text{by} \qquad \psi(x_1,\ldots,x_{n+1}) = \left((x_1,\ldots,x_n), \frac{x_{n+1}}{\sqrt{-t+x_1^2+\ldots+x_n^2}}\right).$$

Since $t<0$, this map is smooth. In fact, it is a diffeomorphism:

$$\psi^{-1}(y_1,\ldots,y_{n+1}) = \big((y_1,\ldots,y_n), \sqrt{-t+x_1^2+\ldots+x_n^2}\,y_{n+1}\big).$$

Since S_t is a submanifold of $\mathbb{R}^{n+1}$, $\psi|_{S_t}$ is also smooth. Furthermore,

$$\psi(S_t) \subset \mathbb{R}^n\times S^0.$$

Since $\mathbb{R}^n\times S^0$ is an *embedded* submanifold of $\mathbb{R}^n\times\mathbb{R}$,

$$\psi|_{S_t}\colon S_t \longrightarrow \mathbb{R}^n\times S^0$$

is smooth by Theorem 1.32. Since ψ is a diffeomorphism,

$$\psi|_{S_t}\colon S_t \longrightarrow \mathbb{R}^n\times S^0$$

is an injective immersion. Since $\psi^{-1}(\mathbb{R}^n \times S^0) \subset S_t$ (i.e. $f(\psi^{-1}(\mathbf{y}))=t$ for all $\mathbf{y} \in \mathbb{R}^n \times S^0$), this map is surjective as well. Thus, by Exercise 6 on p51 (from PS1),

$$\psi|_{S_t}: S_t \longrightarrow \mathbb{R}^n \times S^0$$

is a diffeomorphism.

In summary, S_t is diffeomorphic to $S^{n-1} \times \mathbb{R}$ if $t>0$ and to $\mathbb{R}^n \sqcup \mathbb{R}^n$ if $t<0$.

Remark: The above argument assumed that $n \geq 1$. If $n=0$, S_t is the empty set if $t>0$, consists of one point if $t=0$, and consists of two points if $t<0$. All are zero-dimensional manifolds.

Problem 4 (10pts)

Show that the special unitary group

$$SU_n = \big\{A \in \mathrm{Mat}_n\mathbb{C} \colon \bar{A}^t A = \mathbb{I}_n, \;\; \det A = 1\big\}$$

is a smooth compact manifold. What is its dimension?

We will use the Implicit Function Theorem to show that the unitary group

$$U_n = \big\{A \in \mathrm{Mat}_n\mathbb{C} \colon \bar{A}^t A = \mathbb{I}_n\big\}$$

is a compact embedded submanifold of $\mathrm{Mat}_n\mathbb{C}$ (which is diffeomorphic to $\mathbb{R}^{2n^2}$) and SU_n is a closed embedded submanifold of U_n.

First, for each $B \in \mathrm{Mat}_n\mathbb{C}$, let

$$L_B \colon \mathrm{Mat}_n\mathbb{C} \longrightarrow \mathrm{Mat}_n\mathbb{C}, \qquad L_B(A) = BA,$$

be the left-multiplication map. It is smooth (being a linear transformation) on $\mathbb{C}^{n^2}$.

Let Her_n denote the space of Hermitian $n \times n$ matrices:

$$\mathrm{Her}_n = \big\{A \in \mathrm{Mat}_n\mathbb{C} \colon \bar{A}^t = A\big\}.$$

Since Her_n is a linear subspace of $\mathbb{R}^{2n^2}$ (it is defined by a linear equation on the coefficients), Her_n is an embedded submanifold of $\mathrm{Mat}_n\mathbb{C}$. Define

$$f \colon \mathrm{Mat}_n\mathbb{C} \longrightarrow \mathrm{Mat}_n\mathbb{C} \qquad \text{by} \qquad f(A) = \bar{A}^t A.$$

Since f is a polynomial map in the coefficients of A, f is smooth. Furthermore,

$$f\big(\mathrm{Mat}_n\mathbb{C}\big) \subset \mathrm{Her}_n.$$

Since Her_n is an embedded submanifold of $\mathrm{Mat}_n\mathbb{C}$, the map

$$g \colon \mathrm{Mat}_n\mathbb{C} \longrightarrow \mathrm{Her}_n\mathbb{C}, \qquad g(A) = f(A),$$

is smooth. We will show that $\mathbb{I}_n$ is a regular value for g (it is *not* for f), i.e. $\mathrm{d}_A g$ is surjective for all

$$A \in g^{-1}(\mathbb{I}_n) = U_n.$$

First, we show that

$$\mathrm{d}_{\mathbb{I}_n} g \colon T_{\mathbb{I}_n}\mathrm{Mat}_n\mathbb{C} \longrightarrow T_{\mathbb{I}_n}\mathrm{Her}_n$$

is surjective. For each $B \in \mathrm{Mat}_n\mathbb{C}$, define

$$\alpha_B \colon \mathbb{R} \longrightarrow \mathrm{Mat}_n\mathbb{C} \qquad \text{by} \qquad \alpha_B(s) = \mathbb{I}_n + sB.$$

Then, α_B is a smooth curve in $\mathrm{Mat}_n\mathbb{C}$ so that $\alpha_B(0) = \mathbb{I}_n$. In particular,

$$\alpha'_B(0) = \mathrm{d}_B\alpha\big|_0 \frac{\mathrm{d}}{\mathrm{d}s} \in T_{\mathbb{I}_n}\mathrm{Mat}_n\mathbb{C}.$$

Furthermore,

$$\begin{aligned}
\mathrm{d}_{\mathbb{I}_n} g\big(\alpha'_B(0)\big) &= \mathrm{d}_{\mathbb{I}_n} g\left(\mathrm{d}_0\alpha_B\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)\right) = \mathrm{d}_0(g \circ \alpha_B)\left(\frac{\mathrm{d}}{\mathrm{d}s}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s} g\big(\alpha_B(s)\big)\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{I}_n + s\bar{B}^t)(\mathbb{I}_n + sB)\big|_{s=0} = \bar{B}^t + B \in \mathrm{Her}_n = T_{\mathbb{I}_n}\mathrm{Her}_n.
\end{aligned}$$

In particular, the map

$$\mathrm{d}_{\mathbb{I}_n} g \colon \big\{\alpha'_B(0) \colon B \in \mathrm{Her}_n\big\} \longrightarrow T_{\mathbb{I}_n}\mathrm{Her}_n, \qquad d_{\mathbb{I}_n} g\big(\alpha'_B(0)\big) = 2B,$$

is surjective, and thus so is $d_{\mathbb{I}_n} g$. On the other hand, if $B \in g^{-1}(\mathbb{I}_n)$, then

$$\begin{aligned}
g\big(L_B(A)\big) = g(BA) = \overline{BA}^t(BA) = \bar{A}^t\bar{B}^t BA = \bar{A}^t A = g(A) \quad \forall\, A \in \mathrm{Mat}_n\mathbb{C} \quad &\Longrightarrow \quad g = g \circ L_B \\
\Longrightarrow \qquad \mathrm{d}_{\mathbb{I}_n} g = \mathrm{d}_{L_B(\mathbb{I}_n)} g \circ \mathrm{d}_{\mathbb{I}_n} L_B \colon T_{\mathbb{I}_n}\mathrm{Mat}_n\mathbb{C} \longrightarrow T_B\mathrm{Mat}_n\mathbb{C} \longrightarrow T_{\mathbb{I}_n}\mathrm{Her}_n.&
\end{aligned}$$

Since $\mathrm{d}_{\mathbb{I}_n} g$ is surjective, it follows that so is $\mathrm{d}_B g$, for all $B \in U_n$, i.e. $\mathbb{I}_n$ is a regular value for g. Thus, by the Implicit FT, $U_n = g^{-1}(\mathbb{I}_n)$ is an embedded submanifold of $\mathrm{Mat}_n\mathbb{C}$ of dimension

$$\dim U_n = \dim \mathrm{Mat}_n\mathbb{C} - \dim \mathrm{Her}_n = 2n^2 - \big(2n(n-1)/2 + n\big) = n^2$$

(the condition $\bar{A}^t = A$ defining Her_n means that the $n(n-1)/2$ above-diagonal complex entries can be chosen freely and determine the below-diagram entries, and the diagonal entries must be real). The subspace U_n of $\mathrm{Mat}_n\mathbb{C}$ is compact because it is closed (preimage of a point under a continuous map into a T1-space) and bounded in the *standard* metric on $\mathbb{R}^{2n^2}$ (the condition $\bar{A}^t A = \mathbb{I}_n$ implies that the length of each row and column of A is 1).

We now show that SU_n is an embedded submanifold of U_n. Define

$$\psi \colon \mathrm{Mat}_n\mathbb{C} \longrightarrow \mathbb{C} \qquad \text{by} \qquad \psi(A) = \det\, A.$$

Since ψ is a polynomial in the entries, it is a smooth function. Since U_n is a submanifold of $\mathrm{Mat}_n\mathbb{C}$, $\psi|_{U_n}$ is also smooth. Furthermore,

$$\begin{aligned}
A \in U_n \quad &\Longrightarrow \quad 1 = \det\, \mathbb{I}_n = \det\,(\bar{A}^t A) = \big(\det\, \bar{A}^t\big)\big(\det\, A\big) = (\overline{\det\, A}) \cdot (\det\, A) \\
&\Longrightarrow \quad \det\, A \in S^1 \quad \Longrightarrow \quad \psi(U_n) \subset S^1.
\end{aligned}$$

Since S^1 is an *embedded* submanifold of $\mathbb{C}$, the map

$$\varphi\colon U_n \longrightarrow S^1, \qquad \varphi(A) = \psi(A),$$

is smooth by Theorem 1.32. By definition, $SU_n = \varphi^{-1}(1)$. We will show that 1 is a regular value for φ (but *not* for $\psi|_{U_n}$), i.e. $\mathrm{d}_A\varphi$ is surjective for all $A \in \varphi^{-1}(1)$. First, we show that

$$\mathrm{d}_{\mathbb{I}_n}\varphi\colon T_{\mathbb{I}_n}U_n \longrightarrow T_1 S^1$$

is surjective. Define

$$\alpha\colon \mathbb{R} \longrightarrow \mathrm{Mat}_n\mathbb{C} \qquad \text{by} \qquad \alpha(s) = e^{\mathrm{i}s}\mathbb{I}_n.$$

This map is smooth and $\alpha(\mathbb{R}) \subset U_n$. Since U_n is an *embedded* submanifold of $\mathrm{Mat}_n\mathbb{C}$, the map

$$\beta\colon \mathbb{R} \longrightarrow U_n, \qquad \beta(s) = \alpha(s),$$

is then smooth by Theorem 1.32. Furthermore, $\beta(0) = \mathbb{I}_n$. In particular,

$$\beta'(0) = \mathrm{d}_0\beta\left(\frac{\mathrm{d}}{\mathrm{d}s}\right) \in T_{\mathbb{I}_n}U_n.$$

We have

$$\begin{aligned}\mathrm{d}_{\mathbb{I}_n}\varphi\big(\beta'(0)\big) &= \mathrm{d}_{\mathbb{I}_n}\varphi\left(\mathrm{d}_0\beta\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)\right) = \mathrm{d}_0(\varphi\circ\beta)\left(\frac{\mathrm{d}}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\varphi\big(\beta(s)\big)\big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\det\left(e^{\mathrm{i}s}\mathbb{I}_n\right)\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}e^{\mathrm{i}ns}\big|_{s=0} = \mathfrak{i}\, n \in T_1S^1 \subset T_1\mathbb{C} = \mathbb{C}.\end{aligned}$$

Thus, $\mathrm{d}_{\mathbb{I}_n}\varphi$ is nonzero and must then be surjective (since its target space is one-dimensional). On the other hand, if $B \in \varphi^{-1}(1)$, then $L_B(U_n) \subset U_n$ (i.e. U_n is a subgroup of $\mathrm{GL}_n\mathbb{C}$). Since U_n is an *embedded* submanifold of $\mathrm{Mat}_n\mathbb{C}$, then the map

$$L'_B\colon U_n \longrightarrow U_n, \qquad L'_B(A) = L_B(A),$$

is smooth by Theorem 1.32. Furthermore,

$$\begin{aligned}&\varphi\big(L'_B(A)\big) = \varphi(BA) = \det(BA) = (\det B)(\det A) = \det A = \varphi(A) \quad \forall\, A \in U_n \qquad \Longrightarrow \qquad \varphi = \varphi\circ L'_B \\ &\Longrightarrow \qquad \mathrm{d}_{\mathbb{I}_n}\varphi = \mathrm{d}_{L'_B(\mathbb{I}_n)}\varphi \circ \mathrm{d}_{\mathbb{I}_n}L'_B\colon T_{\mathbb{I}_n}U_n\mathbb{C} \longrightarrow T_BU_n \longrightarrow T_1S^1.\end{aligned}$$

Since $\mathrm{d}_{\mathbb{I}_n}\varphi$ is surjective, it follows that so is $\mathrm{d}_B\varphi$, for all $B \in SU_n$, i.e. 1 is a regular value for φ. Thus, by the Implicit FT, $SU_n = \varphi^{-1}(1)$ is an embedded submanifold of U_n of dimension

$$\dim SU_n = \dim U_n - \dim S^1 = n^2 - 1.$$

Since SU_n is the preimage of a point under a continuous function in a $T1$-space, SU_n is closed subset of U_n and thus compact.

Problem 5 (10pts)

Suppose $f\colon X\longrightarrow M$ and $g\colon Y\longrightarrow M$ are smooth maps that are transverse to each other:

$$T_{f(x)}M = \operatorname{Im} \mathrm{d}_x f + \operatorname{Im} \mathrm{d}_y g \qquad \forall\ (x,y)\in X\times Y \ \text{s.t}\ f(x)=g(y). \tag{1}$$

Show that

$$X\times_M Y \equiv \big\{(x,y)\in X\times Y:\ f(x)=g(y)\big\}$$

is a smooth (embedded) submanifold of $X\times Y$ of codimension equal to the dimension of X and

$$T_{(x,y)}\big(X\times_M Y\big) = \big\{(v,w)\in T_xX\oplus T_yY:\ \mathrm{d}_x f(v)=\mathrm{d}_y g(w)\big\} \qquad \forall\,(x,y)\in X\times_M Y.$$

We need to find a smooth function $h\colon X\times Y\longrightarrow N$ and a submanifold Z of N such that $X\times_M Y = h^{-1}(Z)$ and h is transverse to Z in N. The following is a standard trick for replacing a condition like $f(x)=g(y)\in M$ by $(x,y)\in h^{-1}(Z)$. Let

$$\Delta_M = \big\{(p,p)\in M\times M:\ p\in M\big\} \subset M\times M$$

be the diagonal in $M\times M$. It is the image of M under the smooth map

$$d\colon M \longrightarrow M\times M, \qquad d(p) = (p,p).$$

This map is a topological embedding and an immersion; so $Z=\Delta_M$ is an embedded submanifold of $N=M\times M$ and

$$T_{(p,p)}\Delta_M = \big\{(v,v)\in T_pM\oplus T_pM\big\} \subset T_{(p,p)}(M\times M) = T_pM\oplus T_pM \qquad \forall\, p\in M. \tag{2}$$

Define

$$h\colon X\times Y \longrightarrow M\times M \qquad \text{by} \qquad h(x,y) = \big(f(x),g(y)\big).$$

Since the maps f and g are smooth, so is the map h. Furthermore, $X\times_M Y = h^{-1}(\Delta_M)$.

We will now show that the transversality assumption (eq1) is equivalent to h being transverse to the diagonal:

$$T_{h(x,y)}(M\times M) = \operatorname{Im} \mathrm{d}_{(x,y)}h + T_{h(x,y)}\Delta_M \qquad \forall\,(x,y)\in h^{-1}(\Delta_M); \tag{3}$$

by the Implicit Function Theorem, $h^{-1}(\Delta_M)$ is then a smooth submanifold of $X\times Y$ (we only need to show (eq1) implies (eq3) for this). Suppose $(x,y)\in h^{-1}(\Delta_M)$. Condition (eq3) is equivalent to the condition that for all $v,w\in T_{f(x)}M=T_{g(y)}M$ there exist $x'\in T_xX$ and $y'\in T_yY$ such that

$$v - \mathrm{d}_x f(x') = w - \mathrm{d}_y g(y')$$

because then

$$(v,w) = \big(\mathrm{d}_x f(x'), \mathrm{d}_y g(y')\big) + \big(v-\mathrm{d}_x f(x'), w-\mathrm{d}_y g(y')\big) \in \operatorname{Im} \mathrm{d}_{(x,y)}h + T_{h(x,y)}\Delta_M.$$

This condition is equivalent to (eq1) (just move w to LHS and $\mathrm{d}_x f(x')$ to RHS).

It follows that $X\times_M Y$ is a smooth submanifold of $X\times Y$ of codimension equal to the codimension of Δ_M in M^2, which is the same as the dimension of M. For the last statement, note that

$$T_{(x,y)}\big(X\times_M Y\big) \subset \big\{\mathrm{d}_{(x,y)}h\big\}^{-1}\big(T_{(f(x),f(x))}\Delta_M\big) = \big\{\mathrm{d}_{(x,y)}h\big\}^{-1}\big(\big\{(u,u)\in T_{f(x)}M\oplus T_{f(x)}M:\ u\in T_{f(x)}M\big\}\big),$$

because $h(X\times_M Y)\subset\Delta_M$. By the transversality of h to Δ_M,

$$\begin{aligned}\dim \big\{\mathrm{d}_{(x,y)}h\big\}^{-1}\big(T_{(f(x),f(x))}\Delta_M\big) &= \dim T_xX + \dim T_yY - \big(\dim T_{(f(x),f(x))}M^2 - \dim T_{(f(x),f(x))}\Delta_M\big)\\ &= \dim T_xX + \dim T_yY - \dim M = \dim T_{(x,y)}\big(X\times_M Y\big);\end{aligned}$$

thus, the above inclusion is actually an equality.