MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 1

Problem 1: Chapter 1, #2 (10pts)

Let $\mathcal{F}$ be the (standard) differentiable structure on $\mathbb{R}$ generated by the one-element collection of charts $\mathcal{F}_0 = \{(\mathbb{R}, \mathrm{id})\}$. Let $\mathcal{F}'$ be the differentiable structure on $\mathbb{R}$ generated by the one-element collection of charts

$$\mathcal{F}'_0 = \{(\mathbb{R}, f)\}, \qquad \textit{where} \qquad f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad f(t) = t^3.$$

Show that $\mathcal{F} \neq \mathcal{F}'$, but the smooth manifolds $(\mathbb{R}, \mathcal{F})$ and $(\mathbb{R}, \mathcal{F}')$ are diffeomorphic.

(a) We begin by showing that $\mathcal{F} \neq \mathcal{F}'$. Since $\mathrm{id} \in \mathcal{F}_0 \subset \mathcal{F}$, it is sufficient to show that $\mathrm{id} \notin \mathcal{F}'$, i.e. the overlap map

$$\mathrm{id} \circ f^{-1}\colon f(\mathbb{R} \cap \mathbb{R}) = \mathbb{R} \longrightarrow \mathrm{id}(\mathbb{R} \cap \mathbb{R}) = \mathbb{R}$$

from $f \in \mathcal{F}'_0$ to id is not smooth, in the usual (i.e. calculus) sense:

$$\begin{tikzcd} \mathbb{R} \arrow[rr, dashed, "\mathrm{id}\circ f^{-1}"] & & \mathbb{R} \\ & \mathbb{R} \cap \mathbb{R} \arrow[ul, "f"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

Since $f(t) = t^3$, $f^{-1}(s) = s^{1/3}$, and

$$\mathrm{id} \circ f^{-1}\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad \mathrm{id} \circ f^{-1}(s) = s^{1/3}.$$

This is not a smooth map.

(b) Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be given by $h(t) = t^{1/3}$. It is immediate that h is a homeomorphism. We will show that the map

$$h\colon (\mathbb{R}, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{F}')$$

is a diffeomorphism, i.e. the maps

$$h\colon (\mathbb{R}, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{F}') \qquad \text{and} \qquad h^{-1}\colon (\mathbb{R}, \mathcal{F}') \longrightarrow (\mathbb{R}, \mathcal{F})$$

are smooth. To show that h is smooth, we need to show that it induces smooth maps between the charts in $\mathcal{F}_0$ and $\mathcal{F}'_0$. In this case, there is only one chart in each. So we need to show that the map

$$f \circ h \circ \mathrm{id}^{-1}\colon \mathrm{id}\big(h^{-1}(\mathbb{R}) \cap \mathbb{R}\big) = \mathbb{R} \longrightarrow \mathbb{R}$$

is smooth:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, dashed, "f\circ h\circ \mathrm{id}^{-1}"] & \mathbb{R} \\
(\mathbb{R},\mathcal{F}) \arrow[u, "\mathrm{id}"] \arrow[r, "h"] & (\mathbb{R},\mathcal{F}') \arrow[u, "f"']
\end{tikzcd}$$

Since

$$f\circ h\circ \mathrm{id}^{-1}(t) = f\big(h(t)\big) = f(t^{1/3}) = \big(t^{1/3}\big)^3 = t,$$

this map is indeed smooth, and so is h. To check that h^{-1} is smooth, we need to show that it induces smooth maps between the charts in $\mathcal{F}_0'$ and $\mathcal{F}_0$, i.e. that the map

$$\mathrm{id}\circ h^{-1}\circ f^{-1}\colon f\big(h(\mathbb{R})\cap\mathbb{R}\big)=\mathbb{R}\longrightarrow\mathbb{R}$$

is smooth:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, dashed, "\mathrm{id}\circ h^{-1}\circ f^{-1}"] & \mathbb{R} \\
(\mathbb{R},\mathcal{F}') \arrow[u, "f"] \arrow[r, "h^{-1}"] & (\mathbb{R},\mathcal{F}) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Since

$$\mathrm{id}\circ h^{-1}\circ f^{-1}(t) = h^{-1}\big(f(t)\big) = h^{-1}(t^3) = \big(t^3\big)^{1/3} = t,$$

this map is indeed smooth, and so is h^{-1}. Since h and h^{-1} are smooth maps, we conclude that h is diffeomorphism from $(\mathbb{R},\mathcal{F})$ to $(\mathbb{R},\mathcal{F}')$.

Remarks: (1) Since we know that h is a homeomorphism, it is sufficient to show that h induces *diffeomorphisms* on all charts in $\mathcal{F}_0$ and $\mathcal{F}_0'$. This would imply that h^{-1} is smooth as well, since the maps between charts induced by h^{-1} are inverses of those induced by h.
(2) More generally, every topological manifold of dimension 1, 2, or 3 admits a differentiable structure and any two such structures are diffeomorphic. Up to diffeomorphism, the only connected 1-dimensional manifolds are $\mathbb{R}$ and S^1, with their standard differentiable structures (you can find a proof in the 2.5-page appendix in Milnor's *Topology from Differentiable Viewpoint*). Starting in dimension 4, things get more complicated. Not every topological 4-manifold admits a smooth structure. In a seven-page paper in 1956 (cited in his Fields medal award), Milnor showed that S^7 admits non-diffeomorphic smooth structures. Since then the situation for manifolds in dimensions five and higher has been sorted out; amazingly, 4 is the hard dimension.

Problem 2 (10pts)

Suppose a group G acts properly discontinuously on a smooth n-manifold $\tilde{M}$ by diffeomorphisms. Show that the quotient topological space $M=\tilde{M}/G$ admits a unique smooth structure such that the projection map $\tilde{M}\longrightarrow M$ is a local diffeomorphism.

Since G acts properly discontinuously on $\tilde{M}$, the quotient projection map $\pi\colon \tilde{M}\longrightarrow M$ is a covering projection. The assumption that G acts by diffeomorphism leads to the following key property.

Claim: If V, W are open subset of $\tilde{M}$ such that $\pi|_V$ and $\pi|_W$ are injective, then

$$\pi_{VW} \equiv \{\pi|_V\}^{-1} \circ \pi \colon \{\pi|_W\}^{-1}\big(\pi(V)\big) \longrightarrow \{\pi|_V\}^{-1}\big(\pi(W)\big)$$

is a diffeomorphism.

Proof. By assumption, $\pi : V \longrightarrow \pi(V)$ and $\pi : W \longrightarrow \pi(W)$ are homeomorphisms; thus, so is the map π_{VW} (with the specified domain and range, which are open subsets of $\tilde{M}$). Thus, it is sufficient to show each point $p \in \{\pi|_W\}^{-1}(\pi(V))$ has a neighborhood W_p in $\{\pi|_W\}^{-1}(\pi(V))$ such that $\pi_{VW}|_{W_p}$ is smooth. Let $p' = \{\pi|_V\}^{-1}(\pi(p))$; then $p' = gp$ for a unique $g \in G$. Since $g \colon \tilde{M} \longrightarrow \tilde{M}$ is continuous, $W_p = g^{-1}(V) \cap W$ is an open neighborhood of p in $\{\pi|_W\}^{-1}(\pi(V))$ and $\pi_{VW}|_{W_p} = g|_{W_p}$ (for each $q \in W_p$, $\pi_{VW}(q) = g_q q \in V$ for some $g_q \in G$, $gq \in V$, and there exists (at most) a unique $g' \in G$ such that $g'q \in V$). Since G acts by diffeomorphisms, $g|_{W_p}$ is smooth.

Let $\mathcal{F}_{\tilde{M}}$ be the smooth structure on $\tilde{M}$ and

$$\mathcal{F}_0 = \big\{(\pi(V), \varphi \circ \{\pi|_V\}^{-1}) \colon (V, \varphi) \in \mathcal{F}_{\tilde{M}},\ \pi|_V \text{ is injective}\big\}.$$

Since π is a covering map, $\pi(V) \subset M$ is open whenever $V \subset \tilde{M}$ is. Since for every $p \in \tilde{M}$ there exists $(V, \varphi) \in \mathcal{F}_{\tilde{M}}$ such that $\pi|_V$ is injective, the union of the sets $\pi(V)$ with $(\pi(V), \varphi \circ \{\pi|_V\}^{-1}) \in \mathcal{F}_0$ covers M. If $(\pi(V), \varphi \circ \{\pi|_V\}^{-1}), (\pi(W), \psi \circ \{\pi|_W\}^{-1}) \in \mathcal{F}_0$,

$$\varphi \circ \{\pi|_V\}^{-1} \circ \big(\psi \circ \{\pi|_W\}^{-1}\big)^{-1} = \varphi \circ \pi_{VW} \circ \psi^{-1} \colon \psi\big(\{\pi|_W\}^{-1}(\pi(V))\big) \longrightarrow \varphi\big(\{\pi|_V\}^{-1}(\pi(W))\big)$$

is smooth, because π_{VW} is smooth by the claim and φ and ψ are charts. Thus, $\mathcal{F}_0$ satisfies (i) and (ii) on p5 and thus gives rise to a smooth structure on M.

With respect to this smooth structure, the map $\pi \colon \tilde{M} \longrightarrow M$ is a local diffeomorphism because

$$\varphi \circ \{\pi|_V\}^{-1} \circ \pi|_W \circ \psi^{-1} = \varphi \circ \pi_{VW} \circ \psi^{-1} \colon \psi\big(\{\pi|_W\}^{-1}(\pi(V))\big) \longrightarrow \varphi\big(\{\pi|_V\}^{-1}(\pi(W))\big)$$

is a diffeomorphism whenever $(\pi(V), \varphi \circ \{\pi|_V\}^{-1}), (\pi(W), \psi \circ \{\pi|_W\}^{-1}) \in \mathcal{F}_0$. Conversely, if $\tilde{\mathcal{F}}'$ is any smooth structure on M such that $\pi \colon \tilde{M} \longrightarrow M$ is a local diffeomorphism, then

$$\varphi \circ \{\pi|_V\}^{-1} \circ \big(\psi \circ \{\pi|_W\}^{-1}\big)^{-1} = \varphi \circ \{\pi|_V\}^{-1} \circ \pi|_W \circ \psi^{-1} \colon \psi\big(\{\pi|_W\}^{-1}(\pi(V))\big) \longrightarrow \varphi\big(\{\pi|_V\}^{-1}(\pi(W))\big)$$

is a diffeomorphism whenever $(\pi(V), \varphi \circ \{\pi|_V\}^{-1}), (\pi(W), \psi \circ \{\pi|_W\}^{-1}) \in \mathcal{F}_0$, and so $\mathcal{F}_0 \subset \mathcal{F}'$ and thus $\mathcal{F}' = \mathcal{F}$ by the maximality condition.

Note: this implies that the circle, the infinite Mobius band, the Lens spaces (that are important in 3-manifold topology), the real projective space, and the tautological line bundle over it,

$$S^1 = \mathbb{R}/\mathbb{Z}, \quad s \sim s+1, \qquad MB = (\mathbb{R} \times \mathbb{R})/\mathbb{Z}, \quad (s,t) \sim (s+1, -t),$$
$$L(n,k) = S^3/\mathbb{Z}_n, \quad (z_1, z_2) \sim \big(\mathrm{e}^{2\pi \mathrm{i}/n} z_1, \mathrm{e}^{2\pi \mathrm{i}k/n} z_2\big) \in \mathbb{C}^2,$$
$$\mathbb{R}P^n = S^n/\mathbb{Z}_2, \quad x \sim -x, \qquad \gamma_n = (S^n \times \mathbb{R})/\mathbb{Z}_2, \quad (x,t) \sim (-x, -t),$$

are smooth manifolds in a natural way (k and n are relatively prime in the definition of $L(n,k)$).

Problem 3 (15pts)

(a) *Show that the quotient topologies on $\mathbb{C}P^n$ given by $(\mathbb{C}^{n+1}-0)/\mathbb{C}^*$ and S^{2n+1}/S^1 are the same.*
(b) *Show that $\mathbb{C}P^n$ is a compact topological $2n$-manifold. Furthermore, it admits a structure of a complex (in fact, algebraic) n-manifold, i.e. it can be covered by charts whose overlap maps, $\varphi_\alpha\circ\varphi_\beta^{-1}$, are holomorphic maps between open subsets of $\mathbb{C}^n$ (and rational functions on $\mathbb{C}^n$).*
(c) *Show that $\mathbb{C}P^n$ contains $\mathbb{C}^n$, with its complex structure, as a dense open subset.*

(a) Let

$$p\colon S^{2n+1}\longrightarrow S^{2n+1}/S^1 \qquad\text{and}\qquad q\colon \mathbb{C}^{n+1}-0\longrightarrow(\mathbb{C}^{n+1}-0)/\mathbb{C}^*$$

be the quotient projection maps. Denote by

$$\tilde{i}\colon S^{2n+1}\longrightarrow \mathbb{C}^{n+1}-0 \qquad\text{and}\qquad \tilde{r}\colon \mathbb{C}^{n+1}-0\longrightarrow S^{2n+1}$$

the inclusion map and the natural retraction map, i.e. $\tilde{r}(v)=v/|v|$. We will show that these maps descend to continuous maps on the quotients, i and r,

$$\begin{tikzcd} S^{2n+1} \arrow[r, "\tilde{i}"] \arrow[d, "p"'] & \mathbb{C}^{n+1}-0 \arrow[d, "q"] \\ S^{2n+1}/S^1 \arrow[r, dashed, "i"] & (\mathbb{C}^{n+1}-0)/\mathbb{C}^* \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}^{n+1}-0 \arrow[r, "\tilde{r}"] \arrow[d, "q"'] & S^{2n+1} \arrow[d, "p"] \\ (\mathbb{C}^{n+1}-0)/\mathbb{C}^* \arrow[r, dashed, "r"] & S^{2n+1}/S^1 \end{tikzcd}$$

that are inverses of each other. The map $q\circ\tilde{i}$ is constant on the fibers of p, since if $v,w\in S^{2n+1}$ and $w=g\cdot v$ for some $g\in S^1$, then $\tilde{i}(w)=g'\cdot\tilde{i}(v)$ for some $g'\in\mathbb{C}^*$ (in fact, $g'=g$). Thus, $q\circ\tilde{i}$ induces a map i from the quotient space S^{2n+1}/S^1 (so that the first diagram commutes); since the map $q\circ\tilde{i}$ is continuous, so is the induced map i. Similarly, the map $p\circ\tilde{r}$ is constant on the fibers of q, since if $v,w\in\mathbb{C}^{n+1}-0$ and $w=g\cdot v$ for some $g\in\mathbb{C}^*$, then $\tilde{r}(w)=g'\cdot\tilde{r}(v)$ for some $g'\in S^1$ (in fact, $g'=g/|g|$). Thus, $p\circ\tilde{r}$ induces a map r from the quotient space $(\mathbb{C}^{n+1}-0)/\mathbb{C}^*$; since the map $p\circ\tilde{r}$ is continuous, so is the induced map r. Since $\tilde{r}\circ\tilde{i}=\mathrm{id}_{S^{2n+1}}$, $r\circ i=\mathrm{id}_{S^{2n+1}/S^1}$. Similarly, for all $v\in\mathbb{C}^{n+1}-0$,

$$\tilde{i}\circ\tilde{r}(v)=(1/|v|)v,\quad 1/|v|\in\mathbb{C}^* \quad\Longrightarrow\quad q\big(\tilde{i}\circ\tilde{r}(v)\big)=q(v) \quad\Longrightarrow\quad i\circ r=\mathrm{id}_{(\mathbb{C}^{n+1}-0)/\mathbb{C}^*}.$$

(b-i) Since S^{2n+1} is compact, so is the quotient space $\mathbb{C}P^n=S^{2n+1}/S^1$ (being the image of S^{2n+1} under the continuous map p). Suppose next that $A\subset S^{2n+1}$ is a closed subset. Then,

$$p^{-1}\big(p(A)\big)=S^1\cdot A\equiv\big\{g\cdot v\colon v\in A,\ g\in S^1\big\}.$$

Thus, $p^{-1}(p(A))$ is the image of the closed subset $S^1{\times}A$ in S^{2n+1} under the continuous multiplication map

$$S^1\times S^{2n+1}\longrightarrow S^{2n+1}.$$

Since A is closed in S^{2n+1}, $S^1\times A$ is closed in the compact space $S^1\times S^{2n+1}$ and thus compact. It follows that $p^{-1}(p(A))$ is a compact subset of the Hausdorff space S^{2n+1} and thus closed. We conclude that $p(A)\subset S^{2n+1}/S^1$ is closed for all closed subsets $A\subset S^{2n+1}$, i.e. the quotient map p is a closed map. Since S^{2n+1} is normal, by Lemma 73.3 in Munkres's *Topology* the quotient space

$\mathbb{C}P^n$ is normal as well (and in particular, Hausdorff).

(b-ii) We will now construct a collection of charts $\{(\mathcal{U}_i, \varphi_i)\}_{i=0,1,\dots,n}$ on $\mathbb{C}P^n$ that covers $\mathbb{C}P^n$. Given a point $(X_0, \dots, X_n) \in \mathbb{C}^{n+1} - 0$, we denote its equivalence class in

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$$

by $[X_0, \dots, X_n]$. For $i = 0, 1, \dots, n$, let

$$\mathcal{U}_i = \big\{[X_0, \dots, X_n] \in \mathbb{C}P^n : X_i \neq 0\big\}.$$

Since

$$q^{-1}(\mathcal{U}_i) = \big\{(X_0, \dots, X_n) \in \mathbb{C}^{n+1} - 0 : X_i \neq 0\big\} \equiv \tilde{\mathcal{U}}_i$$

is an open subset of $\mathbb{C}^{n+1} - 0$, $\mathcal{U}_i$ is an open subset of $\mathbb{C}P^n$. Define

$$\tilde{\varphi}_i : \tilde{\mathcal{U}}_i \longrightarrow \mathbb{C}^n = \mathbb{R}^{2n} \qquad \text{by}$$
$$\tilde{\varphi}_i(X_0, \dots, X_n) = \big(X_0/X_i, X_1/X_i, \dots, X_{i-1}/X_i, X_{i+1}/X_i, \dots, X_n/X_i\big).$$

Since $\tilde{\varphi}_i(c \cdot v) = \tilde{\varphi}_i(v)$, the map $\tilde{\varphi}_i$ induces a map φ_i from the quotient space $\mathcal{U}_i$ of $\tilde{\mathcal{U}}_i$:

$$\begin{tikzcd} \tilde{\mathcal{U}}_i \arrow[d, "q"'] \arrow[dr, "\tilde{\varphi}_i"] & \\ \mathcal{U}_i \arrow[r, dashed, "\varphi_i"] & \mathbb{C}^n \end{tikzcd}$$

Since $\tilde{\varphi}_i$ is continuous, so is φ_i. Define

$$\psi_i : \mathbb{C}^n \longrightarrow \mathcal{U}_i \qquad \text{by} \qquad \psi_i(z_1, \dots, z_n) = \big[z_1, \dots, z_i, X_i = 1, z_{i+1}, \dots, z_n\big].$$

Since ψ_i is a composition of two continuous maps, ψ_i is continuous. Since $\psi_i \circ \varphi_i = \mathrm{id}_{\mathcal{U}_i}$ and $\varphi_i \circ \psi_i = \mathrm{id}_{\mathbb{C}^n}$, the map

$$\varphi_i : \mathcal{U}_i \longrightarrow \mathbb{C}^n$$

is a homeomorphism. Note that for every $p \equiv [X_0, \dots, X_n] \in \mathbb{C}P^n$, there exists $i = 0, 1, \dots, n$ such that $X_i \neq 0$, i.e. $p \in \mathcal{U}_i$. Thus, $\{(\mathcal{U}_i, \varphi_i)\}_{i=0,1,\dots,n}$ is a collection of charts on $\mathbb{C}P^n$ that covers $\mathbb{C}P^n$. In particular, $\mathbb{C}P^n$ is locally Euclidean of dimension $2n$. Since this collection of charts is countable (actually, finite), it follows that $\mathbb{C}P^n$ is 2nd-countable (since each open subset $\mathcal{U}_i$ is 2nd-countable).

(b-iii) We now determine the overlap maps

$$\varphi_i \circ \varphi_j^{-1} = \varphi_i \circ \psi_j : \varphi_j(\mathcal{U}_i \cap \mathcal{U}_j) \longrightarrow \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j).$$

Assume that $j < i$. Then,

$$\mathcal{U}_i \cap \mathcal{U}_j = \big\{[X_0, \dots, X_n] \in \mathbb{C}P^n : X_i, X_j \neq 0\big\} \qquad \Longrightarrow$$
$$\varphi_j(\mathcal{U}_i \cap \mathcal{U}_j) = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq 0\big\} \equiv \mathbb{C}^n_i, \quad \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j) = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_{j+1} \neq 0\big\} \equiv \mathbb{C}^n_{j+1};$$

the assumption $j < i$ is used on the second line. By (b-ii), the map

$$\varphi_i \circ \varphi_j^{-1} : \mathbb{C}^n_i \longrightarrow \mathbb{C}^n_{j+1}$$

is given by

$$\varphi_i \circ \varphi_j^{-1}(z_1, \ldots, z_n) = \varphi_i \circ \psi_j(z_1, \ldots, z_n) = \varphi_i\big([z_1, \ldots, z_j, X_j = 1, z_{j+1}, \ldots, z_n]\big)$$
$$= \big(z_1/z_i, \ldots, z_j/z_i, 1/z_i, z_{j+1}/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_i\big).$$

Thus, the overlap map $\varphi_i \circ \varphi_j^{-1}$ is holomorphic on its domain, as is its inverse, $\varphi_j \circ \varphi_i^{-1}$; both maps are given by rational functions on $\mathbb{C}^n$. We conclude that the collection $\mathcal{F}_0 = \{(\mathcal{U}_i, \varphi_i)\}_{i=0,1,\ldots,n}$ determines a complex structure on $\mathbb{C}P^n$.

(c) By part (b), the map

$$\psi_0 \colon \mathbb{C}^n \longrightarrow \mathcal{U}_0 \subset \mathbb{C}P^n, \qquad (z_1, \ldots, z_n) \longrightarrow [1, z_1, \ldots, z_n],$$

is a homeomorphism and

$$\varphi_0^{-1} \circ \psi_0 \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

is the identity map (and thus holomorphic). Since $(\mathcal{U}_0, \varphi_0) \in \mathcal{F}_0$, ψ_0 is a holomorphic embedding. So, $\mathbb{C}P^n$ contains $\mathbb{C}^n$ (as $\mathcal{U}_0$) with its complex structure as an open subset. The subset $\mathcal{U}_0$ is dense in $\mathbb{C}P^n$, since $\tilde{\mathcal{U}}_0 = q^{-1}(\mathcal{U}_0)$ is dense in $\mathbb{C}^{n+1} - 0$.

Remark: We can of course use any of the maps ψ_i in part (c). By part (c), $\mathbb{C}P^n$ is a compactification of $\mathbb{C}^n$ (i.e. $\mathbb{C}^n$ is a dense open subset of the compact Hausdorff space $\mathbb{C}P^n$). In contrast to the one-point compactification S^{2n} of $\mathbb{C}^n$ (for $n > 1$), $\mathbb{C}P^n$ has complex and algebraic structure. If $n = 1$, the two compactifications are the same; $\mathbb{C}P^1$ is the Riemann sphere.

Problem 4: Chapter 1, #6, via 2nd suggested approach (5pts)

Suppose $f \colon M \longrightarrow N$ is a bijective immersion. Show that f is a diffeomorphism.

Let $n = \dim M$ and $k = \dim N$. Since f is an immersion, the differential

$$df|_m \colon T_m M \longrightarrow T_{f(m)} N$$

is injective for all $m \in M$. In particular, $n \le k$. If $n = k$, then $df|_m$ is an isomorphism for all $m \in M$ and f is a local diffeomorphism by the Inverse Function Theorem. Since f is bijective, it then follows that f is a (global) diffeomorphism if $n = k$. Below we show that the case $n < k$ cannot arise.

Suppose $n < k$ and (W, φ) is a coordinate chart on N such that $\varphi(W) = \mathbb{R}^k$. Then, $f^{-1}(W)$ is a smooth n-manifold. It is sufficient to show that the image of $f^{-1}(W)$ under f is not all of W, or equivalently that the smooth map

$$g \equiv \varphi \circ f \colon W \longrightarrow \mathbb{R}^n$$

is not surjective. Let $\{\psi_i : \mathcal{U}_i \longrightarrow V_i\}_{i \in \mathbb{Z}}$ be a collection of charts on $f^{-1}(W)$ that covers $f^{-1}(W)$. Then,

$$g\big(f^{-1}(W)\big) = g\bigg(\bigcup_{i \in \mathbb{Z}} \psi_i^{-1}(V_i)\bigg) = \bigcup_{i \in \mathbb{Z}} g\big(\psi_i^{-1}(V_i)\big) \subset \mathbb{R}^k.$$

Since V_i is an open subset of $\mathbb{R}^n$, $g\circ\psi_i^{-1}\colon V_i\longrightarrow\mathbb{R}^k$ is a smooth map, and $n<k$, the k-measure of $g(\psi_i^{-1}(V_i))$ in $\mathbb{R}^k$ is 0 (for reasons described in detail in the statement of the exercise in the book). Since a countable union of measure 0 subsets of $\mathbb{R}^k$ is of measure 0, it follows that $g(f^{-1}(W))$ is a subset of $\mathbb{R}^k$ of measure 0. In particular,

$$g\big(f^{-1}(W)\big)\subsetneq\mathbb{R}^k,$$

as needed.

Remark: This argument implies that there exist no *smooth* surjective map $f:\mathbb{R}\longrightarrow\mathbb{R}^k$ if $k>1$. Recall from 530 that there does exist a *continuous* surjective map $f:\mathbb{R}\longrightarrow\mathbb{R}^k$ (it can be constructed from the Peano curve).

Problem 5 (5pts)

If $\psi\colon M\longrightarrow N$ is a smooth map and $m\in M$, the differential of ψ at m,

$$d\psi|_m\colon T_mM\longrightarrow T_{\psi(m)}N,$$

is defined by

$$\big\{d\psi|_m v\big\}(f)=v(f\circ\psi)\in\mathbb{R}\qquad\forall\,v\in T_mM,\ \mathbf{f}\in\tilde{F}_{\psi(m)}. \tag{1}$$

Show that $d\psi|_m v$ is indeed a well-defined element of $T_{\psi(m)}N$ for all $v\in T_mM$.

We need to show that $d\psi|_m v$ induces a linear derivation on $\tilde{F}_{\psi(m)}$, i.e. a linear map

$$\tilde{F}_{\psi(m)}\longrightarrow\mathbb{R}$$

satisfying the product rule. Suppose $\mathcal{U}$ and V are (open) neighborhoods of $\psi(m)$ in N, $f\colon\mathcal{U}\longrightarrow\mathbb{R}$ and $g\colon V\longrightarrow\mathbb{R}$ are smooth functions, and $W\subset U\cap V$ is a neighborhood of $\psi(m)$ such that $f|_W=g|_W$, i.e. $\mathbf{f}=\mathbf{g}\in\tilde{F}_{\psi(m)}$. Then, $\psi^{-1}(U)$ and $\psi^{-1}(V)$ are neighborhood of m in M,

$$f\circ\psi\colon\psi^{-1}(U)\longrightarrow\mathbb{R}\qquad\text{and}\qquad g\circ\psi\colon\psi^{-1}(V)\longrightarrow\mathbb{R}$$

are smooth functions, and $\psi^{-1}(W)\subset f^{-1}(U)\cap f^{-1}(V)$ is a neighborhood of m such that

$$\begin{aligned}(f\circ\psi)|_{\psi^{-1}(W)}=(g\circ\psi)|_{\psi^{-1}(W)}\qquad&\Longrightarrow\qquad[f\circ\psi]=[g\circ\psi]\in\tilde{F}_m\\&\Longrightarrow\qquad v(f\circ\psi)=v\big([f\circ\psi]\big)=v\big([g\circ\psi]\big)=v(g\circ\psi),\end{aligned}$$

since $v\in T_mM$. It follows that (1) induces a well-defined map $\tilde{F}_m\longrightarrow\mathbb{R}$ (independent of the choice of representative f for the equivalence class $\mathbf{f}\in\tilde{F}_{\psi(m)}$). If f and g are smooth functions on neighborhoods of $\psi(m)$ in M and $\alpha,\beta\in\mathbb{R}$, then

$$\begin{aligned}\big\{d\psi|_m v\big\}(\alpha f+\beta g)&\equiv v\big((\alpha f+\beta g)\circ\psi\big)=v\big(\alpha\,(f\circ\psi)+\beta\,(g\circ\psi)\big)\\&=\alpha\,v(f\circ\psi)+\beta\,v(g\circ\psi)\equiv\alpha\big\{d\psi|_m v\big\}(f)+\beta\big\{d\psi|_m v\big\}(g),\end{aligned}$$

i.e. $d\psi|_m v$ is a linear map. Finally, with f and g as above,

$$\begin{aligned}\big\{d\psi|_m v\big\}(f\cdot g)&\equiv v\big((f\cdot g)\circ\psi\big)=v\big((f\circ\psi)\cdot(g\circ\psi)\big)\\&=f\circ\psi(m)\,v(g\circ\psi)+g\circ\psi(m)\,v(f\circ\psi)\\&\equiv f\big(\psi(m)\big)\big\{d\psi|_m v\big\}(g)+g\big(\psi(m)\big)\big\{d\psi|_m v\big\}(f),\end{aligned}$$

i.e. $d\psi|_m v$ satisfies the product rule.