MAT 531: Topology&Geometry, II
Spring 2011

Final Exam Solutions

Part I (*choose 2 problems from 1,2, and 3*)

1. *Let $f\colon \mathbb{R}P^3 \longrightarrow T^3 \equiv (S^1)^3$ be a smooth map. Show that f is not an immersion.*
Suppose f is an immersion. Since $\mathbb{R}P^3$ and T^3 have the same dimension, the differential

$$\mathrm{d}_x f\colon T_x\mathbb{R}P^3 \longrightarrow T_{f(x)}T^3$$

is an isomorphism for every $x \in \mathbb{R}P^3$. By the Inverse Function Theorem, f is thus a local diffeomorphism, and so its image is open in T^3. Since $\mathbb{R}P^3$ is compact and T^3 is Hausdorff, $f(\mathbb{R}P^3)$ is closed in T^3. Since T^3 is connected, it follows that f is surjective. Since $\mathbb{R}P^3$ is compact and f is a local diffeomorphism, $f^{-1}(y) \subset \mathbb{R}P^3$ is finite for every $y \in T^3$. Thus, f is a covering projection (the intersection of the images of neighborhoods of elements of $f^{-1}(y)$ on which f is a diffeomorphism is an evenly covered neighborhood of y), and

$$f_*\colon \pi_1(\mathbb{R}P^3, x_0) \longrightarrow \pi_1(T^3, f(x_0))$$

is an injective homomorphism. However, this is impossible, since $\pi_1(\mathbb{R}P^3, x_0) \approx \mathbb{Z}_2$ has torsion, while $\pi_1(T^3, f(x_0)) \approx \mathbb{Z}^3$ is torsion-free.

2. *Let X and Y be the vector fields on $\mathbb{R}^3$ given by*

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\,, \qquad Y = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\,.$$

(a) *Compute the flows φ_s and ψ_t of X and Y (give formulas).*

(b) *Do these flows commute?*

(a) The time s-flow of X through (x_0, y_0, z_0) is the solution to the initial-value problem

$$\begin{cases} x'(s) = 1, \quad y'(s) = x, \quad z'(s) = y, \\ (x(0), y(0), z(0)) = (x_0, y_0, z_0). \end{cases}$$

Solving the first equation, then the second, and finally the third, we find that

$$\big(x(s), y(s), z(s)\big) = \big(x_0 + s, y_0 + x_0 s + \frac{s^2}{2}, z_0 + y_0 s + x_0\frac{s^2}{2} + \frac{s^3}{6}\big).$$

Thus, the time s-flow of X is given by

$$\varphi_s(x, y, z) = \big(x + s, y + sx + \frac{s^2}{2}, z + sy + \frac{s^2}{2}x + \frac{s^3}{6}\big).$$

Similarly, the time t-flow of Y is given by

$$\psi_t(x, y, z) = \big(x + ty + \frac{t^2}{2}z + \frac{t^3}{6}, y + tz + \frac{t^2}{2}, z + t\big),$$

as the roles of x and z in X and Y are interchanged.

(b) Since the Lie bracket of coordinate vector fields is 0,

$$\begin{aligned}[X,Y] &= \left(X(y)\frac{\partial}{\partial x} + X(z)\frac{\partial}{\partial y} + X(1)\frac{\partial}{\partial z}\right) - \left(Y(1)\frac{\partial}{\partial x} + Y(x)\frac{\partial}{\partial y} + Y(y)\frac{\partial}{\partial z}\right) \\ &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\right) - \left(0 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}.\end{aligned}$$

Since $[X,Y]\neq 0$, the flows of X and Y do not commute by PS4 #5.

Alternatively, the first coordinates of $\varphi_s\circ\psi_t$ and $\psi_t\circ\varphi_s$ are given by

$$(x,y,z) \longrightarrow x + ty + \frac{t^2}{2}z + \frac{t^3}{6} + s\,,\, (x+s) + t\big(y+sx+\frac{s^2}{2}\big) + \frac{t^2}{2}\big(z+sy+\frac{s^2}{2}x+\frac{s^3}{6}\big) + \frac{t^3}{6}\,,$$

respectively. Since these are not the same (unless $s=0$ or $t=0$), the flows do not commute.

3. *Let M and N be smooth oriented connected manifolds and $H\colon M\times[0,1]\longrightarrow N$ a smooth map. For each $t\in[0,1]$, define*

$$H_t\colon M \longrightarrow N, \qquad H_t(p) = H(p,t).$$

(a) *Suppose H_t is a diffeomorphism for every $t\in[0,1]$. Show that H_0 is orientation-preserving if and only if H_1 is.*

(b) *Suppose instead that M is compact and H_0, H_1 are diffeomorphisms. Show that H_0 is orientation-preserving if and only if H_1 is.*

(c) *Give an example so that H_0 and H_1 are diffeomorphisms, with H_0 orientation-preserving and H_1 orientation-reversing.*

It can be assumed that the manifolds M and N have the same dimension n. Let $\omega_M\in E^n(M)$ and $\omega_N\in E^n(N)$ be oriented volume forms (nowhere 0 top forms). Let $f\in C^\infty(M\times[0,1])$ and $\gamma\in E^{n-1}(M\times[0,1])$ be such that

$$H^*\omega_N = f\cdot\pi_1^*\omega_M + \gamma\wedge\pi_2^*\mathrm{d}t \qquad\Longrightarrow\qquad H_t^*\omega_N = f_t\omega_M\,,$$

where $f_t\in C^\infty(M)$, $f_t(p)=f(p,t)$.

(a) Since H_t is a diffeomorphism for all t, $f(t,p)=f_t(p)\in\mathbb{R}^*$. Since $M\times[0,1]$ is connected, either $f(t,p)\in\mathbb{R}^+$ for all (t,p) or $f(t,p)\in\mathbb{R}^-$ for all (t,p). Thus, H_0 is orientation-preserving (i.e. $f_0(p)>0$ for all $p\in M$) if and only if H_1 is (i.e. $f_1(p)>0$ for all $p\in M$).

(b) Since the maps $H_0,H_1\colon M\longrightarrow N$ are smoothly homotopic,

$$[H_0^*\omega_N] = [H_1^*\omega_N] \qquad\Longrightarrow\qquad \int_M H_0^*\omega_N = \int_M H_1^*\omega_N\,.$$

Since M is connected, either $f_0(p)>0$ for all $p\in M$ or $f_0(p)<0$ for all $p\in M$; in the first case

$$\int_M H_0^*\omega_N = \int_M f_0\omega_M > 0,$$

while in the second case this integral is negative. The same applies to f_1 and H_1. Since the two integrals are the same, H_0 is orientation-preserving (i.e. $f_0(p)>0$ for all $p\in M$) if and only if H_1 is (i.e. $f_1(p)>0$ for all $p\in M$).

(c) Let $H\colon \mathbb{R}\times[0,1]\longrightarrow\mathbb{R}$ be given by

$$H(p,t) = -tp + (1-t)p.$$

Then, $H_0=\mathrm{id}_{\mathbb{R}}$ is orientation-preserving, while $H_1=-\mathrm{id}_{\mathbb{R}}$ is orientation-reversing.

Note: In this case, $M=\mathbb{R}$ is not compact and $H_{1/2}$ is the constant map sending $\mathbb{R}$ to 0 and so is not a diffeomorphism.

Part II (*choose 2 problems from 4,5, and 6*)

4. *Let M be a smooth manifold obtained by identifying two copies of a Mobius Band, M_1 and M_2, along their boundary circles. Compute $H^*_{\mathrm{deR}}(M)$.*

Since M is 2-manifold, $H^k_{\mathrm{deR}}(M)=0$ for $k\neq 0,1,2$. Since M is connected, $H^0_{\mathrm{deR}}(M)\approx\mathbb{R}$. Since the interior of M_1 is a non-orientable open subset of M, M is also non-orientable, and so $H^2_{\mathrm{deR}}(M)\approx 0$. It remains to compute $H^1_{\mathrm{deR}}(M)$.

This can be done using Mayer-Vietoris. Let $U\subset M$ be a small tubular neighborhood of M_1 (or the complement of the "equator" in M_2) and $V\subset M$ a small tubular neighborhood of M_2 (or the complement of the "equator" in M_1). Thus, U and V are open Mobius Bands, while $U\cap V$ is an open cylinder. Since all three are homotopic to a circle, by the homotopy invariance of de Rham cohomology,

$$H^k_{\mathrm{deR}}(U), H^k_{\mathrm{deR}}(V), H^k_{\mathrm{deR}}(U\cap V) \approx H^k_{\mathrm{deR}}(S^1) \approx \begin{cases} \mathbb{R}, & \text{if } k=0,1; \\ 0, & \text{otherwise.} \end{cases}$$

The MV long sequence in this case is

$$\begin{aligned} 0 \longrightarrow H^0_{\mathrm{deR}}(M) &\longrightarrow H^0_{\mathrm{deR}}(U)\oplus H^0_{\mathrm{deR}}(V) \longrightarrow H^0_{\mathrm{deR}}(U\cap V) \\ \xrightarrow{\delta_0} H^1_{\mathrm{deR}}(M) &\longrightarrow H^1_{\mathrm{deR}}(U)\oplus H^1_{\mathrm{deR}}(V) \longrightarrow H^1_{\mathrm{deR}}(U\cap V) \longrightarrow H^2_{\mathrm{deR}}(M). \end{aligned}$$

Plugging in for the known groups, we obtain

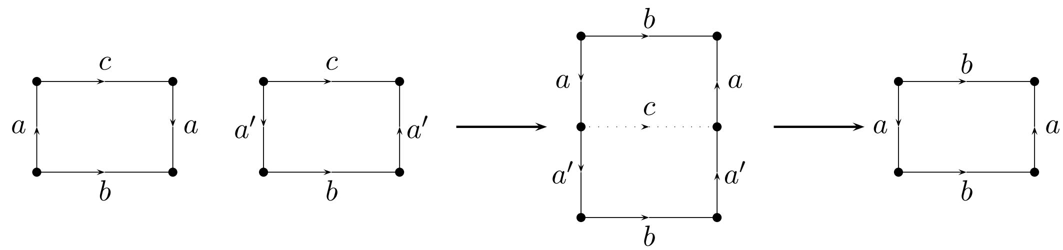
$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}\oplus\mathbb{R} \longrightarrow \mathbb{R} \xrightarrow{\delta_0} H^1_{\mathrm{deR}}(M) \longrightarrow \mathbb{R}\oplus\mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0.$$

Thus, δ_0 is the zero homomorphism, and the sequence

$$0 \longrightarrow H^1_{\mathrm{deR}}(M) \longrightarrow \mathbb{R}\oplus\mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0$$

is exact. So, $H^1_{\mathrm{deR}}(M) \approx \mathbb{R}$.

Alternatively, the manifold M is homeomorphic to the Klein bottle, as can be seen from the following diagram (see Chapter 8 in Munkres):



By the last diagram, Hurewicz Theorem, Universal Coefficient Theorem, and de Rham Theorem,

$$\begin{aligned}\pi_1(M) = \langle a, b | abab^{-1} \rangle \quad &\Longrightarrow \quad H_1(M;\mathbb{Z}) \approx \mathrm{Abel}(\pi_1(M)) \approx \mathbb{Z}_2 \oplus \mathbb{Z} \\ &\Longrightarrow \quad H_1(M;\mathbb{R}) \approx H_1(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \approx \mathbb{R} \quad \Longrightarrow \quad H^1_{\mathrm{deR}}(M) \approx H_1(M;\mathbb{R})^* \approx \mathbb{R}.\end{aligned}$$

5. *Let M be a smooth manifold admitting an open cover $\{U_i\}_{i=1,\dots,m}$ such that every intersection $U_{i_1} \cap \ldots \cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^n$. Show that*

(a) if $m=2$, $H^p_{deR}(M) = 0$ for all $p \neq 0$;

(b) if $m \geq 2$, $H^p_{deR}(M) = 0$ for all $p \geq m-1$.

(a) By Mayer-Vietoris, there is a long exact

$$\begin{aligned}0 \longrightarrow H^0_{\mathrm{deR}}(M) &\longrightarrow H^0_{\mathrm{deR}}(U_1) \oplus H^0_{\mathrm{deR}}(U_2) \longrightarrow H^0_{\mathrm{deR}}(U_1 \cap U_2) \\ \xrightarrow{\delta_0} H^1_{\mathrm{deR}}(M) &\longrightarrow H^1_{\mathrm{deR}}(U_1) \oplus H^1_{\mathrm{deR}}(U_2) \longrightarrow H^1_{\mathrm{deR}}(U_1 \cap U_2) \\ \vdots \\ \xrightarrow{\delta_p} H^{p+1}_{\mathrm{deR}}(M) &\longrightarrow H^{p+1}_{\mathrm{deR}}(U_1) \oplus H^{p+1}_{\mathrm{deR}}(U_2) \longrightarrow H^{p+1}_{\mathrm{deR}}(U_1 \cap U_2)\end{aligned}$$

Since $H^p_{\mathrm{deR}}(U_1 \cap U_2), H^{p+1}_{\mathrm{deR}}(U_1), H^{p+1}_{\mathrm{deR}}(U_2) = 0$ for all $p \geq 1$, $H^{p+1}_{\mathrm{deR}}(M) = 0$ for all $p \geq 1$. If $U_1 \cap U_2 = \emptyset$, then this statement applies for $p=0$ as well. If $U_1 \cap U_2 \neq \emptyset$, M is connected, and the sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \xrightarrow{\delta_0} H^1_{\mathrm{deR}}(M) \longrightarrow 0$$

is exact. So, $\delta_0 = 0$ and $H^1_{\mathrm{deR}}(M) = 0$.

(b) Suppose the statement holds for some $m \geq 2$ (part (a) is the $m=2$ case); we use Mayer-Vietoris to show that it holds for $m+1$. Let

$$U = U_1 \cup U_2 \cup \ldots \cup U_m, \qquad V = U_{m+1}.$$

By MV, the sequence

$$H^p_{\mathrm{deR}}(U \cap V) \xrightarrow{\delta_p} H^{p+1}_{\mathrm{deR}}(M) \longrightarrow H^{p+1}_{\mathrm{deR}}(U) \oplus H^{p+1}_{\mathrm{deR}}(V)$$

is exact. By the inductive assumption, $H^p(U), H^p(U \cap V) = 0$ for all $p \geq m-1$. Since $H^p_{\mathrm{deR}}(V) = 0$ for $p \geq 1$, the outer terms of the above exact sequence vanish if $p \geq m-1$. Thus, $H^{p+1}_{\mathrm{deR}}(M) = 0$ if $p+1 \geq (m+1)-1$, as needed for the inductive step.

6. (a) *Explain why* $\mathbb{R}P^2\times\mathbb{R}P^4$ *is not orientable.*
(b) *Describe the orientable double cover* M *of* $\mathbb{R}P^2\times\mathbb{R}P^4$.
(c) *Determine the de Rham cohomology of* M.

(a) If M and N are smooth nonempty manifolds, $M\times N$ is orientable if and only if M and N are orientable; see MT06 #5. The even-dimensional projective spaces, $\mathbb{R}P^2$ and $\mathbb{R}P^4$, are not orientable.

(b) The universal cover of $\mathbb{R}P^2\times\mathbb{R}P^4$ is $\tilde{M}=S^2\times S^4$ (because the latter is connected and simply connected and admits a covering map to the former). The group of deck transformations is

$$G=\{\mathrm{id}\times\mathrm{id}, a_1\times\mathrm{id}, \mathrm{id}\times a_2, a_1\times a_2\}\approx\pi_1(M)=\mathbb{Z}_2\oplus\mathbb{Z}_2\,,$$

where $a_1\colon S^2\longrightarrow S^2$ and $a_2\colon S^4\longrightarrow S^4$ are the antipodal maps. The orientable double cover M is the quotient of $\tilde{M}$ by a subgroup of G of index 2 and thus of order 2. There are three such subgroups. The quotients of $\tilde{M}$ by $\{\mathrm{id}\times\mathrm{id}, a_1\times\mathrm{id}\}$ and $\{\mathrm{id}\times\mathrm{id}, \mathrm{id}\times a_2\}$ are $\mathbb{R}P^2\times S^4$ and $S^2\times\mathbb{R}P^4$; these are non-orientable manifolds, since one of the components in each product is non-orientable. Thus,

$$M=\tilde{M}\big/\{\mathrm{id}\times\mathrm{id}, a_1\times a_2\}\equiv S^2\times S^4,\quad (x,y)\sim(-x,-y).$$

(c) Since M is a 6-manifold, $H^k_{\mathrm{deR}}(M)=0$ unless $0\le k\le 6$. Since M is compact connected and orientable, $H^0_{\mathrm{deR}}(M), H^6_{\mathrm{deR}}(M)\approx\mathbb{R}$. By PS7 #5,

$$H^k_{\mathrm{deR}}(M)\approx H^k_{\mathrm{deR}}(\tilde{M})^{\mathbb{Z}_2}\equiv\big\{[\tilde{\alpha}]\in H^k_{\mathrm{deR}}(\tilde{M})\colon\ \{a_1\times a_2\}^*[\tilde{\alpha}]=[\tilde{\alpha}]\big\}.$$

By Kunneth's formula, the homomorphism

$$\bigoplus_{p+q=k} H^p_{\mathrm{deR}}(S^2)\otimes H^q_{\mathrm{deR}}(S^4)\longrightarrow H^k_{\mathrm{deR}}(S^2\times S^4),\qquad [\beta]\otimes[\gamma]\longrightarrow\big[\pi_1^*\beta\wedge\pi_2^*\gamma\big],$$

is an isomorphism. In particular,

$$H^1_{\mathrm{deR}}(\tilde{M}), H^3_{\mathrm{deR}}(\tilde{M}), H^5_{\mathrm{deR}}(\tilde{M})=0\qquad\Longrightarrow\qquad H^1_{\mathrm{deR}}(M), H^3_{\mathrm{deR}}(M), H^5_{\mathrm{deR}}(M)=0.$$

On the other hand, let $[\omega_1]$ and $[\omega_2]$ be the generators of $H^2_{\mathrm{deR}}(S^2)\approx\mathbb{R}$ and $H^4_{\mathrm{deR}}(S^4)\approx\mathbb{R}$, respectively. By the solution to PS6 #6a, $a_1^*[\omega_1]=(-1)^{2+1}[\omega_1]$ and so

$$\{a_1\times a_2\}^*\pi_1^*[\omega_1]=\{\pi_1\circ a_1\times a_2\}^*[\omega_1]=\{a_1\circ\pi_1\}^*[\omega_1]=\pi_1^*a_1^*[\omega_1]=-\pi_1^*[\omega_1].$$

Similarly, $\{a_1\times a_2\}^*\pi_2^*[\omega_2]=-\pi_2^*[\omega_2]$. Thus,

$$H^2_{\mathrm{deR}}(M)\approx H^2_{\mathrm{deR}}(\tilde{M})^{\mathbb{Z}_2}=0,\qquad H^4_{\mathrm{deR}}(M)\approx H^4_{\mathrm{deR}}(\tilde{M})^{\mathbb{Z}_2}=0.$$

Part III (*choose 1 problem from 7 and 8*)

7. *Let $V, W \longrightarrow S^1$ be smooth real vector bundles. Show that at least one of the vector bundles*

$$V, W, V \oplus W \longrightarrow S^1$$

is orientable.

This is equivalent to showing that at least one of the real line bundles

$$\Lambda^{\mathrm{top}} V, \Lambda^{\mathrm{top}} W, \Lambda^{\mathrm{top}}(V \oplus W) \approx \Lambda^{\mathrm{top}} V \otimes \Lambda^{\mathrm{top}} W \longrightarrow S^1$$

is trivial; see Lemma 15.1 in *Lecture Notes*. The set of isomorphism classes of real line bundles with the tensor product is an abelian group isomorphic to $\check{H}^1(S^1; \mathbb{Z}_2)$, with the trivial line bundle corresponding to $0 \in \check{H}^1(S^1; \mathbb{Z}_2)$; see PS9 #2a. The latter group is isomorphic to

$$H^1(S^1; \mathbb{Z}_2) \approx \mathrm{Hom}\big(H_1(S^1; \mathbb{Z}_2), \mathbb{Z}_2\big) \approx \mathrm{Hom}\big(H_1(S^1; \mathbb{Z}), \mathbb{Z}_2\big) \approx \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_2) \approx \mathbb{Z}_2\,,$$

since $H_1(S^1; \mathbb{Z}) \approx \mathrm{Abel}(\pi_1(S^1))$. Thus, for any $v, w \in \check{H}^1(S^1; \mathbb{Z}_2)$, at least one of the three elements $v, w, v+w$ is zero.

Here is a direct argument. Let $\{U_i\}_{i=1,2,\ldots,k}$, with $k \geq 4$, be a cover of S^1 by open intervals such that $U_i \cap U_j = \emptyset$ unless $i=j$ or $i \equiv j \pm 1 \mod n$,

$$h_i^V : V|_{U_i} \longrightarrow U_i \times \mathbb{R}^l \qquad \text{and} \qquad h_i^W : W|_{U_i} \longrightarrow U_i \times \mathbb{R}^m$$

trivializations of V and W, and

$$g_{ij}^V : U_i \cap U_j \longrightarrow \mathrm{GL}_l \mathbb{R} \qquad \text{and} \qquad g_{ij}^W : U_i \cap U_j \longrightarrow \mathrm{GL}_m \mathbb{R}$$

the corresponding transition data. The maps

$$g_{i,j}^{V \oplus W} = g_{i,j}^V \oplus g_{i,j}^W :: U_i \cap U_j \longrightarrow \mathrm{GL}_{l+m} \mathbb{R}$$

are then transition data for $V \oplus W$. By our assumptions, $U_i \cap U_j$ is a connected interval and thus $\det g_{ij}$ does not change sign on $U_i \cap U_j$. By negating the first component of h_{i+1}^V and h_{i+1}^W if necessary, we can assume that

$$\det g_{i,i+1}^V, \det g_{i,i+1}^W > 0 \qquad \forall\, i = 1, 2, \ldots, n-1.$$

If $\det g_{n,1}^V > 0$, then V is orientable; see Lemma 15.1 in *Lecture Notes*. If $\det g_{n,1}^V, \det g_{n,1}^W < 0$, then

$$\det g_{i,j}^{V \oplus W} = \det g_{i,j}^V \cdot \det g_{i,j}^W > 0 \;\; \forall\, i,j = 1, 2, \ldots, n.$$

So, if $V, W \longrightarrow S^1$ are not orientable, then $V \oplus W \longrightarrow S^1$ is orientable.

8. *Let $\pi\colon V\longrightarrow M$ be a smooth vector bundle. A* connection *in V is an $\mathbb{R}$-linear map*

$$\nabla\colon\Gamma(M;V)\longrightarrow\Gamma(M;T^*M\otimes V)\quad\text{s.t.}\quad\nabla(fs)=\mathrm{d}f\otimes s+f\nabla s\quad\forall\,f\in C^\infty(M),\ s\in\Gamma(M;V).$$

(a) *Show that ∇ is a first-order differential operator.*

(b) *What is the symbol of ∇?*

(c) *Under what conditions (on M and/or V) is ∇ elliptic?*

(a) First, ∇ is a local operator, i.e. the value of ∇s at a point $p\in M$ depends only on the restriction of s to any neighborhood U of p. If f is a smooth function on M supported in U such that $f(p)=1$, then

$$\nabla s\big|_p=\nabla\big(fs)\big|_p-\mathrm{d}_pf\otimes s(p),$$

by the product-rule condition. The right-hand side of this expression depends only on $s|_U$.

Let $\varphi\equiv(x_1,\ldots,x_n):U\longrightarrow\mathbb{R}^n$ be a chart on M. An isomorphism $\psi: V|_U\longrightarrow\mathbb{R}^n\times\mathbb{R}^k$ of vector bundles covering φ induces such an isomorphism for the bundle $T^*M\otimes V$:

$$\Psi\colon T^*M\otimes V\big|_U\longrightarrow\mathbb{R}^n\times(\mathbb{R}^k)^n\,,\qquad\eta\longrightarrow\bigg(p,\eta\bigg(\frac{\partial}{\partial x_1}\bigg|_p\bigg),\ldots,\eta\bigg(\frac{\partial}{\partial x_n}\bigg|_p\bigg)\bigg)\quad\forall\,\eta\in T_p^*M\otimes V_p,\,p\in U.$$

For each $i=1,2,\ldots,k$, define

$$s_i\in\Gamma(U;V)\qquad\text{by}\qquad s_i(p)=\psi^{-1}\big(\varphi_i(p),e_i\big)\ \ \forall\,p\in U,$$

where $e_i\in\mathbb{R}^k$ is the i-th standard coordinate vector. The homomorphisms

$$\tilde\psi\colon C^\infty(\mathbb{R}^n;\mathbb{R}^k)\longrightarrow\Gamma(U;V)\qquad\big\{\tilde\psi(f_1,\ldots,f_k)\big\}(p)=\sum_{i=1}^{i=k}f_i\big(\phi(p)\big)s_i(p),$$

$$\tilde\Psi\colon C^\infty(\mathbb{R}^n;(\mathbb{R}^k)^n)\longrightarrow\Gamma(U;T^*M\otimes V)\quad\big\{\tilde\Psi((f_{j,l})_{j=1,\ldots,n;l=1,2,\ldots,k})\big\}(p)=\sum_{j=1}^{j=n}\sum_{j=1}^{l=k}f_{j,l}\big(\phi(p)\big)\mathrm{d}_px_j\otimes s_l(p),$$

are then isomorphisms. By definition of ∇, there exist

$$\theta^i_{j,l}\in C^\infty(U)\qquad\text{s.t.}\qquad\nabla s_i\big|_p=\sum_{j=1}^{j=n}\sum_{l=1}^{l=k}\theta^i_{j,l}(p)\mathrm{d}_px_j\otimes s_l(p)\qquad\forall\,p\in U.$$

By the product-rule condition on ∇,

$$\begin{aligned}\nabla\big(\tilde\psi(f_1,\ldots,f_k))\big|_p&=\sum_{i=1}^{i=k}\mathrm{d}_p(f_i\circ\phi)\otimes s_i(p)+\sum_{i=1}^{i=k}\sum_{j=1}^{j=n}\sum_{j=1}^{l=k}\theta^i_{j,l}(p)f_i\big(\phi(p)\big)\mathrm{d}_px_j\otimes s_l(p)\\&=\sum_{j=1}^{j=n}\sum_{j=1}^{l=k}\bigg(\frac{\partial(f_l\circ\phi)}{\partial x_j}\bigg|_p+\sum_{i=1}^{i=k}\theta^i_{j,l}(p)f_i\big(\phi(p)\big)\bigg)\mathrm{d}_px_j\otimes s_l(p).\end{aligned}$$

Thus, the operator $\nabla|_U$ in the local coordinates (φ,ψ,Ψ) on $(U,V|_U,T^*M\otimes V|_U)$ is given by

$$\tilde{\Psi}^{-1}\circ\nabla\circ\tilde{\psi}\colon C^\infty(\mathbb{R}^n;\mathbb{R}^k)\longrightarrow C^\infty(\mathbb{R}^n;(\mathbb{R}^k)^n),$$
$$(f_i)_{i=1,2,\ldots,k}\longrightarrow\left(\frac{\partial f_l}{\partial x_j}+\sum_{i=1}^{i=k}\theta^i_{j,l}\circ\varphi^{-1}\cdot f_i\right)_{j=1,2,\ldots,n;l=1,2,\ldots,k}.$$

Since this is a first-order differential operator on functions on $\mathbb{R}^n$, ∇ is a first-order differential operator on vector-bundle sections over M.

(b) Let $p\in M$, $\alpha\in T^*_pM$, $f\in C^\infty(M)$ be such that $f(p)=0$ and $\mathrm{d}_pf=\alpha$, and $s\in\Gamma(M;V)$. By the product-rule condition on ∇,

$$\nabla(fs)\big|_p=\mathrm{d}_pf\otimes s+f(p)\otimes(\nabla s)|_p=\alpha\otimes s(p).$$

Thus, the symbol of ∇ is given by

$$\sigma_\nabla\colon T^*M\longrightarrow\mathrm{Hom}(V,T^*M\otimes V),\qquad\big\{\sigma_\nabla(\alpha)\big\}(v)=\alpha\otimes v\quad\forall\,\alpha\in T^*_pM,\ v\in V_p,\ p\in M.$$

(c) The operator ∇ is elliptic if and only if the homomorphism

$$\sigma_\nabla(\alpha)\colon V_p\longrightarrow T^*_pM\otimes V_p$$

is an isomorphism for all $\alpha\in T^*_pM-0$ and $p\in M$. If this is the case (and V has positive rank), then

$$\mathrm{rk}\,V=\mathrm{rk}\,T^*M\otimes V\qquad\Longrightarrow\qquad\dim M=1.$$

Conversely, if $\dim M=1$, $\sigma_\nabla(\alpha)$ is an isomorphism for all $\alpha\in T^*_pM-0$ and $p\in M$. Thus, ∇ is elliptic if and only if $\dim M=1$ (or $\mathrm{rk}\,V=0$).

Bonus Problem

Let $\gamma\longrightarrow\mathbb{C}P^1$ be the tautological (complex) line bundle. Compute

$$\int_{\mathbb{C}P^1}c_1(\gamma^*),$$

where $\mathbb{C}P^1$ has its canonical orientation as a complex manifold and $c_1(\gamma^*)$ is the image of γ^* under the composition

$$\check{H}^1(\mathbb{C}P^1;\mathfrak{C}^\infty(\mathbb{C}^*))\longrightarrow\check{H}^2(\mathbb{C}P^1;\underline{\mathbb{Z}})\longrightarrow\check{H}^2(\mathbb{C}P^1;\underline{\mathbb{C}})\longrightarrow H^2_{\mathrm{deR}}(\mathbb{C}P^1;\mathbb{C}),$$

$\mathfrak{C}^\infty(\mathbb{C}^*)\longrightarrow\mathbb{C}P^1$ is the sheaf of germs of $\mathbb{C}^*$-valued smooth functions, the first homomorphism is induced by the exponential short exact sequence of sheaves, and the last homomorphism is the de Rham isomorphism (using $\mathbb{C}$ instead of $\mathbb{R}$-coefficients simplifies the computation).

We find a representative $\omega \in E^2(\mathbb{C}P^1)$ for $c_1(\gamma^*) \in H^2_{\mathrm{deR}}(\mathbb{C}P^1)$ by unwinding the definitions. Let

$$U_0 = \big\{[X_0, X_1] \in \mathbb{C}P^1 \colon X_0 \neq 0\big\}, \qquad U_1 = \big\{[X_0, X_1] \in \mathbb{C}P^1 \colon X_1 \neq 0\big\},$$

be the usual open subsets isomorphic to $\mathbb{C}$. The bundle maps

$$\gamma|_{U_0} \xrightarrow{h_0} U_0 \times \mathbb{C}, \qquad (\ell, c_0, c_1) \longrightarrow c_0,$$
$$\gamma|_{U_1} \xrightarrow{h_1} U_1 \times \mathbb{C}, \qquad (\ell, c_0, c_1) \longrightarrow c_1,$$

are the trivializations of γ with the overlap map

$$h_0 \circ h_1^{-1} \colon U_0 \cap U_1 \times \mathbb{C} \longrightarrow U_0 \cap U_1 \times \mathbb{C}, \qquad \big([X_0, X_1], c_1\big) \longrightarrow \big([X_0, X_1], c_0 = (X_0/X_1)c_1\big).$$

Thus, the corresponding transition data for γ is given by

$$U_0 \cap U_1 \longrightarrow \mathbb{C}^*, \qquad [X_0, X_1] \longrightarrow \frac{X_0}{X_1}.$$

The induced transition data for γ^* is described by

$$g \in \check{Z}^1\big(\{U_0, U_1\}; \mathfrak{C}^\infty(\mathbb{C}^*)\big), \qquad g_{01}\big([X_0, X_1]\big) = \frac{X_1}{X_0},$$

with $g_{10} = 1/g_{01}$, $g_{00}, g_{11} \equiv 1$ (as functions on $U_0 \cap U_0$ and $U_1 \cap U_1$). It determines elements

$$[g] \in \check{H}^1(\{U_0, U_1\}; \mathfrak{C}^\infty(\mathbb{C}^*)), \qquad \big[[g]\big] \in \check{H}^1(\mathbb{C}P^1; \mathfrak{C}^\infty(\mathbb{C}^*)).$$

The short exact sequence of sheaves inducing the first arrow in the statement of the problem is

$$\underline{0} \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathfrak{C}^\infty(\mathbb{C}) \xrightarrow{\exp} \mathfrak{C}^\infty(\mathbb{C}^*) \longrightarrow \underline{0}$$
$$f \longrightarrow e^{2\pi \mathrm{i} f}$$

In order to find the image of γ^* (or equivalently of $\big[[g]\big]$) in $\check{H}^2(\mathbb{C}P^1; \underline{\mathbb{Z}})$, apply the Snake Lemma construction to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{C}^2\big(\mathfrak{U}'; \underline{\mathbb{Z}}\big) & \xrightarrow{i_2} & \check{C}^2\big(\mathfrak{U}'; \mathfrak{C}^\infty(\mathbb{C})\big) & \xrightarrow{\exp_2} & \check{C}^2\big(\mathfrak{U}'; \mathfrak{C}^\infty(\mathbb{C}^*)\big) & \longrightarrow & 0 \\
 & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \\
0 & \longrightarrow & \check{C}^1\big(\mathfrak{U}'; \underline{\mathbb{Z}}\big) & \xrightarrow{i_1} & \check{C}^1\big(\mathfrak{U}'; \mathfrak{C}^\infty(\mathbb{C})\big) & \xrightarrow{\exp_1} & \check{C}^1\big(\mathfrak{U}'; \mathfrak{C}^\infty(\mathbb{C}^*)\big) & \longrightarrow & 0
\end{array}$$

for a refinement $\mathfrak{U}'$ of $\{U_0, U_1\}$. Since $g_{01} \in C^\infty(U_0 \cap U_1; \mathbb{C}^*)$ does not have a well-defined logarithm (g_{01} corresponds to $z \longrightarrow z$ on $\mathbb{C}^*$ under the usual identification of U_0 with $\mathbb{C}$),

$$g \in \check{Z}^1(\{U_0, U_1\}; \mathfrak{C}^\infty(\mathbb{C}^*)) \subset \check{C}^1(\{U_0, U_1\}; \mathfrak{C}^\infty(\mathbb{C}^*))$$

is not in the image of the homomorphism $\exp_1$. Thus, we need to take a proper refinement $\mathfrak{U}'$ of $\{U_0, U_1\}$ and choose a refining map μ. Let

$$U'_0 = \big\{[X_0, X_1] \in \mathbb{C}P^1 \colon |X_0| > |X_1|\big\},$$
$$U'_+ = U_1 - \big\{[r, 1] \in U_1 \colon r \in [1, \infty)\big\},$$
$$U'_- = U_1 - \big\{[r, 1] \in U_1 \colon r \in (-\infty, -1]\big\},$$

$$\mathfrak{U}' = \big\{U'_0, U'_+, U'_-\big\},$$
$$\mu \colon (0, +, -) \longrightarrow (0, 1, 1).$$

Thus, $(\mu^*g)_{0\pm}=g_{01}|_{U_0'\cap U_\pm'}$, $(\mu^*g)_{+-}\equiv 1$, and $\mu^*g=\exp_1(\tilde g)$, with $\tilde g\in\check C^1(\mathfrak U';\mathfrak C^\infty(\mathbb C))$ described by

$$\tilde g_{0\pm}\big([X_0,X_1]\big)=\frac{1}{2\pi\mathfrak i}\ln\left(\frac{X_1}{X_0}\right),\quad \operatorname{Im}\tilde g_{0+}\in(0,1),\quad \operatorname{Im}\tilde g_{0-}\in(-1/2,1/2),$$
$$\tilde g_{\pm 0}=-\tilde g_{0\pm},\qquad \tilde g_{00},\tilde g_{\pm\pm}\equiv 0.$$

By the proof of the Snake Lemma, there exists $h\in\check Z^2(\mathfrak U';\mathbb Z)$ such that $i_2(h)=\delta_1(\tilde g)$. By the Snake Lemma construction, the image of $\big[[g]\big]\in\check H^1(\mathbb CP^1;\mathfrak C^\infty(\mathbb C^*))$ under the boundary homomorphism in the corresponding long exact sequence of modules is $\big[[h]\big]\in\check H^2(\mathbb CP^1;\underline{\mathbb Z})$.

Via the inclusion $\mathbb Z\longrightarrow\mathbb C$, $\big[[h]\big]\in\check H^2(\mathbb CP^1;\underline{\mathbb C})$. It remains to compute its image in $H^2_{\mathrm{deR}}(\mathbb CP^1;\mathbb C)$ under the de Rham isomorphism. In this case, this involves going through two boundary homomorphisms. The first arises from the Snake Lemma construction for the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check C^2\big(\mathfrak U';\underline{\mathbb C}\big) & \longrightarrow & \check C^2\big(\mathfrak U';\mathfrak C^\infty(\mathbb C)\big) & \xrightarrow{\mathrm d} & \check C^2\big(\mathfrak U';\mathcal Z^1\big) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & \check C^1\big(\mathfrak U';\underline{\mathbb C}\big) & \longrightarrow & \check C^1\big(\mathfrak U';\mathfrak C^\infty(\mathbb C)\big) & \xrightarrow{\mathrm d} & \check C^1\big(\mathfrak U';\mathcal Z^1\big) & \longrightarrow & 0
\end{array}$$

where $\mathcal Z_1\subset\mathcal E^1$ is the sheaf of germs of closed $\mathbb C$-valued 1-forms. By the previous paragraph, the construction of the Snake Lemma maps the element

$$\alpha\in\check Z^1\big(\mathfrak U';\mathcal Z^1\big)\subset\check C^1\big(\mathfrak U';\mathcal Z^1\big),\qquad \alpha_{**}\equiv\mathrm d\tilde g_{**}\,,$$

to h. Let $\beta\in\check Z^1(\{U_0,U_1\};\mathcal Z^1)$ be given by

$$\beta_{01}\in E^1(U_0\cap U_1),\qquad \beta_{01}(z)=\frac{1}{2\pi\mathfrak i}\frac{\mathrm dz}{z}\,,\quad\text{where}\quad z=\frac{X_1}{X_0}\,.$$

Since $\mu^*\beta=\alpha$, the boundary homomorphism for the short exact sequence

$$\underline 0\longrightarrow\underline{\mathbb C}\longrightarrow\mathfrak C^\infty(\mathbb C)\xrightarrow{\mathrm d}\mathcal Z^1\longrightarrow\underline 0$$

takes $\big[[\beta]\big]\in\check H^1(\mathbb CP^1;\mathcal Z^1)$ to $\big[[h]\big]\in\check H^2(\mathbb CP^1;\underline{\mathbb C})$.

Finally, we need to find a preimage $\omega\in\check H^0(\mathbb CP^1;\mathcal Z^2)=\mathcal Z^2(\mathbb CP^1)=E^2(\mathbb CP^1)$ of $\big[[\beta]\big]$ under the boundary homomorphism for the short exact sequence

$$\underline 0\longrightarrow\mathcal Z^1\longrightarrow\mathcal E^1\xrightarrow{\mathrm d}\mathcal Z^2\longrightarrow\underline 0$$

of sheaves over $\mathbb CP^1$. This involves applying the Snake Lemma to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check C^1\big(\{U_0,U_1\};\mathcal Z^1\big) & \longrightarrow & \check C^1\big(\{U_0,U_1\};\mathcal E^1\big) & \xrightarrow{\mathrm d} & \check C^1\big(\{U_0,U_1\};\mathcal Z^2\big) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & \check C^0\big(\{U_0,U_1\};\mathcal Z^1\big) & \longrightarrow & \check C^0\big(\{U_0,U_1\};\mathcal E^1\big) & \xrightarrow{\mathrm d} & \check C^0\big(\{U_0,U_1\};\mathcal Z^2\big) & \longrightarrow & 0
\end{array}$$

Let $\phi\in C^\infty(\mathbb{C}P^1)$ be such that $\phi([X_0,X_1])=1$ if $|X_0|<|X_1|$ and $\phi([X_0,X_1])=0$ if $|X_0|>2|X_1|$. Thus,

$$\eta\in\check{C}^0\big(\{U_0,U_1\};\mathcal{E}^1\big), \qquad \eta_0=-\phi\beta_{01}\in E^1(U_0), \quad \eta_1=(1-\phi)\beta_{01}\in E^1(U_1),$$

is well-defined. Since $\mathrm{d}\eta_0=\mathrm{d}\eta_1$ on $U_0\cap U_1$, there is a unique 2-form $\omega\in E^2(\mathbb{C}P^1)$ such $\omega|_{U_i}=\mathrm{d}\eta_i$ on U_i. Since

$$(\delta\eta)_{01}\equiv\eta_1\big|_{U_0\cap U_1}-\eta_0\big|_{U_0\cap U_1}=\beta_{01},$$

$\omega\in\check{Z}^0(\{U_0,U_1\};\mathcal{Z}^2)$ is mapped to β by the Snake Lemma. Thus,

$$[\omega]\in H^2_{\mathrm{deR}}(\mathbb{C}P^1)\equiv\frac{E^2(\mathbb{C}P^1)}{\mathrm{d}\,E^1(\mathbb{C}P^1)}=\frac{\check{H}^0(\mathbb{C}P^1;\mathcal{Z}^2)}{\mathrm{d}\,\check{H}^0(\mathbb{C}P^1;\mathcal{E}^1)}$$

corresponds to $\big[[\beta]\big]\in\check{H}^1(\mathbb{C}P^1;\mathcal{Z}^1)$ and $\big[[h]\big]\in\check{H}^2(\mathbb{C}P^1;\underline{\mathbb{C}})$ under the isomorphisms factoring the de Rham isomorphism and to the image of γ^*.

Using Stokes' Theorem, we now obtain

$$\int_{\mathbb{C}P^1}c_1(\gamma^*)=\int_{\mathbb{C}P^1}\omega=\int_{U_0'}\omega=-\frac{1}{2\pi\mathfrak{i}}\int_{\bar{U}_0'}\mathrm{d}\bigg(\phi\frac{\mathrm{d}z}{z}\bigg)=-\frac{1}{2\pi\mathfrak{i}}\oint_{S^1}\phi\frac{\mathrm{d}z}{z}=-\frac{1}{2\pi\mathfrak{i}}\oint_{S^1}\frac{\mathrm{d}z}{z}=-1.$$

Remark: With the "correct" definition of c_1, the answer should be 1. Thus, $c_1(L)$ should really be defined to be the negative of the image of L under the above composition of homomorphism. In the note for PS9 #2, I repeated a mistake from G&H. Their proof that their incorrect definition of $c_1(L)$ is the correct one (i.e. satisfies 2. in Proposition on p141) contains an error. The relation between θ_α and θ_β *worked out in Section 5 Chapter 0* (the last displayed expression on p72) is the opposite of the third equation in the proof on p141; this would change the sign in the relation. The seemingly natural isomorphism between the Čech and de Rham cohomologies in G&H and Warner is actually not the natural one from a certain perspective. In particular, there is a separate isomorphism on each level, i.e. between $\check{H}^p$ and H^p_{deR}. They can be unified by forming a double complex, $\check{C}^p(\mathfrak{U};\mathcal{E}^q)$, with the differential $D_{p,q}=\delta+(-1)^p d$, where δ and d are the usual Čech and de Rham differentials; the sign is needed to insure that $D^2=0$. The Čech and de Rham complexes then inject into this double complex, inducing isomorphisms in cohomology. The induced isomorphism between $\check{H}^p$ and H^p_{deR} is then $(-1)^{p(p+1)/2}$ times the isomorphism in G&H, correcting the sign error in the definition of $c_1(L)$ in the de Rham cohomology.