MAT 531: Topology&Geometry, II
Spring 2010

Final Exam Solutions

Part I (*choose 2 problems from 1,2, and 3*)

1. *Let M and N be compact oriented connected n-manifolds and $f\colon M \longrightarrow N$ a smooth map. Show that there exists a unique number* $(\deg\, f)\in\mathbb{R}$ *such that*

$$\int_M f^*\omega = (\deg\, f)\cdot\int_N \omega \qquad \forall\ \omega\in E^n(N).$$

Note: there are two parts to this problem.

Both integrals are well-defined because M and N are compact and oriented. Since N is a compact oriented connected n-manifold, $H^n_{\mathrm{deR}}(M)\approx\mathbb{R}$ is one-dimensional and generated by $[\omega_N]$, where $\omega_N\in E^n(M)$ is any top form such that $\int_N\omega_N\neq 0$. By the last condition, there exists a unique number $(\deg\, f)\in\mathbb{R}$ such that

$$\int_M f^*\omega_N = (\deg\, f)\cdot\int_N \omega_N\,.$$

Since $H^n_{\mathrm{deR}}(M)$ is generated by $[\omega_N]$, for every $\omega\in E^n(M)$ there exists c such that

$$[\omega]=c[\omega_N]=[c\,\omega_N]\in H^n_{\mathrm{deR}}(N) \qquad\Longrightarrow\qquad [f^*\omega]=[f^*(c\omega_N)]=[cf^*\omega_N]\in H^n_{\mathrm{deR}}(M). \tag{1}$$

By the second statement above,

$$\int_M f^*\omega=\int_M cf^*\omega_N=c\int_M f^*\omega_N=c\,(\deg\, f)\cdot\int_N\omega_N.$$

By the first statement in (1),

$$\int_N\omega=\int_N c\,\omega_N=c\int_N\omega_N \qquad\Longrightarrow\qquad \int_M f^*\omega=(\deg\, f)\cdot\int_N\omega.$$

2. *Let X and Y be the vector fields on $\mathbb{R}^3$ given by*

$$X=\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\,, \qquad Y=y\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\,.$$

(a) *Compute* $[X,Y]$.

(b) *Is there a coordinate chart $\varphi=(x_1,x_2,x_3)\colon U\longrightarrow\mathbb{R}^3$ on a neighborhood of the origin in $\mathbb{R}^3$ such that*

$$X|_U=\frac{\partial}{\partial x_1}\,, \qquad Y|_U=\frac{\partial}{\partial x_2}\,?$$

(a) Since the Lie bracket of coordinate vector fields is 0,

$$\begin{aligned}[X,Y] &= \left(X(y)\frac{\partial}{\partial x} + X(z)\frac{\partial}{\partial y} + X(1)\frac{\partial}{\partial z}\right) - \left(Y(1)\frac{\partial}{\partial x} + Y(x)\frac{\partial}{\partial y} + Y(y)\frac{\partial}{\partial z}\right) \\ &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\right) - \left(0 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}.\end{aligned}$$

(b) No, since $\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right] = 0$, while $[X,Y]$ does not vanish identically on any neighborhood of the origin.

Alternatively, since X and Y are linearly independent on some neighborhood W of the origin in $\mathbb{R}^3$ (but not everywhere on $\mathbb{R}^3$), it is sufficient to show that the distribution

$$\mathcal{D} \equiv \{fX + gY \colon f, g \in C^\infty(V)\}$$

is not integrable on any neighborhood V of the origin in U (however, $\mathcal{D}$ being integrable would imply only existence of a chart so that

$$\{fX + gY \colon f, g \in C^\infty(U)\} = \{f\frac{\partial}{\partial x_1} + g\frac{\partial}{\partial x_2} \colon f, g \in C^\infty(U)\},$$

not necessarily yes to the question). By Frobenius Theorem, the distribution $\mathcal{D}$ is integrable on an open subset V of W if and only if for some $f, g \in C^\infty(V)$

$$[X,Y]_V = fX + gY \qquad \Longrightarrow \qquad \begin{cases} f + yg = x \\ xf + zg = 0 \\ yf + g = z \end{cases}$$

The first two equations give $f = xz/(z-yx)$; this is not a smooth function on any neighborhood of the origin. Thus, the distribution $\mathcal{D}$ is not integrable.

3. *Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin.*

(a) *Why is the line bundle $\Lambda^2(TS^2) \longrightarrow S^2$ trivial?*

(b) *Describe an explicit isomorphism $\Lambda^2(TS^2) \longrightarrow S^2 \times \mathbb{R}$ of real line bundles over S^2 (give a formula).*

(a) The line bundle $\Lambda^2(TS^2) \longrightarrow S^2$ is trivial because S^2 is orientable and thus the vector bundle $TS^2 \longrightarrow S^2$ is orientable; the latter is equivalent to the line bundle $\Lambda^2(TS^2) \longrightarrow S^2$ being orientable and thus trivial. Alternatively, the normal bundle NS^2 of $S^2 \subset \mathbb{R}^3$ is trivial and orientable (it has a nowhere zero section given by the unit outward normal $\vec{n}$); since

$$T\mathbb{R}^3|_{S^2} \approx TS^2 \oplus NS^2$$

is orientable, TS^2 is orientable. Another explanation is provided by (b).

(b) The normal bundle NS^2 can be trivialized by the unit normal vector, which at a point $u \in S^2$ is just u:

$$NS^2 \equiv \big\{(u,cu) \in S^2 \times \mathbb{R}^3 \colon c \in \mathbb{R}\big\} \longrightarrow S^2 \times \mathbb{R}, \qquad (u,v) \longrightarrow \langle u,v\rangle\,, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner-product on $\mathbb{R}^3$. This isomorphism can be written without identifying the normal bundle $NS^2 \equiv T\mathbb{R}^3|_{S^2}/TS^2$ with the orthogonal complement of TS^2 in $T\mathbb{R}^3|_{S^2}$:

$$T\mathbb{R}^3|_{S^2}/TS^2 \longrightarrow S^2 \times \mathbb{R}, \qquad v \in T_u\mathbb{R}^3|_{S^2} \longrightarrow \mathrm{d}_u r(v), \tag{3}$$

where $r \colon \mathbb{R}^3 - 0 \longrightarrow \mathbb{R}$ is the distance to the origin. Since this bundle homomorphism vanishes on the subbundle $TS^2 \subset T\mathbb{R}^3|_{S^2}$ (because the function r is constant in S^2), it descends to the quotient bundle NS^2. Since the homomorphism $\mathrm{d}_u r$ is surjective for all $u \in S^2$ and NS^2 is a line bundle, the bundle homomorphism (3) is an isomorphism.

In order to find the required bundle isomorphism, it is thus sufficient to find a bundle isomorphism $\Lambda^2(TS^2) \longrightarrow NS^2$ over S^2. Such a bundle isomorphism is equivalent to a surjective bundle map

$$h\colon TS^2 \oplus TS^2 \longrightarrow NS^2 \qquad \text{s.t.} \qquad h(v,w) = -h(w,v)$$

which is bilinear on each fiber. Such a map is provided by the cross-product of vectors on $\mathbb{R}^3$: $h(v,w) = v \times w$, with $v,w \in T_uS^2 \approx \mathbb{R}^3$. Combining this bundle map with the isomorphism (2), we obtain a required isomorphism of line bundles over S^2:

$$\Lambda^2(TS^2) \equiv \big\{(u, v\wedge w)\colon\ u,v,w \in \mathbb{R}^3,\ |u|=1,\ v,w \perp u\big\} \longrightarrow S^2 \times \mathbb{R}, \quad (u, v\wedge w) = \langle u, v \times w\rangle = \det\big(u\ v\ w\big).$$

Part II (*choose 2 problems from 4,5, and 6*)

4. *Show that there exist a closed 1-form α on $\mathbb{R}P^n$ and a smooth function $f\colon [0,1] \longrightarrow \mathbb{R}P^n$ so that*

$$f(0) = f(1) \qquad \text{and} \qquad \int_{[0,1]} f^*\alpha \neq 0$$

if and only if $n=1$.

If $n=1$, $\mathbb{R}P^n = S^1$; $\alpha = \mathrm{d}\theta$ and $f(t) = \mathrm{e}^{2\pi \mathrm{i} t}$ then give a nonzero integral, 2π. If $n<0$, $\mathbb{R}P^n = \emptyset$; if $n=0$, $\mathbb{R}P^n = \{pt\}$. In either of these two cases, there are no nonzero one-forms.

Suppose $n \geq 2$. In this case, $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$; so $H_1(\mathbb{R}P^n;\mathbb{R}) = 0$ and $H^1_{\mathrm{deR}}(\mathbb{R}P^n) = 0$. Thus, if α is a closed one-form on $\mathbb{R}P^n$, then α is exact, i.e. $\alpha = \mathrm{d}h$ for some $h \in E^0(\mathbb{R}P^n) \equiv C^\infty(\mathbb{R}P^n)$. If $f\colon [0,1] \longrightarrow \mathbb{R}P^n$ is any smooth function such that $f(0) = f(1)$, then

$$\int_{[0,1]} f^*\alpha = \int_{[0,1]} f^*\mathrm{d}h = \int_{[0,1]} \mathrm{d}(f^*h) = \int_{[0,1]} \mathrm{d}(h \circ f) = \int_{\partial[0,1]} h \circ f = h \circ f\Big|_0^1 = h(f(1)) - h(f(0)).$$

Alternatively, a closed one-form α on $\mathbb{R}P^n$ defines a linear functional

$$H_1(\mathbb{R}P^n;\mathbb{R}) \longrightarrow \mathbb{R}, \qquad [\sigma] \longrightarrow \int_\sigma \alpha,$$

while a smooth function $f\colon [0,1] \longrightarrow \mathbb{R}P^n$ with $f(0)=f(1)$ defines an element of $H_1(\mathbb{R}P^n;\mathbb{R})$. Since $H_1(\mathbb{R}P^n;\mathbb{R})=0$,

$$\int_{[0,1]} f^*\alpha = \int_f \alpha$$

must be 0.

5. *Let $M=(S^1\times\mathbb{R}P^n\times\mathbb{R}P^n)/\sim$, where $n\in\mathbb{Z}^+$, $(x,y,z)\sim(-x,z,y)$, and S^1 is viewed as the unit circle in $\mathbb{C}$ (so $x\in\mathbb{C}$). Show that M is not orientable and describe the orientable double cover of M.* *Hint: both parts require some care.*

The finite group $\mathbb{Z}_2$ acts freely on the manifold $\tilde{M}\equiv S^1\times\mathbb{R}P^n\times\mathbb{R}P^n$ by

$$(-1)\cdot(x,y,z)=(-x,z,y).$$

Thus, the quotient $M=\tilde{M}/\mathbb{Z}_2$ is a smooth manifold.

(a) If n is even, $\mathbb{R}P^n$ is not orientable (see PS6 #6). Thus, $\tilde{M}$ is not orientable (see MT06 #5), and neither is M. If n is odd, $\mathbb{R}P^n$ is orientable, and so is $S^1\times\mathbb{R}P^n\times\mathbb{R}P^n$. A volume form on the latter is given by $\Omega=\pi_1^*\mathrm{d}\theta\wedge\pi_2^*\omega\wedge\pi_3^*\omega$, where ω is a volume form on $\mathbb{R}P^n$. The action of $(-1)\in\mathbb{Z}_2$ preserves $\pi_1^*\mathrm{d}\theta$ and interchanges $\pi_2^*\omega$ and $\pi_3^*\omega$; since $n\cdot n$ is odd, it thus sends Ω to $-\Omega$. So, the $\mathbb{Z}_2$-action is not orientation-preserving and thus the quotient is not orientable (see PS6 #6).

(b) If n is odd, $\tilde{M}$ is orientable and thus is the orientable double cover of M. If n is even, M is the quotient of the orientable manifold, $\widetilde{\widetilde{M}}=S^1\times S^n\times S^n$ by the action of the group $G\approx D_4$ (or D_8 depending on one's notation) generated by the diffeomorphisms

$$g,a_2,a_3\colon \widetilde{\widetilde{M}}\longrightarrow\widetilde{\widetilde{M}}, \qquad g(x,y,z)=(-x,z,y), \quad a_2(x,y,z)=(x,-y,z), \quad a_3(x,y,z)=(x,y,-z);$$

since $ga_2=a_3g$, G is generated just by a_2 and g. The subgroup G^+ of the orientation-preserving diffeomorphisms must be of index 2 and is generated by g and a_2a_3 (g is orientation-preserving because n is even and a_2a_3 is orientation-preserving because a_2 and a_3 are both orientation-reversing). The orientable double cover $\tilde{M}'$ of M is $\widetilde{\widetilde{M}}/G^+$. Since

$$a_2a_3\colon S^1\times S^n\times S^n\longrightarrow S^1\times S^n\times S^n, \qquad (x,y,z)\longrightarrow(x,-y,-z),$$

$\tilde{M}'=(S^1\times M')/\sim$, where

$$M'=(S^n\times S^n)/\sim', \quad (y,z)\sim'(-y,-z), \qquad \big(x,[y,z]\big)\sim\big(-x,[z,y]\big).$$

6. *Let $M=\mathbb{R}^3/\sim$, where*

$$(x,y,z)\sim\big(x+k,y+m,z+ky+n\big) \qquad \forall\ (x,y,z)\in\mathbb{R}^3,\ (k,m,n)\in\mathbb{Z}^3.$$

(a) *Show that this is an equivalence relation and M is a connected compact orientable 3-manifold.*

(b) *Determine the de Rham cohomology of M (as graded vector space).*

(a) The relation $\sim$ is reflexive because

$$(x,y,z) \sim \big(x+0, y+0, z+0y+0\big) \qquad \forall\, (x,y,z)\in\mathbb{R}^3.$$

It is symmetric because if $(x',y',z')=(x+k, y+m, z+ky+n)$, then

$$(x,y,z) = \big(x'+(-k), y'+(-m), z'+(-k)y'+(km-n)\big).$$

It is transitive because

$$(x_2,y_2,z_2) = \big(x_1+k_1, y_1+m_1, z_1+k_1y_1+n_1\big), \quad (x_3,y_3,z_3) = \big(x_2+k_2, y_2+m_2, z_2+k_2y_2+n_2\big)$$
$$\Longrightarrow \qquad (x_3,y_3,z_3) = \big(x_1+(k_1+k_2), y_1+(m_1+m_2), z_1+(k_1+k_2)y_1+(n_1+n_2+k_2m_1)\big),$$

By the last statement, $M=\mathbb{R}^3/G$, where G is the group such that $G=\mathbb{Z}^3$ as sets and

$$(k_1,m_1,n_1)\cdot(k_2,m_2,n_2) = (k_1+k_2, m_1+m_2, n_1+n_2+k_2m_1).$$

Since $\mathbb{R}^3$ is connected, so is M. Since M is an S^1-bundle over T^2, it is compact; alternatively, $\mathbb{R}^3 = G[0,1]^3$ and the fundamental domain $[0,1]$ is compact. If $(x',y',z') = g(x,y,z)$ for some $g\in G-id$, then either $|x-x'|\ge 1$, or $|y-y'|\ge 1$, or $|z-z'|\ge 1$. Thus, G acts properly discontinuously on $\mathbb{R}^3$ and the projection $\mathbb{R}^3\longrightarrow M$ is a covering map. The quotient M is a smooth orientable 3-manifold because G acts by orientation-preserving diffeomorphisms; in fact, G preserves the standard volume form of $\mathbb{R}^3$:

$$g^*\big(\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\big) = \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \qquad \forall\, g\in G.$$

(b) Since M is a compact connected orientable 3-manifold,

$$H^p_{\mathrm{deR}}(M) = 0 \ \ \forall\, p<0,\, p>3, \qquad H^0_{\mathrm{deR}}(M)\approx\mathbb{R}, \qquad H^3_{\mathrm{deR}}(M)\approx\mathbb{R}.$$

Since $\mathbb{R}^3\longrightarrow M$ is the universal cover, $\pi_1(M)=G$ and

$$H_1(M;\mathbb{Z}) \approx G/[G,G] \approx \mathbb{Z}^2;$$

the second isomorphism is induced by the projection on the first two components, $\mathbb{Z}^3\longrightarrow\mathbb{Z}^2$. Thus,

$$H^1_{\mathrm{deR}}(M)\approx\mathbb{R}^2, \qquad H^2_{\mathrm{deR}}(M)\approx\mathbb{R}^2\,;$$

the second statement follows from the first by Poincare duality.

Note: $S^1\times M$ *is a symplectic manifold that admits no Kahler structure.* It is a symplectic manifold because the symplectic form

$$\omega = \mathrm{d}\theta\wedge\mathrm{d}x + \mathrm{d}y\wedge\mathrm{d}z \in E^2(S^1\times\mathbb{R}^3)$$

is preserved by the action of G (acting trivially on the first component), i.e. $g^*\omega=\omega$ for all $g\in G$; thus, ω induces a symplectic form $\bar\omega\in E^2(S^1\times M)$. It admits no Kahler structure because $H^1_{\mathrm{deR}}(S^1\times M)\approx\mathbb{R}^3$ by the above and the Kunneth formula, while odd cohomology of a Kahler manifold is even-dimensional by the Hodge diamond (more in MAT 545).

Part III (*choose 1 problem from 7 and 8*)

7. *Let $T^3 = S^1 \times S^1 \times S^1$ be the 3-torus and X the complement of two disjoint closed balls in T^3. The boundary of $\bar{X}$ consists of two copies of S^2, S_0 and S_1, which inherit an orientation from the removed closed balls (this orientation is opposite to the orientation as the boundary of $\bar{X}$). The boundary of $S^2 \times [0,1]$ also consists of two copies of S^2, $S^2 \times 0$ and $S^2 \times 1$, which inherit an orientation from the standard orientation of S^2 (on $S^2 \times 0$ this orientation is opposite to its orientation as boundary of $S^2 \times [0,1]$). Let M be the smooth 3-manifold obtained by joining $\bar{X}$ and $S^2 \times [0,1]$ along their boundaries so that S_i is identified with $S^2 \times i$ by an orientation-preserving diffeomorphism for $i = 0, 1$. Show that M is not orientable and determine its de Rham cohomology (as graded vector space).*

Since M is a 3-manifold, $H^p_{\mathrm{deR}}(M) = 0$ if $p < 0$ or $p > 3$. Since M is connected, $H^0_{\mathrm{deR}}(M) \approx \mathbb{R}$.

We show that M admits no continuously varying orientation, as represented by an ordered basis at each point. If there is such an orientation, we can assume that it agrees with the restriction of the standard orientation on T^3 to $\bar{X}$ (otherwise, take the opposite orientation). If $\{u, v\}$ is an oriented basis for S_i (with S_i oriented as the boundary of the corresponding 3-ball removed from T^3, not of $\bar{X}$) and $\vec{n}_{\mathrm{in}}$ is an *in*ward normal at the same point of S_i, then $\{\vec{n}_{\mathrm{in}}, u, v\}$ is an oriented basis for $\bar{X}$. Since the identification of S_i with $S^2 \times i$ is orientation-preserving (with respect to the standard orientation on $S^2 \times i$) and takes $\vec{n}_{\mathrm{in}}$ to an outward normal $\vec{n}'_{\mathrm{out}}$ for $S^2 \times i \subset S^2 \times [0,1]$, if $\{u', v'\}$ is an oriented basis for $S^2 \times i$, then $\{\vec{n}'_{\mathrm{out}}, u', v'\}$ is an oriented basis for $S^2 \times [0,1]$ in the orientation induced from the orientation of M. Since the orientation on $S^2 \times 1$ is its orientation as a boundary component of $S^2 \times [0,1]$ in the standard orientation of the latter, the restriction of the orientation on M to $S^2 \times [0,1]$ must be the standard orientation on the latter. However, the orientation on $S^2 \times 0$ is its orientation as a boundary component of $S^2 \times [0,1]$ in the opposite orientation of the latter, and thus the restriction of the orientation on M to $S^2 \times [0,1]$ must be the opposite orientation on the latter. This is a contradiction; so M is not orientable. Since M is connected and non-orientable, $H^3_{\mathrm{deR}}(M) = 0$. Mayer-Vietoris provides another approach to showing that M is non-orientable; see below.

Remark: The manifold M being orientable or not depends on how the boundary components are identified. For example, a two-dimensional connected manifold M can be formed by joining $\bar{X} = S^1 \times [0,1]$ with $\bar{Y} = S^1 \times [0,1]$. Join $S^1 \times 1 \subset \bar{X}$ and $S^1 \times 1 \subset \bar{Y}$ in an "obvious" way inside of $\mathbb{R}^3$. If $S^1 \times 0 \subset \bar{X}$ and $S^1 \times 0 \subset \bar{Y}$ are also joined in an "obvious" way inside of $\mathbb{R}^3$, then M is a 2-torus T^2, which is an orientable manifold. However, if $S^1 \times 0 \subset \bar{X}$ and $S^1 \times 0 \subset \bar{Y}$ are joined in a "non-obvious" way, by taking the $\bar{Y}$ end "inside" of the $\bar{X}$-piece, M is a Klein bottle K, which is not orientable.

We use Mayer-Vietoris to first compute $H^*_{\mathrm{deR}}(X)$ and then $H^*_{\mathrm{deR}}(M)$. Since X is a connected and non-compact 3-manifold,

$$H^0_{\mathrm{deR}}(X) \approx \mathbb{R}, \qquad H^3_{\mathrm{deR}}(X) = 0.$$

We first apply Mayer-Vietoris to T^3 with $U = X$ and V the union of two disjoint open balls in T^3

containing the two removed balls:

$$\begin{aligned}
0 \longrightarrow H^0_{\mathrm{deR}}(T^3) \longrightarrow H^0_{\mathrm{deR}}(U) \oplus H^0_{\mathrm{deR}}(V) \longrightarrow H^0_{\mathrm{deR}}(U \cap V) \longrightarrow \\
\longrightarrow H^1_{\mathrm{deR}}(T^3) \longrightarrow H^1_{\mathrm{deR}}(U) \oplus H^1_{\mathrm{deR}}(V) \longrightarrow H^1_{\mathrm{deR}}(U \cap V) \longrightarrow \\
\longrightarrow H^2_{\mathrm{deR}}(T^3) \longrightarrow H^2_{\mathrm{deR}}(U) \oplus H^2_{\mathrm{deR}}(V) \longrightarrow H^2_{\mathrm{deR}}(U \cap V) \longrightarrow \\
\longrightarrow H^3_{\mathrm{deR}}(T^3) \longrightarrow H^3_{\mathrm{deR}}(U) \oplus H^3_{\mathrm{deR}}(V)
\end{aligned}$$

Since $U \cap V$ is homotopy-equivalent to two copies of S^2 and $H^*_{\mathrm{deR}}(T^3)$ is given by the Kunneth formula, plugging in for the known groups we obtain

$$\begin{aligned}
0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \xrightarrow{\delta_0} \mathbb{R}^3 \longrightarrow H^1_{\mathrm{deR}}(X) \oplus 0 \longrightarrow 0 \longrightarrow \\
\longrightarrow \mathbb{R}^3 \longrightarrow H^2_{\mathrm{deR}}(X) \oplus 0 \xrightarrow{g_2} \mathbb{R}^2 \xrightarrow{\delta_2} \mathbb{R} \longrightarrow 0 \oplus 0
\end{aligned} \tag{4}$$

By exactness of the sequence, δ_0 is the zero homomorphism and so

$$H^1_{\mathrm{deR}}(X) \approx \mathbb{R}^3, \qquad H^2_{\mathrm{deR}}(X) \approx \mathbb{R}^4.$$ [1]

We next apply Mayer-Vietoris to M with $U = X$ and V being a small extension of $S^2 \times [0,1]$ into T^3, which is still homotopy-equivalent to S^2:

$$\begin{aligned}
0 \longrightarrow H^0_{\mathrm{deR}}(M) \longrightarrow H^0_{\mathrm{deR}}(U) \oplus H^0_{\mathrm{deR}}(V) \longrightarrow H^0_{\mathrm{deR}}(U \cap V) \longrightarrow \\
\longrightarrow H^1_{\mathrm{deR}}(M) \longrightarrow H^1_{\mathrm{deR}}(U) \oplus H^1_{\mathrm{deR}}(V) \longrightarrow H^1_{\mathrm{deR}}(U \cap V) \longrightarrow \\
\longrightarrow H^2_{\mathrm{deR}}(M) \longrightarrow H^2_{\mathrm{deR}}(U) \oplus H^2_{\mathrm{deR}}(V) \longrightarrow H^2_{\mathrm{deR}}(U \cap V) \longrightarrow \\
\longrightarrow H^3_{\mathrm{deR}}(M) \longrightarrow H^3_{\mathrm{deR}}(U) \oplus H^3_{\mathrm{deR}}(V)
\end{aligned}$$

Since $U \cap V$ is homotopy-equivalent to two copies of S^2 as before, plugging in for the known groups we obtain

$$\begin{aligned}
0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow H^1_{\mathrm{deR}}(M) \longrightarrow \mathbb{R}^3 \oplus 0 \longrightarrow 0 \longrightarrow \\
\longrightarrow H^2_{\mathrm{deR}}(M) \longrightarrow \mathbb{R}^4 \oplus \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow H^3_{\mathrm{deR}}(M) \longrightarrow 0 \oplus 0
\end{aligned}$$

By exactness of the sequence, $H^1_{\mathrm{deR}}(M) \approx \mathbb{R}^4$ $(4 = 1 + 2 + 3 - 2)$ and the sequence

$$0 \longrightarrow H^2_{\mathrm{deR}}(M) \longrightarrow \mathbb{R}^4 \oplus \mathbb{R} \xrightarrow{g_2} \mathbb{R}^2 \longrightarrow H^3_{\mathrm{deR}}(M) \longrightarrow 0 \tag{5}$$

is exact. Since M is non-orientable, $H^3_{\mathrm{deR}}(M) = 0$ and so $H^2_{\mathrm{deR}}(M) \approx \mathbb{R}^3$ $(3 = 5 - 2)$. Alternatively, we show below that $H^3_{\mathrm{deR}}(M) = 0$ (and thus M is non-orientable) by showing that the restrictions of the homomorphism g_2 above to each of the two factors have different one-dimensional images in $\mathbb{R}^2$; this implies that g_2 is onto.

[1] the alternating sum of dimensions in an exact sequence with 0's on the ends must be 0; in this case, $4 = 3 + 2 - 1$

In both MV sequences, $U\cap V$ is the *same* open subset of T^3 and M which is diffeomorphic to two copies of $S^2\times(0,1)$. Orient both spheres, S_0 and S_1, as the boundaries of the oriented balls in T^3 that are removed and choose $\omega_i\in E^2(S_i)$ so that $\int_{S_i}\omega_i=1$. These give rise to closed forms

$$\tilde{\omega}_i = \pi_1^*\omega_i \in E^2\big(S_i\times(0,1)\big)$$

that restrict to ω_i on S_i. The connecting homomorphism δ_2 in (4) is given by

$$\delta_2([\tilde{\omega}_i]) = \big[\mathrm{d}\eta_V\wedge\tilde{\omega}_i\big],$$

where $\eta_V: T^3\longrightarrow[0,1]$ is any smooth function such that $\mathrm{supp}\,\eta_V\subset V$ and $\mathrm{supp}(1-\eta_V)\subset U$ (see PS7 #2). Since T^3 is oriented, a 3-form on T^3 is exact (zero in $H^3_{\mathrm{deR}}(T^3)$) if and only if it integrates to zero. By Stokes Theorem,

$$\int_{T^3}\mathrm{d}\eta_V\wedge\tilde{\omega}_i = \int_U \mathrm{d}(\eta_V\tilde{\omega}_i) = -\int_{S_i}\tilde{\omega}_i = -1.$$

Thus, the kernel of the homomorphism δ_2 in (4) is the linear span of $[\tilde{\omega}_0|_{U\cap V}-\tilde{\omega}_1|_{U\cap V}]$. This is also the image of the homomorphism g_2 in (4) (by exactness) and the image of the restriction of the homomorphism g_2 in (5) to the first factor (this restriction is the *same* homomorphism as in (4)). On the other hand, if V is as in (5), $H^2_{\mathrm{deR}}(V)$ is generated by $[\pi_1^*\omega]$, where $\omega\in E^2(S^2)$ is any 2-form such that $\int_{S^2}\omega=1$. With respect to the orientations of $S^2\times 0$ and $S^2\times 1$ induced by S^2,

$$\int_{S^2\times 0}\omega = 1 \qquad\text{and}\qquad \int_{S^2\times 0}\omega = 1.$$

Since $S^2\times i$ is identified with S_i by an orientation-preserving diffeomorphism, it follows

$$\big[\pi_1^*\omega|_{U\cap V}\big] = \big[\tilde{\omega}_0|_{U\cap V}+\tilde{\omega}_1|_{U\cap V}\big] \in H^2_{\mathrm{deR}}(U\cap V).$$

Thus, the images of the restrictions of g_2 in (5) to the two components of the domain are different one-dimensional subspaces of the image.

We can also compute $H^1_{\mathrm{deR}}(M)$ by computing $\pi_1(M)$ via van Kampen. In this case, we need open subsets U and V of M such that $M=U\cup V$ *and* U, V, and $U\cap V$ are connected. Choose open subset W' and W of T^3 diffeomorphic to $\mathbb{R}^3$ which contain the two 3-balls to be removed and so that $\bar{W}'\subset W$. Let $U=T^3-\bar{W}'\subset M$ and let $V\subset M$ be the open subset W with $S^2\times[0,1]$ attached in place of the two balls. Thus, $U\cap V=W-\bar{W}'$ is homotopy-equivalent to S^2, while the inclusion $U\longrightarrow T^3$ induces an isomorphism $\pi_1(U)\longrightarrow\pi_1(T^3)$ (every loop in T^3 can be taken around $W'\subset\mathbb{R}^3$)[2]; so $\pi_1(U)\approx\mathbb{Z}^3$ and $\pi_1(U\cap V)=0$. Since V is homotopy-equivalent to $S^2\times[0,1]$ with $x\times 0$ identified with $x\times 1$ for some $x\in S^2$, $\pi_1(V)\approx\mathbb{Z}$.[3] Since $\pi_1(U\cap V)=0$, $\pi_1(M)$ is the free product of $\pi_1(U)\approx\mathbb{Z}^3$ and $\pi_1(V)\approx\mathbb{Z}$. Since M is connected,

$$\begin{aligned} H_1(M;\mathbb{Z})\approx \mathrm{Abel}\big(\pi_1(M)\big)\approx\mathbb{Z}^4 \qquad &\Longrightarrow \qquad H_1(M;\mathbb{R})\approx H_1(M;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\approx\mathbb{R}^4\\ &\Longrightarrow \qquad H^1_{\mathrm{deR}}(M)\approx H^1(M;\mathbb{R})\approx H_1(M;\mathbb{R})^*\approx\mathbb{R}^4\,. \end{aligned}$$

[2] apply van Kampen to $U=T^3-\bar{W}'$ and $V=W$; then $U\cap V\approx\mathbb{R}^3-0$ and $\pi_1(V)=0=\pi_1(U\cap V)$

[3] The universal cover of the last space is a string of spheres: $S^2\times\mathbb{Z}$ with (x_+,i) identified with $(x_-,i+1)$ for all $i\in\mathbb{Z}$ and some fixed distinct $x_+,x_-\in S^2$. This string is indeed a cover (just project each $S^2\times i$ to S^2 sending $i\times x_\pm$ to $[x,0]=[x,1]$). Since the string is simply-connected (each S^2 is), it is thus the universal cover. The group of deck transformations (and thus $\pi_1(V)$) is $\mathbb{Z}$, acting on $\mathbb{Z}$ by addition.

Since M is a non-orientable 3-manifold, Poincare duality cannot be used to determine $H^2_{\mathrm{deR}}(M)$ from $H^1_{\mathrm{deR}}(M)$. However, the euler characteristic (alternating sum of the dimensions of the cohomology groups) of any odd-dimensional compact manifold is 0.[4] From the 3 de Rham cohomology groups already computed, this gives $H^2_{\mathrm{deR}}(M)\approx\mathbb{R}^3$ $(3=4+0-1)$.

8. *Let X be a smooth vector field on a manifold M and define*

$$P\colon \Gamma(M;TM)\longrightarrow\Gamma(M;TM) \qquad \text{by} \qquad P(Y)=[X,Y].$$

(a) *Show that P is a first-order differential operator.*

(b) *What is the symbol of P?*

(c) *Under what conditions (on M and/or X) is P elliptic?*

(a) First, P is a local operator because if $U\subset M$ is an open subset and $Y,Y'\in\Gamma(M;TM)$ are vector fields on M such that $Y|_U=Y'|_U$, then $[X,Y]|_U=[X,Y']|_U$ (this is by definition of $[X,Y]$ in Warner's 1.44). In a coordinate chart, $\varphi=(x_1,\ldots,x_n)\colon U\longrightarrow\mathbb{R}^n$,

$$X|_U=\sum_{j=1}^{i=n}a_i\frac{\partial}{\partial x_i}\,, \qquad Y|_U=\sum_{j=1}^{j=n}f_j\frac{\partial}{\partial x_j}$$

for some $a_i,f_j\in C^\infty(U)$ and

$$[X,Y]|_U=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\left(a_i\frac{\partial f_j}{\partial x_i}\frac{\partial}{\partial x_j}-f_j\frac{\partial a_i}{\partial x_j}\frac{\partial}{\partial x_i}\right)=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\left(a_j\frac{\partial f_i}{\partial x_j}-f_j\frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}\,.$$

Thus, in these local coordinate P corresponds to the operator

$$P_\varphi=\left\{\left(\sum_{j=1}^{j=n}a_j\frac{\partial}{\partial x_j}\right)\mathbb{I}_n-\begin{pmatrix}\frac{\partial a_1}{\partial x_1}&\cdots&\frac{\partial a_1}{\partial x_n}\\ \vdots&&\vdots\\ \frac{\partial a_n}{\partial x_1}&\cdots&\frac{\partial a_n}{\partial x_n}\end{pmatrix}\right\}: C^\infty(\mathbb{R}^n;\mathbb{R}^n)\longrightarrow C^\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{6}$$

This is a first-order differential operator, as needed.

(b) Given $p\in M$, $\alpha\in T_p^*M$, and $v\in T_pM$, choose $f\in C^\infty(M;\mathbb{R})$ and $Y\in\Gamma(M;TM)$ such that $f(p)=0$, $\mathrm{d}_pf=\alpha$, and $Y(p)=v$. Since P is a first-order differential operator,

$$\{\sigma_P(\alpha)\}\equiv P(f^1Y)\big|_p=[X,fY]_p=f(p)[X,Y]_p+X(f)|_pY_p=0\cdot[X,Y]_p+\alpha(X)v.$$

[4]This is true for an orientable odd-dimensional manifold by Poincare duality. For a non-orientable manifold, one can pass to the orientable double cover, and show that $\chi(\tilde M)=k\chi(M)$ for any covering map $\tilde M\longrightarrow M$ of degree k between compact manifolds; this can be done using triangulations. Alternatively, one can show that *any* compact manifold M satisfies Poincare duality with $\mathbb{Z}_2$-coefficients, again using triangulations (or a good cover as in PS11 #3), while $\chi(M)$ is independent of the choice of field.

So the symbol of P is the bundle map

$$\sigma_P\colon T^*M \longrightarrow \mathrm{Hom}(TM,TM), \qquad \big\{\sigma_P(\alpha)\big\}(v) = \alpha(X_p)v \quad \forall\, p\in M,\ \alpha\in T^*M,\ v\in T_pM. \tag{7}$$

Since P is a first-order differential operator, the bundle map σ_P is in fact a vector-bundle homomorphism, i.e. it is linear on each fiber (in general $\sigma_P(c\alpha)=c^k\sigma(\alpha)$, where k is the order of P).

(c) The operator P is elliptic if for all $p\in M$ and $\alpha\in T^*M-0$ the vector-space homomorphism

$$\sigma_P(\alpha)\colon T_pM \longrightarrow T_pM$$

is an isomorphism. By (7), this is the case if and only if $\alpha(X_p)\neq 0$ for all $\alpha\in T_p^*M-0$, i.e. the homomorphism

$$T_p^*M \longrightarrow \mathbb{R}, \qquad \alpha \longrightarrow \alpha(X_p),$$

is injective for all p. This is the case if and only if M is one-dimensional and $X\in\Gamma(M;TM)$ is a nowhere zero vector field (or M is zero-dimensional).

The first definition of *elliptic* in Warner's 6.28 can also be used. Since P_φ is a first-order differential operator, drop the second (matrix) term in (6). Given $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$, let $P_\varphi(\xi)$ be the matrix-valued function obtained by replacing each $\partial/\partial x_j$ by ξ_j:

$$P_\varphi(\xi) = \bigg(\sum_{j=1}^{j=n} a_j\xi_j\bigg)\mathbb{I}_n\,.$$

The operator P is elliptic if this matrix is invertible everywhere on U for all $\xi\neq 0$ and for all charts φ. This is the case if and only if the coefficient in front of $\mathbb{I}_n$ above is never zero for $\xi\neq 0$. This is the case if and only if $n=1$ and the function a_1 does not vanish (or $n=0$); since $X=a_1\partial/\partial x_1$ if $n=1$, the last condition is equivalent to the vector field X never vanishing.

Bonus Problem

Let $\gamma_n \longrightarrow \mathbb{C}P^n$ be the tautological complex line bundle, where $n \ge 1$. Show that for every $k \in \mathbb{Z}^+$, the complex line bundles

$$\gamma_n^{\otimes k} \equiv \underbrace{\gamma_n \otimes \ldots \otimes \gamma_n}_{k} \longrightarrow \mathbb{C}P^n \qquad \textit{and} \qquad \gamma_n^{\otimes(-k)} \equiv \underbrace{\gamma_n^* \otimes \ldots \otimes \gamma_n^*}_{k} \longrightarrow \mathbb{C}P^n$$

are not trivial (not isomorphic to $\mathbb{C}P^n \times \mathbb{C}$ as complex line bundles over $\mathbb{C}P^n$).
Hint: there is a short solution, but it connects several different things encountered in class and thus requires a solid understanding of what is going on.

The manifold $\mathbb{C}P^1$ embeds into $\mathbb{C}P^n$ as $\{[X_0, X_1, 0, \ldots, 0]\}$ and $\gamma_n|_{\mathbb{C}P^1} = \gamma_1$ as line bundles over $\mathbb{C}P^1$ and so $(\gamma_n^{\otimes k})|_{\mathbb{C}P^1} = \gamma_1^{\otimes k}$. Since any restriction of a trivial line bundle is trivial, it is sufficient to consider the $\mathbb{C}P^1$ case of the problem. The set $\mathrm{LB}_{\mathbb{C}}(\mathbb{C}P^1)$ of isomorphism classes of line bundles over $\mathbb{C}P^1$ forms an abelian group under the tensor product. By PS8 #3, there is a group isomorphism

$$c_1 \colon \mathrm{LB}_{\mathbb{C}}(\mathbb{C}P^1) \longrightarrow H^2(\mathbb{C}P^1;\mathbb{Z});$$

in particular,

$$c_1\big(\gamma_1^{\otimes k}\big) = k \cdot c_1(\gamma_1).$$

Since $\gamma_1 \longrightarrow \mathbb{C}P^1$ is a non-trivial complex line bundle by PS2 #5, $c_1(\gamma_1) \neq 0 \in H^2(\mathbb{C}P^1;\mathbb{Z})$. On the other hand, $H^2(\mathbb{C}P^1;\mathbb{Z}) \approx \mathbb{Z}$ because $\mathbb{C}P^1 = S^2$.[5] Thus,

$$c_1\big(\gamma_1^{\otimes k}\big) = k \cdot c_1(\gamma_1) \neq 0 \in H^2(\mathbb{C}P^1;\mathbb{Z}) \qquad \forall\ k \in \mathbb{Z}-0.$$

Since c_1 is a group isomorphism, it follows that $\gamma_1^{\otimes k} \longrightarrow \mathbb{C}P^1$ is non-trivial line bundle for all $k \in \mathbb{Z}$–0.

Note: In fact, $c_1(\gamma)$ is a generator for $H^2(\mathbb{C}P^n;\mathbb{Z})$; this is shown for example in Milnor-Stasheff's *Characteristic Classes*. Thus, $c_1(\gamma^*) = -c_1(\gamma)$ is also a generator for $H^2(\mathbb{C}P^n;\mathbb{Z})$; this is generally the preferred generator, often called the positive generator, because $\int_{\mathbb{C}P^1} c_1(\gamma^*) = 1$ if $\mathbb{C}P^1$ has its natural orientation as a complex manifold (more in MAT 545). Since $c_1(\gamma)$ is a generator for $H^2(\mathbb{C}P^n;\mathbb{Z})$, the *only* complex line bundles over are $\gamma_n^{\otimes k}$ with $k \in \mathbb{Z}$.

[5] $H^*(S^n;\mathbb{Z})$ is computable via Mayer-Vietoris just like $H^*_{\mathrm{deR}}(S^n)$ on PS7 #3.