MAT 531: Topology&Geometry, II
Spring 2010

Final Exam

Instructions

- Give concise proofs, quoting established facts as appropriate; no treatises.
- The problems are worth 20 points each, but are not necessarily of the same difficulty. Parts of a problem may not carry equal weight.
- Your final exam score will be based on your work on 5 problems: 2 (highest-scoring) from Part I, 2 from Part II, and 1 from Part III. The points earned on the bonus problem will be added to your final-exam score, even if the total exceeds 100. However, *NO extra credit will be awarded for solutions to any of the other three problems.*
- Please start each problem on a new sheet of paper. When you are finished, please assemble your solutions in order and attach them to the cover sheet. Do *not* attach this problem sheet.

Part I (*choose 2 problems from 1,2, and 3*)

1. Let M and N be compact oriented connected n-manifolds and $f\colon M \longrightarrow N$ a smooth map. Show that there exists a unique number $(\deg\, f)\in\mathbb{R}$ such that

$$\int_M f^*\omega = (\deg\, f)\cdot\int_N \omega \qquad \forall\ \omega\in E^n(N).$$

Note: there are two parts to this problem.

2. Let X and Y be the vector fields on $\mathbb{R}^3$ given by

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\,, \qquad Y = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\,.$$

(a) Compute $[X,Y]$.

(b) Is there a coordinate chart $\varphi = (x_1,x_2,x_3)\colon U \longrightarrow \mathbb{R}^3$ on a neighborhood of the origin in $\mathbb{R}^3$ such that

$$X|_U = \frac{\partial}{\partial x_1}\,, \qquad Y|_U = \frac{\partial}{\partial x_2}\,?$$

3. Let $S^2\subset\mathbb{R}^3$ be the unit sphere centered at the origin.

(a) Why is the line bundle $\Lambda^2(TS^2)\longrightarrow S^2$ trivial?

(b) Describe an explicit isomorphism $\Lambda^2(TS^2)\longrightarrow S^2\times\mathbb{R}$ of real line bundles over S^2 (give a formula).

Part II (*choose 2 problems from 4,5, and 6*)

4. Show that there exist a closed 1-form α on $\mathbb{R}P^n$ and a smooth function $f\colon [0,1]\longrightarrow\mathbb{R}P^n$ so that

$$f(0)=f(1) \qquad \text{and} \qquad \int_{[0,1]} f^*\alpha\neq 0$$

if and only if $n=1$.

5. Let $M=(S^1\times\mathbb{R}P^n\times\mathbb{R}P^n)/\sim$, where $n\in\mathbb{Z}^+$, $(x,y,z)\sim(-x,z,y)$, and S^1 is viewed as the unit circle in $\mathbb{C}$ (so $x\in\mathbb{C}$). Show that M is not orientable and describe the orientable double cover of M. *Hint:* both parts require some care.

6. Let $M=\mathbb{R}^3/\sim$, where

$$(x,y,z)\sim\big(x+k,y+m,z+ky+n\big) \qquad \forall\ (x,y,z)\in\mathbb{R}^3,\ (k,m,n)\in\mathbb{Z}^3\,.$$

(a) Show that this is an equivalence relation and M is a connected compact orientable 3-manifold.

(b) Determine the de Rham cohomology of M (as graded vector space).

Part III (*choose 1 problem from 7 and 8*)

7. Let $T^3=S^1\times S^1\times S^1$ be the 3-torus and X the complement of two disjoint closed balls in T^3. The boundary of $\bar{X}$ consists of two copies of S^2, S_0 and S_1, which inherit an orientation from the removed closed balls (this orientation is *opposite* to the orientation as the boundary of $\bar{X}$). The boundary of $S^2\times[0,1]$ also consists of two copies of S^2, $S^2\times 0$ and $S^2\times 1$, which inherit an orientation from the standard orientation of S^2 (on $S^2\times 0$ this orientation is opposite to its orientation as boundary of $S^2\times[0,1]$). Let M be the smooth 3-manifold obtained by joining $\bar{X}$ and $S^2\times[0,1]$ along their boundaries so that S_i is identified with $S^2\times i$ by an orientation-preserving diffeomorphism for $i=0,1$. Show that M is not orientable and determine its de Rham cohomology (as graded vector space).

8. Let X be a smooth vector field on a manifold M and define

$$P\colon \Gamma(M;TM)\longrightarrow\Gamma(M;TM) \qquad \text{by} \qquad P(Y)=[X,Y].$$

(a) Show that P is a first-order differential operator.

(b) What is the symbol of P?

(c) Under what conditions (on M and/or X) is P elliptic?

Bonus Problem

Let $\gamma_n\longrightarrow\mathbb{C}P^n$ be the tautological complex line bundle, where $n\geq 1$. Show that for every $k\in\mathbb{Z}^+$, the complex line bundles

$$\gamma_n^{\otimes k}\equiv\underbrace{\gamma_n\otimes\ldots\otimes\gamma_n}_{k}\longrightarrow\mathbb{C}P^n \qquad \text{and} \qquad \gamma_n^{\otimes(-k)}\equiv\underbrace{\gamma_n^*\otimes\ldots\otimes\gamma_n^*}_{k}\longrightarrow\mathbb{C}P^n$$

are not trivial (not isomorphic to $\mathbb{C}P^n\times\mathbb{C}$ as complex line bundles over $\mathbb{C}P^n$).
Hint: there is a short solution, but it connects several different things encountered in class and thus requires a solid understanding of what is going on.