Notes on Lectures 1-5

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March 16, 2010

Contents

Chapter 0

Notation and Terminology

If M is a topological space and $p \in M$, a neighborhood of p in M is an open subset U of M that contains p.

The identity element in the groups $\mathrm{GL}_k\mathbb{R}$ and $\mathrm{GL}_k\mathbb{C}$ of invertible $k \times k$ real and complex matrices will be denoted $\mathbb{I}_k$.

Chapter 1

Smooth Manifolds and Maps

1 Smooth Manifolds: Definition and Examples

Definition 1.1. *A topological space M is a topological m-manifold if*

(TM1) *M is Hausdorff and second-countable, and*

(TM2) *every point $p \in M$ has a neighborhood U homeomorphic to $\mathbb{R}^m$.*

A chart around p on M is a pair (U, φ), where U is a neighborhood of p in M and $\varphi\colon U \longrightarrow U'$ is a homeomorphism onto an open subset of $\mathbb{R}^m$.

Thus, the set of rational numbers, $\mathbb{Q}$, in the discrete topology is a 0-dimensional topological manifold. However, the set of real numbers, $\mathbb{R}$, in the discrete topology is not a 0-dimensional manifold because it does not have a countable basis. On the other hand, $\mathbb{R}$ with its standard topology is a 1-dimensional topological manifold, since

(TM1:$\mathbb{R}$) $\mathbb{R}$ is Hausdorff (being a metric space) and second-countable;

(TM2:$\mathbb{R}$) the map $\varphi = \mathrm{id}\colon U = \mathbb{R} \longrightarrow \mathbb{R}$ is a homeomorphism; thus, $(\mathbb{R}, \mathrm{id})$ is a chart around every point $p \in \mathbb{R}$.

A topological space satisfying (TM2) in Definition 1.1 is called locally Euclidean; such a space is made up of copies of $\mathbb{R}^m$ glued together; see Figure 1.1. While every point in a locally Euclidean space has a neighborhood which is homeomorphic to $\mathbb{R}^m$, the space itself need not be Hausdorff; see Example 1.2 below. A Hausdorff locally Euclidean space is easily seen to be regular, while a regular second-countable space is normal [5, Theorem 32.1], metrizable (Urysohn Metrization Theorem [5, Theorem 34.1]), paracompact [5, Theorem 41.4], and thus admits partitions of unity (see Definition 5.12 below).

Example 1.2. Let $M = (0 \times \mathbb{R} \sqcup 0' \times \mathbb{R})/\sim$, where $(0, s) \sim (0', s)$ for all $s \in \mathbb{R} - 0$. As sets, $M = \mathbb{R} \sqcup \{0'\}$. Let $\mathcal{B}$ be the collection of all subsets of $\mathbb{R} \sqcup \{0'\}$ of the form

$$(a, b) \subset \mathbb{R}, \quad a, b \in \mathbb{R}, \qquad (a, b)' \equiv \big((a, b) - 0\big) \sqcup \{0'\} \ \text{ if } a < 0 < b.$$

This collection $\mathcal{B}$ forms a basis for the quotient topology on M. Note that

(TO1) any neighborhoods U of 0 and U' of $0'$ in M intersect, and thus M is not Hausdorff;

locally Euclidean space　　　　　　　　line with two origins

Figure 1.1: A locally Euclidean space M, such as an m-manifold, consists of copies of $\mathbb{R}^m$ glued together. The line with two origins is a non-Hausdorff locally Euclidean space.

(TO2) the subsets $M-0'$ and $M-0$ of M are open in M and homeomorphic to $\mathbb{R}$; thus, M is locally Euclidean.

This example is illustrated in the right diagram in Figure 1.1. The two thin lines have length zero: $\mathbb{R}^-$ continues through 0 and $0'$ to $\mathbb{R}^+$. Since M is not Hausdorff, it cannot be topologically embedded into $\mathbb{R}^m$ (and thus cannot be accurately depicted in a diagram). Note that the quotient map

$$q\colon 0\times\mathbb{R}\sqcup 0'\times\mathbb{R} \longrightarrow M$$

is open (takes open sets to open sets); so open quotient maps do not preserve separation properties. In contrast, the image of a *closed* quotient map from a normal topological space is still normal [5, Lemma 73.3].

Definition 1.3. *A smooth m-manifold is a pair $(M,\mathcal{F})$, where M is a topological m-manifold and $\mathcal{F}=\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of charts on M such that*

(SM1) $M=\bigcup\limits_{\alpha\in\mathcal{A}} U_\alpha$,

(SM2) $\varphi_\alpha\circ\varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha\cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha\cap U_\beta)$ *is a smooth map (between open subsets of* $\mathbb{R}^m$*) for all* $\alpha,\beta\in\mathcal{A}$*;*

(SM3) $\mathcal{F}$ *is maximal with respect to* (SM2)*.*

The collection $\mathcal{F}$ is called a smooth structure on M.

Since the maps φ_α and φ_β in Definition 1.3 are homeomorphisms, $\varphi_\beta(U_\alpha\cap U_\beta)$ and $\varphi_\alpha(U_\alpha\cap U_\beta)$ are open subsets of $\mathbb{R}^m$, and so the notion of a *smooth* map between them is well-defined; see Figure 1.2. Since $\{\varphi_\alpha\circ\varphi_\beta^{-1}\}^{-1}=\varphi_\beta\circ\varphi_\alpha^{-1}$, *smooth map* in (SM2) can be replaced by *diffeomorphism.* If $\alpha=\beta$,

$$\varphi_\alpha\circ\varphi_\beta^{-1}=\mathrm{id}: \varphi_\beta(U_\alpha\cap U_\beta)=\varphi_\alpha(U_\alpha) \longrightarrow \varphi_\alpha(U_\alpha\cap U_\beta)=\varphi_\alpha(U_\alpha)$$

is of course a smooth map, and so it is sufficient to verify the smoothness requirement of (SM2) only for $\alpha\neq\beta$.

It is hardly ever practical to specify a smooth structure $\mathcal{F}$ on a manifold M by listing all elements of $\mathcal{F}$. Instead $\mathcal{F}$ can be specified by describing a collection of charts $\mathcal{F}_0=\{(U,\varphi)\}$ satisfying (SM1) and (SM2) in Definition 1.3 and setting

$$\mathcal{F}=\big\{\text{chart } (V,\psi) \text{ on } M \big|\ \varphi\circ\psi^{-1}\colon \psi(U\cap V)\longrightarrow\varphi(U\cap V) \text{ is diffeomorphism } \forall\,(U,\varphi)\in\mathcal{F}_0\big\}. \quad (1.1)$$

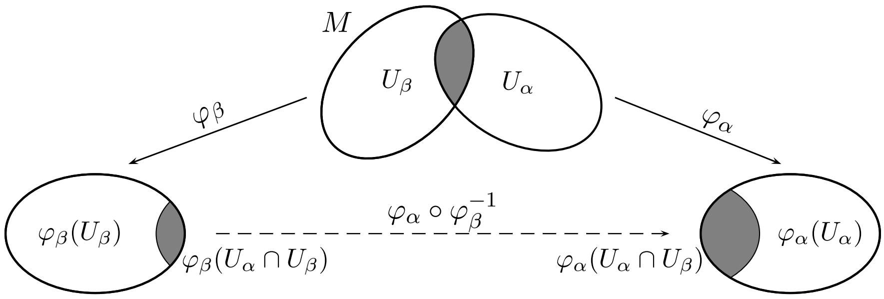


Figure 1.2: The overlap map between two charts is a map between open subsets of $\mathbb{R}^m$.

Example 1.4. The map $\varphi = \mathrm{id}\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a homeomorphism, and thus the pair $(\mathbb{R}^m, \mathrm{id})$ is a chart around every point in the topological m-manifold $M = \mathbb{R}^m$. So, the single-element collection $\mathcal{F}_0 = \{(\mathbb{R}^m, \mathrm{id})\}$ satisfies (SM1) and (SM2) in Definition 1.3. It thus induces a smooth structure $\mathcal{F}$ on $\mathbb{R}^m$; this smooth structure is called the standard smooth structure on $\mathbb{R}^m$.

Example 1.5. Every finite-dimensional vector space V has a canonical topology specified by the requirement that any vector-space isomorphism $\varphi\colon V \longrightarrow \mathbb{R}^m$, where $m = \dim V$, is a homeomorphism (with respect to the standard topology on $\mathbb{R}^m$). If $\psi : V \longrightarrow \mathbb{R}^m$ is another vector-space isomorphism, then the map

$$\varphi \circ \psi^{-1}\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m \tag{1.2}$$

is an invertable linear transformation; thus, it is a diffeomorphism and in particular a homeomorphism. So, two different isomorphisms $\varphi, \psi : V \longrightarrow \mathbb{R}^m$ determine the same topology on V. Each pair (V, φ) is then a chart on V, and the one-element collection $\mathcal{F}_0 = \{(V, \varphi)\}$ determines a smooth structure $\mathcal{F}$ on V. Since the map (1.2) is a diffeomorphism, $\mathcal{F}$ is independent of the choice of vector-space isomorphism $\varphi\colon V \longrightarrow \mathbb{R}^m$. Thus, every finite-dimensional vector space carries a canonical smooth structure.

Example 1.6. The map $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(t) = t^3$, is a homeomorphism, and thus the pair $(\mathbb{R}, \varphi)$ is a chart around every point in the topological 1-manifold $M = \mathbb{R}$. So, the single-element collection $\mathcal{F}_0' = \{(\mathbb{R}, \varphi)\}$ satisfies (SM1) and (SM2) in Definition 1.3. It thus induces a smooth structure $\mathcal{F}'$ on $\mathbb{R}$. While $\mathcal{F}' \neq \mathcal{F}$, where $\mathcal{F}$ is the standard smooth structure on $\mathbb{R}^1$ described in Example 1.4, the smooth manifolds $(\mathbb{R}^1, \mathcal{F})$ and $(\mathbb{R}^1, \mathcal{F}')$ are diffeomorphic in the sense of (2) in Definition 2.1 below.

Example 1.7. Let $M = S^1$ be the unit circle in the complex (s,t)-plane,

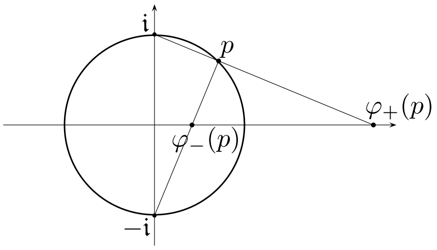
$$U_+ = S^1 - \{\mathrm{i}\}, \qquad U_- = S^1 - \{-\mathrm{i}\}\,.$$

For each $p \in U_\pm$, let $\varphi_\pm(p) \in \mathbb{R}$ be the s-coordinate of the intersection of the s-axis with the line through the points $\pm\mathrm{i}$ and $p \neq \pm\mathrm{i}$; see Figure 1.3. The maps $\varphi_\pm : U_\pm \longrightarrow \mathbb{R}$ are homeomorphisms and $S^1 = U_+ \cup U_-$. Since

$$U_+ \cap U_- = S^1 - \{\mathrm{i}, -\mathrm{i}\} = U_+ - \{-\mathrm{i}\} = U_- - \{\mathrm{i}\}$$

and $\varphi_\pm(U_+ \cap U_-) = \mathbb{R} - 0 \equiv \mathbb{R}^*$, the overlap map is

$$\varphi_+ \circ \varphi_-^{-1}\colon \varphi_-(U_+ \cap U_-) = \mathbb{R}^* \longrightarrow \varphi_+(U_+ \cap U_-) = \mathbb{R}^*;$$



$$\varphi_+(s,t) = \frac{s}{1-t}$$

$$\varphi_-(s,t) = \frac{s}{1+t}$$

$$\varphi_+ \circ \varphi_-^{-1} \colon \mathbb{R}^* \longrightarrow \mathbb{R}^*, \quad \varphi_+ \circ \varphi_-^{-1}(s) = 1/s$$

Figure 1.3: A pair of charts on S^1 determining a smooth structure.

by a direct computation, this map is $s \longrightarrow s^{-1}$. Since this map is a diffeomorphism between open subsets of $\mathbb{R}^1$, the collection

$$\mathcal{F}_0 = \big\{(U_+, \varphi_+), (U_-, \varphi_-)\big\}$$

determines a smooth structure $\mathcal{F}$ on S^1.

A smooth structure on the unit sphere $M = S^m \subset \mathbb{R}^{m+1}$ can be defined similarly: take $U_\pm \subset S^m$ to be the complement of the point $q_\pm \in S^m$ with the last coordinate ± 1 and $\varphi_\pm(p) \in \mathbb{R}^m$ the intersection of the line through $q_\pm$ and $p \neq q_\pm$ with $\mathbb{R}^m = \mathbb{R}^m \times 0$. This smooth structure is the unique one with which S^m is a submanifold of $\mathbb{R}^{m+1}$; see Definition 4.1 and Corollary 4.6.

Example 1.8. Let $\mathrm{MB} = ([0,1] \times \mathbb{R})/\sim$, $(0,t) \sim (1,-t)$, be the infinite Mobius Band,

$$U_0 = (0,1) \times \mathbb{R} \subset \mathrm{MB}, \qquad \varphi_0 = \mathrm{id} \colon U_0 \longrightarrow (0,1) \times \mathbb{R},$$

$$\varphi_{1/2} \colon U_{1/2} = \mathrm{MB} - \{1/2\} \times \mathbb{R} \longrightarrow (0,1) \times \mathbb{R}, \quad \varphi_{1/2}([s,t]) = \begin{cases} (s-1/2, t), & \text{if } s \in (1/2, 1], \\ (s+1/2, -t), & \text{if } s \in [0, 1/2), \end{cases}$$

where $[s,t]$ denotes the equivalence class of $(s,t) \in [0,1] \times \mathbb{R}$ in MB. The pairs (U_0, φ_0) and $(U_{1/2}, \varphi_{1/2})$ are then charts on the topological 1-manifold MB. The overlap map between them is

$$\varphi_{1/2} \circ \varphi_0^{-1} \colon \varphi_0(U_0 \cap U_{1/2}) = \big((0,1/2) \cup (1/2,1)\big) \times \mathbb{R} \longrightarrow \varphi_{1/2}(U_0 \cap U_{1/2}) = \big((0,1/2) \cup (1/2,1)\big) \times \mathbb{R},$$

$$\varphi_{1/2} \circ \varphi_0^{-1}(s,t) = \begin{cases} (s+1/2, -t), & \text{if } s \in (0, 1/2); \\ (s-1/2, t), & \text{if } s \in (1/2, 1); \end{cases}$$

see Figure 1.4. Since this map is a diffeomorphism between open subsets of $\mathbb{R}^2$, the collection

$$\mathcal{F}_0 = \big\{(U_0, \varphi_0), (U_{1/2}, \varphi_{1/2})\big\}$$

determines a smooth structure $\mathcal{F}$ on MB.

Example 1.9. The real projective space of dimension n, denoted $\mathbb{R}P^n$, is the space of real one-dimensional subspaces of $\mathbb{R}^{n+1}$ (or lines through the origin in $\mathbb{R}^{n+1}$) in the natural quotient topology. In other words, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a nonzero vector in $\mathbb{R}^{n+1}$, i.e. an element of $\mathbb{R}^{n+1} - 0$. Two such vectors determine the same one-dimensional subspace in $\mathbb{R}^{n+1}$ and the same element of $\mathbb{R}P^n$ if and only if they differ by a non-zero scalar. Thus, as sets

$$\mathbb{R}P^n = \big(\mathbb{R}^{n+1} - 0\big)/\mathbb{R}^* \equiv \big(\mathbb{R}^{n+1} - 0\big)/\sim, \qquad \text{where}$$
$$c \cdot v = cv \in \mathbb{R}^{n+1} - 0 \ \ \forall\, c \in \mathbb{R}^*, \, v \in \mathbb{R}^{n+1} - 0, \qquad v \sim cv \ \ \forall\, c \in \mathbb{R}^*, \, v \in \mathbb{R}^{n+1} - 0.$$

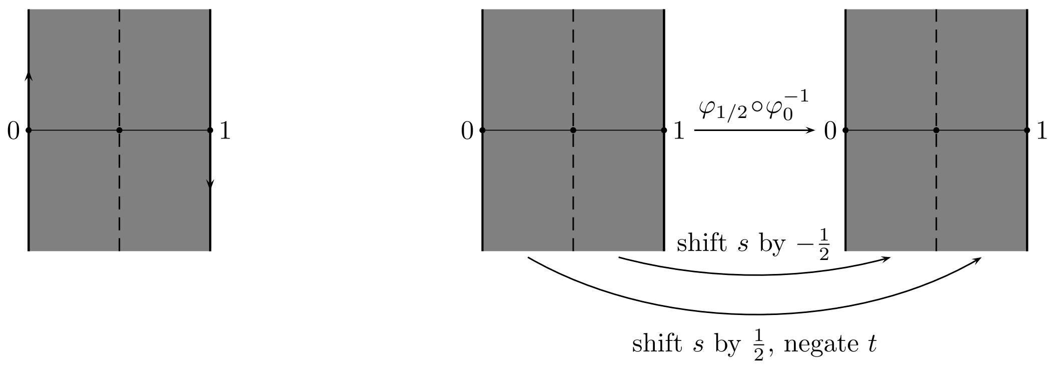


Figure 1.4: The infinite Mobius band MB is obtained from an infinite strip by identifying the two infinite edges in opposite directions, as indicated by the arrows in the first diagram. The two charts on MB of Example 1.8 overlap smoothly.

Alternatively, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a unit vector in $\mathbb{R}^{n+1}$, i.e. an element of S^n. Two such vectors determine the same element of $\mathbb{R}P^n$ if and only if they differ by a non-zero scalar, which in this case must necessarily be ±1. Thus, as sets

$$\begin{gathered}\mathbb{R}P^n = S^n/\mathbb{Z}_2 \equiv S^n/\sim, \qquad \text{where} \\ \mathbb{Z}_2=\{\pm1\}, \quad c\cdot v = cv\in S^n \quad \forall\, c\!\in\!\mathbb{Z}_2,\, v\!\in\! S^n, \quad v\sim cv \quad \forall\, c\!\in\!\mathbb{Z}_2,\, v\!\in\! S^2.\end{gathered} \tag{1.3}$$

Thus, as sets,

$$\mathbb{R}P^n = \big(\mathbb{R}^{n+1}-0\big)/\mathbb{R}^* = S^n/\mathbb{Z}_2.$$

It follows that $\mathbb{R}P^n$ has two natural quotient topologies; these two topologies are the same, however. The space $\mathbb{R}P^n$ has a natural smooth structure, induced from that of $\mathbb{R}^{n+1}$–0 and S^n. It is generated by the $n+1$ charts

$$\begin{gathered}\varphi_i\colon U_i\equiv\big\{[X_0,X_1,\ldots,X_n]\colon\, X_i\!\neq\!0\big\}\longrightarrow\mathbb{R}^n, \\ \big[X_0,X_1,\ldots,X_n\big]\longrightarrow\bigg(\frac{X_0}{X_i},\ldots,\frac{X_{i-1}}{X_i},\frac{X_{i+1}}{X_i},\ldots,\frac{X_n}{X_i}\bigg).\end{gathered}$$

Note that $\mathbb{R}P^1\!=\!S^1$.

Example 1.10. The complex projective space of dimension n, denoted $\mathbb{C}P^n$, is the space of complex one-dimensional subspaces of $\mathbb{C}^{n+1}$ in the natural quotient topology. Similarly to the real case of Example 1.9,

$$\begin{gathered}\mathbb{C}P^n = \big(\mathbb{C}^{n+1}-0\big)/\mathbb{C}^* = S^{2n+1}/S^1, \qquad \text{where} \\ S^1=\big\{c\!\in\!\mathbb{C}^*\colon |c|\!=\!1\big\}, \qquad S^{2n+1}=\big\{v\!\in\!\mathbb{C}^{n+1}-0\colon |v|\!=\!1\big\}, \\ c\cdot v = cv\in\mathbb{C}^{n+1}-0 \quad \forall\, c\!\in\!\mathbb{C}^*,\, v\!\in\!\mathbb{C}^{n+1}-0.\end{gathered}$$

The two quotient topologies on $\mathbb{C}P^n$ arising from these quotients are again the same. The space $\mathbb{C}P^n$ has a natural complex structure, induced from that of $\mathbb{C}^{n+1}-0$.

There are a number of canonical ways of constructing new smooth manifolds.

Proposition 1.11. *(1) If $(M,\mathcal{F})$ is a smooth m-manifold, $U\subset M$ is open, and*

$$\mathcal{F}|_U \equiv \big\{(U_\alpha\cap U, \varphi_\alpha|_{U_\alpha\cap U})\colon (U_\alpha,\varphi_\alpha)\in\mathcal{F}\big\} = \big\{(U_\alpha,\varphi_\alpha)\in\mathcal{F}\colon\, U_\alpha\subset U\big\}, \tag{1.4}$$

then $(U,\mathcal{F}|_U)$ is also a smooth m-manifold.
(2) If $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ are smooth manifolds, then the collection

$$\mathcal{F}_0 = \big\{(U_\alpha\times V_\beta, \varphi_\alpha\times\psi_\beta)\colon\, (U_\alpha,\varphi_\alpha)\in\mathcal{F}_M,\, (V_\beta,\psi_\beta)\in\mathcal{F}_N\big\} \tag{1.5}$$

satisfies (SM1) *and* (SM2) *of Definition 1.3 and thus induces a smooth structure on $M\times N$.*

It is immediate that the second collection in (1.4) is contained in the first. The first collection is contained in the second because $\mathcal{F}$ is maximal with respect to (SM2) in Definition 1.3 and the restriction of a smooth map from an open subset of $\mathbb{R}^m$ to a smaller open subset is still smooth. Since every element $(U_\alpha,\varphi_\alpha)$ of $\mathcal{F}$ is a chart on M, every such element with $U_\alpha\subset U$ is also a chart on U. Since $\{U_\alpha\colon\, (U_\alpha,\varphi_\alpha)\in\mathcal{F}\}$ is an open cover of M, $\{U_\alpha\cap U\colon\, (U_\alpha,\varphi_\alpha)\in\mathcal{F}\}$ is an open cover of U. Since $\mathcal{F}$ satisfies (SM2) in Definition 1.3, so does its subcollection $\mathcal{F}|_U$. Since $\mathcal{F}$ is maximal with respect to (SM2) in Definition 1.3, so is its subcollection $\mathcal{F}|_U$. Thus, $\mathcal{F}|_U$ is indeed a smooth structure on U.

Let $m=\dim M$ and $n=\dim N$. Since each $(U_\alpha,\varphi_\alpha)\in\mathcal{F}_M$ is a chart on M and each $(V_\beta,\psi_\beta)\in\mathcal{F}_N$ is a chart on N,

$$\varphi_\alpha\times\psi_\beta\colon U_\alpha\times V_\beta \longrightarrow \varphi_\alpha(U_\alpha)\times\psi_\beta(V_\beta)\subset\mathbb{R}^m\times\mathbb{R}^n=\mathbb{R}^{m+n}$$

is a homeomorphism between an open subset of $M\times N$ (in the product topology) and an open subset of $\mathbb{R}^{m+n}$. Since the collections $\{U_\alpha\colon\, (U_\alpha,\varphi_\alpha)\in\mathcal{F}_M\}$ and $\{V_\beta\colon\, (V_\beta,\psi_\beta)\in\mathcal{F}_N\}$ cover M and N, respectively, the collection

$$\big\{U_\alpha\times V_\beta\colon\, (U_\alpha,\varphi_\alpha)\in\mathcal{F}_M,\, (V_\beta,\psi_\beta)\in\mathcal{F}_N\big\}$$

covers $M\times N$. If $(U_\alpha\times V_\beta,\varphi_\alpha\times\psi_\beta)$ and $(U_{\alpha'}\times V_{\beta'},\varphi_{\alpha'}\times\psi_{\beta'})$ are elements of the collection (1.5),

$$\begin{aligned}
U_\alpha\times V_\beta\cap U_{\alpha'}\times V_{\beta'} &= \big(U_\alpha\cap U_{\alpha'}\big)\times\big(V_\beta\cap V_{\beta'}\big),\\
\big\{\varphi_\alpha\times\psi_\beta\big\}\big(U_\alpha\times V_\beta\cap U_{\alpha'}\times V_{\beta'}\big) &= \varphi_\alpha\big(U_\alpha\cap U_{\alpha'}\big)\times\psi_\beta\big(V_\beta\cap V_{\beta'}\big)\subset\mathbb{R}^{m+n},\\
\big\{\varphi_{\alpha'}\times\psi_{\beta'}\big\}\big(U_\alpha\times V_\beta\cap U_{\alpha'}\times V_{\beta'}\big) &= \varphi_{\alpha'}\big(U_\alpha\cap U_{\alpha'}\big)\times\psi_{\beta'}\big(V_\beta\cap V_{\beta'}\big)\subset\mathbb{R}^{m+n},
\end{aligned}$$

and the overlap map,

$$\big\{\varphi_\alpha\times\psi_\beta\big\}\circ\big\{\varphi_{\alpha'}\times\psi_{\beta'}\big\}^{-1} = \big\{\varphi_\alpha\circ\varphi_{\alpha'}^{-1}\big\}\times\big\{\varphi_\beta\circ\varphi_{\beta'}^{-1}\big\},$$

is the product of the overlap maps for M and N; thus, it is smooth. So the collection (1.5) satisfies the requirements (SM1) and (SM2) of Definition 1.3 and thus induces a smooth structure on $M\times N$, called the product smooth structure.

Corollary 1.12. *The general linear group,*

$$\mathrm{GL}_n\mathbb{R} = \big\{A\in\mathrm{Mat}_{n\times n}\mathbb{R}\colon\, \det A\neq 0\big\},$$

is a smooth manifold of dimension n^2.

The map

$$\det\colon \mathrm{Mat}_{n\times n}\mathbb{R}\approx\mathbb{R}^{n^2}\longrightarrow\mathbb{R}$$

is continuous. Since $\mathbb{R}-0$ is an open subset of $\mathbb{R}$, its pre-image under det, $\mathrm{GL}_n\mathbb{R}$, is an open subset of $\mathbb{R}^{n^2}$ and thus is a smooth manifold of dimension n^2 by (1) of Proposition 1.11.

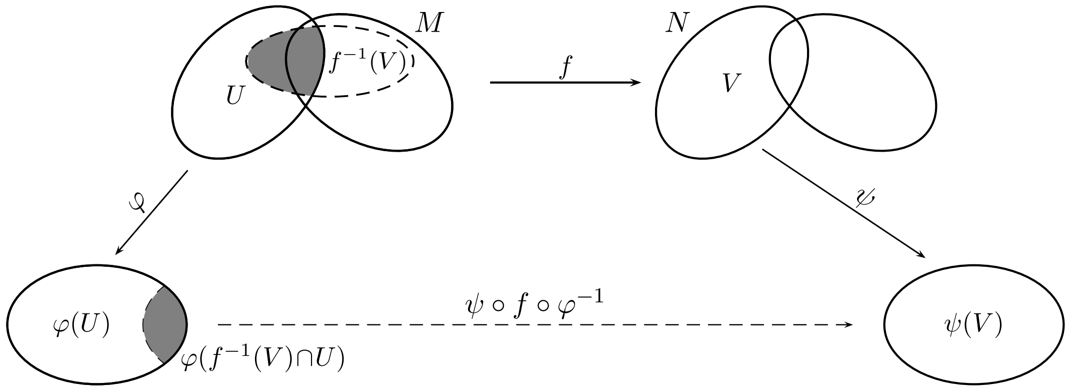


Figure 1.5: A continuous map f between manifolds is smooth if it induces smooth maps between open subsets of Euclidean spaces via the charts.

2 Smooth Maps: Definition and Examples

Definition 2.1. *Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth manifolds.*

(1) A continuous map $f: M \longrightarrow N$ is a smooth map *between $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ if for all $(U,\varphi)\in\mathcal{F}_M$ and $(V,\psi)\in\mathcal{F}_N$ the map*

$$\psi\circ f\circ\varphi^{-1}: \varphi\big(f^{-1}(V)\cap U\big) \longrightarrow \psi(V) \tag{2.1}$$

is a smooth map (between open subsets of Euclidean spaces).

(2) A smooth bijective map $f: (M,\mathcal{F}_M) \longrightarrow (N,\mathcal{F}_N)$ is a diffeomorphism *if the inverse map, $f^{-1}: (N,\mathcal{F}_N)\longrightarrow(M,\mathcal{F}_M)$, is also smooth.*

(3) A smooth map $f: (M,\mathcal{F}_M)\longrightarrow(N,\mathcal{F}_N)$ is a local diffeomorphism *if for every $p\in M$ there are open neighborhoods U_p of p in M and $V_{f(p)}$ of $f(p)$ in N such that $f|_{U_p}: U_p \longrightarrow V_{f(p)}$ is a diffeomorphism between the smooth manifolds $(U_p,\mathcal{F}_M|_{U_p})$ and $(V_{f(p)},\mathcal{F}_N|_{V_{f(p)}})$.*

If $f: M\longrightarrow N$ is a continuous map and $(V,\psi)\in\mathcal{F}_N$, $f^{-1}(V)\subset M$ is open and $\psi(V)\subset\mathbb{R}^n$ is open, where $n=\dim N$. If in addition $(U,\varphi)\in\mathcal{F}_M$, then $\varphi(f^{-1}(V)\cap U)$ is an open subset of $\mathbb{R}^m$, where $m=\dim M$. Thus, (2.1) is a map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and so the notion of a *smooth* map between them is well-defined; see Figure 1.5. If $\mathcal{F}_{M;0}$ and $\mathcal{F}_{N;0}$ are collections of charts on M and N, respectively, that generate $\mathcal{F}_M$ and $\mathcal{F}_N$ in the sense of (1.1), then $f: (M,\mathcal{F}_M)\longrightarrow(N,\mathcal{F}_N)$ is a smooth map if and only if (2.1) is a smooth map for every $(U,\varphi)\in\mathcal{F}_{M;0}$ and all $(V,\psi)\in\mathcal{F}_{N;0}$ covering $f(U)$. Similarly, $f: M\longrightarrow N$ is a local diffeomorphism if and only if (2.1) is a local diffeomorphism for every $(U,\varphi)\in\mathcal{F}_{M;0}$ and all $(V,\psi)\in\mathcal{F}_{N;0}$ covering $f(U)$.

A bijective local diffeomorphism is a diffeomorphism, and vice versa. In particular, the identity map $\mathrm{id}: (M,\mathcal{F})\longrightarrow(M,\mathcal{F})$ on any manifold is a diffeomorphism, since for all $(U,\varphi),(V,\psi)\in\mathcal{F}_M$ the map (2.1) is simply

$$\psi\circ\varphi^{-1}: \varphi\big(U\cap V\big) \longrightarrow \psi\big(U\cap V\big) \subset \psi(V);$$

it is smooth by (SM2) in Definition 1.3. For the same reason, the map

$$\varphi: \big(U,\mathcal{F}_M|_U\big) \longrightarrow \varphi(U) \subset \mathbb{R}^m$$

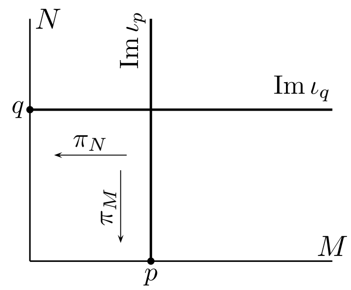


Figure 1.6: A horizontal slice $M\times q=\operatorname{Im}\iota_q$, a vertical slice $p\times N=\operatorname{Im}\iota_p$, and the two component projection maps $M\times N\longrightarrow M,N$

is a diffeomorphism for every $(U,\varphi)\in\mathcal{F}_M$. A composition of two smooth maps (local diffeomorphisms, diffeomorphisms) is again smooth (a local diffeomorphism, a diffeomorphism).

Example 2.2. Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth manifolds and $\mathcal{F}_{M\times N}$ the product smooth structure on $M\times N$ of Proposition 1.11. Let $\mathcal{F}_0$ be as in (1.5).

(1) For each $q\in N$, the inclusion as a "horizontal" slice,

$$\iota_q\colon M\longrightarrow M\times N,\qquad p\longrightarrow (p,q),$$

is smooth, since for every $(U,\varphi)\in\mathcal{F}_M$ and $(U\times V,\varphi\times\psi)\in\mathcal{F}_0$ with $q\in V$ the map

$$\{\varphi\times\psi\}\circ\iota_q\circ\varphi^{-1}=\mathrm{id}\times\psi(q)\colon \varphi\big(\iota_q^{-1}(U\times V)\cap U\big)=\varphi(U)\longrightarrow\{\varphi\times\psi\}\big(U\times V\big)=\varphi(U)\times\psi(V)$$

is smooth and $\iota_q(U)\subset U\times V$. Similarly, for each $p\in M$, the inclusion as a "vertical" slice,

$$\iota_p\colon N\longrightarrow M\times N,\qquad q\longrightarrow (p,q),$$

is also smooth.

(2) The projection map onto the first component,

$$\pi_1=\pi_M\colon M\times N\longrightarrow M,\qquad (p,q)\longrightarrow p,$$

is smooth, since for every $(U\times V,\varphi\times\psi)\in\mathcal{F}_0$ and $(U,\varphi)\in\mathcal{F}_M$ the map

$$\varphi\circ\pi_M\circ\{\varphi\times\psi\}^{-1}=\pi_1\colon\{\varphi\times\psi\}\big(\pi_M^{-1}(U)\cap U\times V\big)=\varphi(U)\times\psi(V)\longrightarrow\varphi(U)$$

is smooth (being the restriction of the projection $\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}^m$ to an open subset) and $\pi_M(U\times V)\subset U$. Similarly, the projection map onto the second component,

$$\pi_2=\pi_N\colon M\times N\longrightarrow N,\qquad (p,q)\longrightarrow q,$$

is also smooth.

The following two lemmas and a proposition provide additional ways of constructing smooth structures. Lemma 2.3 can be used in the proof of Proposition 2.5; Lemma 2.4 gives rise to manifold structures on the tangent and cotangent bundles of a smooth manifold, as indicated in Example 6.5.

Lemma 2.3. *Let M be a Hausdorff second-countable topological space and $\{\varphi_\alpha : U_\alpha \longrightarrow M_\alpha\}$ a collection of homeomorphisms from open subsets U_α of M to m-manifolds M_α such that*

$$\varphi_\alpha\circ\varphi_\beta^{-1}: \varphi_\beta(U_\alpha\cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha\cap U_\beta) \tag{2.2}$$

is a smooth map for all $\alpha,\beta\in\mathcal{A}$. If the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M, then M admits a unique smooth structure such that each map φ_α is a diffeomorphism.

Lemma 2.4. *Let M be a set and $\{\varphi_\alpha: U_\alpha \longrightarrow M_\alpha\}$ a collection of bijections from subsets U_α of M to m-manifolds M_α such that*

$$\varphi_\alpha\circ\varphi_\beta^{-1}: \varphi_\beta(U_\alpha\cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha\cap U_\beta) \tag{2.3}$$

is a smooth map between open subsets of M_β and M_α, respectively, for all $\alpha,\beta\in\mathcal{A}$.

(NMS1) *If the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M, then M admits a unique topology $\mathcal{T}_M$ such that each map φ_α is a homeomorphism.*

(NMS2) *If in addition $\mathcal{T}_M$ is Hausdorff and second-countable, then $(M,\mathcal{T}_M)$ admits a unique smooth structure such that each map φ_α is a diffeomorphism.*

Proposition 2.5. *If a group G acts properly discontinuously on a smooth m-manifold $(\tilde{M},\mathcal{F}_{\tilde{M}})$ by diffeomorphisms and $\pi: \tilde{M}\longrightarrow M=\tilde{M}/G$ is the quotient projection map, then*

$$\mathcal{F}_0 = \big\{(\pi(U), \varphi\circ\{\pi|_U\}^{-1}):\ (U,\varphi)\in\mathcal{F}_{\tilde{M}},\ \pi|_U \text{ is injective}\big\}$$

is a collection of charts on the quotient topological space M that satisfies (SM1) *and* (SM2) *in Definition 1.3 and thus induces a smooth structure $\mathcal{F}_M$ on M. This smooth structure on M is the unique one satisfying either of the following two properties:*

(QSM1) *the projection map $\tilde{M}\longrightarrow M$ is a local diffeomorphism;*

(QSM2) *if N is a smooth manifold, a continuous map $f: M\longrightarrow N$ is smooth if and only if the map $f\circ\pi: \tilde{M}\longrightarrow N$ is smooth.*

A basis for the topology $\mathcal{T}_M$ of Lemma 2.4 consists of the subsets $U\subset M$ such that $U\subset U_\alpha$ and $\varphi_\alpha(U_\alpha)\subset M_\alpha$ is open for some $\alpha\in\mathcal{A}$. In the case of Lemma 2.3, $\varphi_\alpha(U_\alpha\cap U_\beta)$ is an open subset of M_α because U_α and U_β are open subsets of M and φ_α is a homeomorphism; thus, smoothness for the map (2.2) is a well-defined requirement in light of (1) of Proposition 1.11 and (1) of Definition 2.1. In the case of Lemma 2.4, $\varphi_\alpha(U_\alpha\cap U_\beta)$ need not be a priori open in M_α, and so this must be one of the assumptions. In both cases, the requirement that $\varphi_\alpha\circ\varphi_\beta^{-1}$ be smooth can be replaced by the requirement that it be a diffeomorphism. We leave proofs of Lemmas 2.3 and 2.4 and Proposition 2.5 as exercises.

The smooth structure $\mathcal{F}_M$ on M of Proposition 2.5 is called the quotient smooth structure on M. For example, the group $\mathbb{Z}$ acts on $\mathbb{R}$ and on $\mathbb{R}\times\mathbb{R}$ by

$$\mathbb{Z}\times\mathbb{R}\longrightarrow\mathbb{R}, \qquad (m,s)\longrightarrow s+m, \tag{2.4}$$

$$\mathbb{Z}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}\times\mathbb{R}, \qquad (m,s,t)\longrightarrow \big(s+m,(-1)^m t\big). \tag{2.5}$$

Both of these actions satisfy the assumptions of Proposition 2.5 and thus give rise to quotient smooth structures on $S^1=\mathbb{R}/\mathbb{Z}$ and $\mathrm{MB}=(\mathbb{R}\times\mathbb{R})/\mathbb{Z}$. These smooth structures are the same as those of Examples 1.7 and 1.8, respectively.

Example 1.6 is a special case of the following phenomenon. If $(M,\mathcal{F})$ is a smooth manifold and $h\colon M\longrightarrow M$ is a homeomorphism, then

$$h^*\mathcal{F}\equiv\big\{\big(h^{-1}(U),\varphi\circ h\big)\colon\,(U,\varphi)\in\mathcal{F}\big\}$$

is also a smooth structure on M, since the overlap maps are the same as for the collection $\mathcal{F}$. The smooth structures $\mathcal{F}$ and $h^*\mathcal{F}$ are the same if and only if $h\colon(M,\mathcal{F})\longrightarrow(M,\mathcal{F})$ is a diffeomorphism. However, in all cases, the map $h^{-1}\colon(M,\mathcal{F})\longrightarrow(M,h^*\mathcal{F})$ is a diffeomorphism; so if a topological manifold admits a smooth structure, it admits many smooth equivalent (diffeomorphic) smooth structures.

This raises the question of which topological manifolds admit smooth structures and if so how many inequivalent ones. Since every connected component of a topological manifold is again a topological manifold, it is sufficient to study this question for connected topological manifolds.

dim=0: every connected topological 0-manifold M consists of a single point, $M=\{pt\}$; the only smooth structure on such a topological manifold is the single-element collection $\{(M,\varphi)\}$, where φ is the unique map $M\longrightarrow\mathbb{R}^0$.

dim=1: every connected topological (smooth) 1-manifold is homeomorphic (diffeomorphic) to either $\mathbb{R}$ or S^1 in the standard topology (and with standard smooth structure); a short proof of the smooth statement is given in [2, Appendix].

dim=2: every topological 2-manifold admits a unique smooth structure; every compact topological 2-manifold is homeomorphic (and thus diffeomorphic) to either a "torus" with $g\geq 0$ handles or to a connected sum of such a "torus" with $\mathbb{R}P^2$ [5, Chapter 8]; every such manifold admits a smooth structure as it is the quotient of either S^2 or $\mathbb{R}^2$ by a group acting properly discontinuously by diffeomorphisms.

dim=3: every topological 3-manifold admits a unique smooth structure [3].

dim=4: there are lots of topological 4-manifolds that admit no smooth structure and lots of other topological 4-manifolds (including $\mathbb{R}^4$) that admit many (even uncountably many) smooth structures.

The first known example of a topological manifold admitting non-equivalent smooth structures is the 7-sphere [1]. Since then the situation in dimensions 5 or greater has been sorted out by topological arguments.

Remark 2.6. While topology studies the topological category $\mathcal{TC}$, differential geometry studies the smooth category $\mathcal{SC}$. The objects in the latter are smooth manifolds, while the morphisms are smooth maps. The composition of two morphisms is the usual composition of maps (which is still a smooth map). For each object $(M,\mathcal{F}_M)$, the identity morphism is just the identity map id_M on M (which is a smooth map). The "forgetful map"

$$\mathcal{SC}\longrightarrow\mathcal{TC},\qquad (M,\mathcal{F}_M)\longrightarrow M,\qquad \big(f\colon(M,\mathcal{F}_M)\longrightarrow(N,\mathcal{F}_N)\big)\longrightarrow\big(f: M\longrightarrow N\big),$$

is a functor from the smooth category to the topological category.

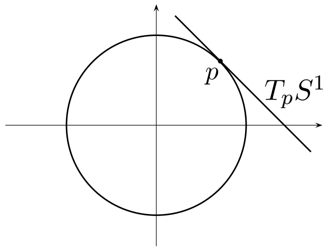


Figure 1.7: The tangent space of S^1 at p viewed as a subspace of $\mathbb{R}^2$.

3 Tangent Vectors

This section defines tangent vectors and related concepts for a smooth manifold. These are often used in describing properties of smooth maps between manifolds as well as in studying specific manifolds.

If M is an m-manifold embedded in $\mathbb{R}^n$, with $m \le n$, and $\alpha\colon(-\epsilon,\epsilon)\longrightarrow M$ is a smooth map (curve on M) such that $\alpha(0)=p\in M$, then $\alpha'(0)\in\mathbb{R}^n$ should be a tangent vector of M at p. The set of such vectors is an m-dimensional linear subspace of $\mathbb{R}^n$; it is often thought of as having the 0-vector at p, at the origin of $\mathbb{R}^n$; see Figure 1.7. However, this presentation of the tangent space T_pM of M at p depends on the embedding of M in $\mathbb{R}^n$, and not just on M and p.

On the other hand, the tangent space at a point $p\in\mathbb{R}^m$ should be $\mathbb{R}^m$ itself, but based (with the origin) at p. Each vector $v\in\mathbb{R}^m$ acts on smooth functions f defined near p by

$$\partial_v|_p f = \lim_{t\longrightarrow 0}\frac{f(p+tv)-f(p)}{t}\,. \tag{3.1}$$

If $v=e_i$ is the i-th coordinate vector on $\mathbb{R}^m$, then $\partial_{e_i}|_p f$ is just the i-th partial derivative $\partial_i f|_p$ of f at p. The map $\partial_v|_p$ defined by (3.1) takes each smooth function defined on a neighborhood of p in $\mathbb{R}^m$ to $\mathbb{R}$ and satisfies:

(TV1) if $f: U\longrightarrow\mathbb{R}$ and $g: V\longrightarrow\mathbb{R}$ are smooth functions on neighborhoods of p such that $f|_W=g|_W$ for some neighborhood W of p in $U\cap V$, then $\partial_v|_p f=\partial_v|_p g$;

(TV2) if $f\colon U\longrightarrow\mathbb{R}$ and $g\colon V\longrightarrow\mathbb{R}$ are smooth functions on neighborhoods of p and $a,b\in\mathbb{R}$, then

$$\partial_v|_p\big(af+bg\big) = a\,\partial_v|_p f + b\,\partial_v|_p g\,,$$

where $af+bg$ is the smooth function on the neighborhood $U\cap V$ given by

$$\{af+bg\}(q) = af(q)+bg(q)\,;$$

(TV3) if $f\colon U\longrightarrow\mathbb{R}$ and $g\colon V\longrightarrow\mathbb{R}$ are smooth functions on neighborhoods of p, then

$$\partial_v|_p(fg) = f(p)\partial_v|_p g + g(p)\partial_v|_p f\,,$$

where fg is the smooth function on the neighborhood $U\cap V$ given by $\{fg\}(q) = f(q)g(q)$.

It turns out every $\mathbb{R}$-valued map on the space of smooth functions defined on neighborhood of p satisfying (TV1)-(TV3) is $\partial_v|_p$ for some $v \in \mathbb{R}^m$; see Proposition 3.4 below. At the same time, these three conditions make sense for any smooth manifold, and this approach indeed leads to an intrinsic definition of tangent vectors for smooth manifolds.

The space of functions defined on various neighborhoods of a point does not have a very nice structure. In order to study the space of operators satisfying (TV1)-(TV3) it is convenient to put an equivalence relation on this space.

Definition 3.1. *Let M be a smooth manifold and $p \in M$.*

(1) Functions $f: U \longrightarrow \mathbb{R}$ and $g: V \longrightarrow \mathbb{R}$ defined on neighborhoods of p in M are p-equivalent, or $f \sim_p g$, if there exists a neighborhood W of p in $U \cap V$ such that $f|_W = g|_W$.

(2) The set of p-equivalence classes of smooth functions is denoted $\tilde{F}_p$; the p-equivalence class of a smooth function $f: U \longrightarrow \mathbb{R}$ on a neighborhood of p is called the germ of f at p and is denoted $\underline{f}_p$.

The set $\tilde{F}_p$ has a natural $\mathbb{R}$-algebra structure:

$$a\underline{f}_p + b\underline{g}_p = \underline{af+bg}_p, \quad \underline{f}_p \cdot \underline{g}_p = \underline{fg}_p, \qquad \forall\, \underline{f}_p, \underline{g}_p \in \tilde{F}_p,\ a, b \in \mathbb{R},$$

where $af+bg$ and fg are functions defined on $U \cap V$ if f and g are defined on U and V, respectively. There is a well-defined valuation homomorphism,

$$\mathrm{ev}_p: \tilde{F}_p \longrightarrow \mathbb{R}, \qquad \underline{f}_p \longrightarrow f(p).$$

Let $F_p = \ker \mathrm{ev}_p$; this subset of $\tilde{F}_p$ consists of the germs at p of the smooth functions defined on neighborhoods of p in M that vanish at p. Since ev_p is an $\mathbb{R}$-algebra homomorphisms, F_p is an ideal in $\tilde{F}_p$; this can also be seen directly: if $f(p)=0$, then $\{fg\}(p)=0$. Let $F_p^2 \subset F_p$ be the ideal in $\tilde{F}_p$ consisting of all finite linear combinations of elements of the form $\underline{f}_p\underline{g}_p$ with $\underline{f}_p, \underline{g}_p \in F_p$. If $c \in \mathbb{R}$, let $\underline{c}_p \in \tilde{F}_p$ denote the germ at p of the constant function with value c on M.

Lemma 3.2. *Let M be a smooth manifold and $p \in M$. If v is a derivation on $\tilde{F}_p$ relative to the valuation ev_p,[1] then*

$$v|_{F_p^2} \equiv 0, \qquad v(\underline{c}_p) = 0 \ \ \forall\, c \in \mathbb{R}. \tag{3.2}$$

If $\underline{f}_p, \underline{g}_p \in F_p$, then $f(p), g(p)=0$ and thus

$$v\big(\underline{f}_p\underline{g}_p\big) = f(p)v(\underline{g}_p) + g(p)v(\underline{f}_p) = 0;$$

so v vanishes identically on F_p^2. If $c \in \mathbb{R}$,

$$\begin{aligned} v(\underline{c}_p) = v(\underline{1}_p\underline{c}_p) = 1(p) \cdot v(\underline{c}_p) + c(p) \cdot v(\underline{1}_p) &= 1 \cdot v(\underline{c}_p) + c \cdot v(\underline{1}_p) \\ &= v(\underline{c}_p) + v(c \cdot \underline{1}_p) = v(\underline{c}_p) + v(\underline{c}_p); \end{aligned}$$

so $v(\underline{c}_p) = 0$.

[1] i.e. $v: \tilde{F}_p \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-linear map such that

$$v(\underline{f}_p\underline{g}_p) = \mathrm{ev}_p(\underline{f}_p)v(\underline{g}_p) + \mathrm{ev}_p(\underline{g}_p)v(\underline{f}_p), \qquad \forall\, \underline{f}_p, \underline{g}_p \in \tilde{F}_p\,.$$

Corollary 3.3. *If M be a smooth manifold and $p\in M$, the map $v\longrightarrow v|_{F_p}$ induces an isomorphism from the vector space* $\mathrm{Der}(\tilde F_p,\mathrm{ev}_p)$ *of derivations on $\tilde F_p$ relative to the valuation* ev_p *to*

$$\big\{L\in\mathrm{Hom}(F_p,\mathbb{R})\colon\ L|_{F_p^2}\equiv 0\big\}\approx\big(F_p/F_p^2\big)^*.$$

The set $\mathrm{Der}(\tilde F_p,\mathrm{ev}_p)$ of derivations on $\tilde F_p$ relative to the valuation ev_p indeed forms a vector space:

$$\big\{av+bw\big\}\big(\underline{f}_p\big)=av\big(\underline{f}_p\big)+bw\big(\underline{f}_p\big)\qquad\forall\ v,w\in\mathrm{Der}(\tilde F_p,\mathrm{ev}_p),\ a,b\in\mathbb{R},\ \underline{f}_p\in\tilde F_p.$$

If $v\in\mathrm{Der}(\tilde F_p,\mathrm{ev}_p)$, the restriction of v to $F_p\subset\tilde F_p$ is a homomorphism to $\mathbb{R}$ that vanishes on F_p^2 by Lemma 3.2. Conversely, if $L\colon F_p\longrightarrow\mathbb{R}$ is a linear homomorphism vanishing on F_p^2, define

$$v_L\colon\tilde F_p\longrightarrow\mathbb{R}\qquad\text{by}\qquad v_L\big(\underline{f}_p\big)=L\big(\underline{f-f(p)}_p\big);$$

since the function $f-f(p)$ vanishes at p, $\underline{f-f(p)}_p\in F_p$ and so v_L is well-defined. It is immediate that v_L is a homomorphism of vector spaces. Furthermore, for all $\underline{f}_p,\underline{g}_p$,

$$\begin{aligned}v_L\big(\underline{f}_p\underline{g}_p\big)&=L\big(\underline{fg-f(p)g(p)}_p\big)=L\big(f(p)\underline{g-g(p)}_p+g(p)\underline{f-f(p)}_p+\underline{f-f(p)}_p\underline{g-g(p)}_p\big)\\&=f(p)L\big(\underline{g-g(p)}_p\big)+g(p)L\big(\underline{f-f(p)}_p\big)+L\big(\underline{f-f(p)}_p\underline{g-g(p)}_p\big)\\&=f(p)v_L\big(\underline{g}_p\big)+g(p)v_L\big(\underline{f}_p\big)+0,\end{aligned}$$

since L vanishes on F_p^2; so v_L is a derivation with respect to the valuation ev_p. It is also immediate that the maps

$$\begin{aligned}\mathrm{Der}(\tilde F_p,\mathrm{ev}_p)&\longrightarrow\big\{L\in\mathrm{Hom}(F_p,\mathbb{R})\colon\ L|_{F_p^2}\equiv 0\big\}, &\qquad v&\longrightarrow L_v\equiv v|_{F_p},\\ \big\{L\in\mathrm{Hom}(F_p,\mathbb{R})\colon\ L|_{F_p^2}\equiv 0\big\}&\longrightarrow\mathrm{Der}(\tilde F_p,\mathrm{ev}_p), & L&\longrightarrow v_L,\end{aligned}\tag{3.3}$$

are homomorphisms of vector spaces. If $L\in\mathrm{Hom}(F_p,\mathbb{R})$ and $L|_{F_p^2}\equiv 0$, the restriction of v_L to F_p is L, and so $L_{v_L}=L$. If $v\in\mathrm{Der}(\tilde F_p,\mathrm{ev}_p)$ and $\underline{f}_p\in\tilde F_p$, by the second statement in (3.2)

$$v\big(\underline{f}_p\big)=v\big(\underline{f}_p\big)-v\big(\underline{f(p)}_p\big)=v\big(\underline{f-f(p)}_p\big)=L_v\big(\underline{f-f(p)}_p\big)=v_{L_v}\big(\underline{f}_p\big);$$

so $v_{L_v}=v$ and the two homomorphisms in (3.3) are inverses of each other. This completes the proof of Corollary 3.3.

Proposition 3.4. *If $p\in\mathbb{R}^m$, the vector space F_p/F_p^2 is m-dimensional and the homomorphism*

$$\mathbb{R}^m\longrightarrow\mathrm{Der}\big(\tilde F_p,\mathrm{ev}_p\big)\approx\big(F_p/F_p^2\big)^*,\qquad v\longrightarrow\partial_v|_p\,,\tag{3.4}$$

induced by (3.1), is an isomorphism.

By (TV1), $\partial_v|_p$ induces a well-defined map $\tilde F_p\longrightarrow\mathbb{R}$. By (TV2), $\partial_v|_p$ is a vector-space homomorphism. By (TV3), $\partial_v|_p$ is a derivation with respect to the valuation ev_p. Thus, the map (3.4) is well-defined and is clearly a vector-space homomorphism. If $\pi_j\colon\mathbb{R}^m\longrightarrow\mathbb{R}$ is the projection on the j-th component,

$$\partial_{e_i}|_p\big(\pi_j-\pi_j(p)\big)=\big(\partial_i(\pi_j-\pi_j(p))\big)_p=\delta_{ij}\equiv\begin{cases}1, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases}\tag{3.5}$$

Thus, the homomorphism (3.4) is injective, and the set $\{\underline{\pi_j - \pi_j(p)}_p\}$ is linearly independent in F_p/F_p^2. On the other hand, Lemma 3.5 below implies that

$$f(p+x) = f(p) + \sum_{i=1}^{i=m} (\partial_i f)_p x_i + \sum_{i,j=1}^{i,j=m} x_i x_i \int_0^1 (1-t)(\partial_i \partial_j f)_{p+tx} \mathrm{d}t \tag{3.6}$$

for every smooth function f defined on a open ball U around p in $\mathbb{R}^m$ and for all $p+x \in U$. Thus, the set $\{\underline{\pi_j - \pi_j(p)}_p\}$ spans F_p/F_p^2; so F_p/F_p^2 is m-dimensional and the homomorphism (3.4) is an isomorphism.

Note that the inverse of the isomorphism (3.4) is given by

$$\mathrm{Der}\big(\tilde{F}_p, \mathrm{ev}_p\big) \longrightarrow \mathbb{R}^m, \qquad v \longrightarrow \big(v(\underline{\pi_1}_p), \ldots, v(\underline{\pi_m}_p)\big); \tag{3.7}$$

by (3.5), this is a right inverse and thus must be the inverse.

Lemma 3.5. *If $h\colon U \longrightarrow \mathbb{R}$ is a smooth function defined on an open ball U around a point p in $\mathbb{R}^m$, then*

$$h(p+x) = h(p) + \sum_{i=1}^{i=m} x_i \int_0^1 (\partial_i h)_{p+tx} \mathrm{d}t$$

for all $p+x \in U$.

This follows from the Fundamental Theorem of Calculus:

$$\begin{aligned} h(p+x) = h(p) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} h(p+tx) \mathrm{d}t &= h(p) + \int_0^1 \sum_{i=1}^{i=m} (\partial_i h)_{p+tx} x_i \mathrm{d}t \\ &= h(p) + \sum_{i=1}^{i=m} x_i \int_0^1 (\partial_i h)_{p+tx} \mathrm{d}t. \end{aligned}$$

Corollary 3.6. *If M be a smooth m-manifold and $p \in M$, the vector space $\mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p)$ is m-dimensional.*

If $\varphi\colon U \longrightarrow \mathbb{R}^m$ is a chart around $p \in M$, the map $f \longrightarrow f \circ \varphi$ induces an $\mathbb{R}$-algebra isomorphism

$$\varphi^*\colon \tilde{F}_{\varphi(p)} \longrightarrow \tilde{F}_p, \qquad \underline{f}_{\varphi(p)} \longrightarrow \underline{f \circ \varphi}_p\,. \tag{3.8}$$

Since $\mathrm{ev}_{\varphi(p)} = \mathrm{ev}_p \circ \varphi^*$, φ^* restricts to an isomorphism $F_{\varphi(p)} \longrightarrow F_p$ and descends to an isomorphism

$$F_{\varphi(p)}/F_{\varphi(p)}^2 \longrightarrow F_p/F_p^2\,. \tag{3.9}$$

Thus, Corollary 3.6 follows from Corollary 3.3 and Proposition 3.4.

Definition 3.7. *Let M be a smooth manifold and $p \in M$.*

(1) The tangent space *of M at p is the vector space $T_pM = \mathrm{Der}(\tilde{F}_p, \mathrm{ev}_p)$; a* tangent vector *of M at p is an element of T_pM.*

(2) The cotangent space *of M at p is $T_p^*M \equiv (T_pM)^* \equiv \mathrm{Hom}(T_pM, \mathbb{R})$.*

By Corollary 3.6, T_pM and T_p^*M are m-dimensional vector spaces if M is an m-dimensional manifold. By Proposition 3.4, $T_p\mathbb{R}^m$ is canonically isomorphic to $\mathbb{R}^m$ for every $p\in\mathbb{R}^m$. By Corollary 3.3, $T_p^*M\approx F_p/F_p^2$; an element $\underline{f}_p+F_p^2$ of F_p/F_p^2 determines the vector-space homomorphism

$$T_pM \longrightarrow \mathbb{R}, \qquad v \longrightarrow v\big(\underline{f}_p\big). \tag{3.10}$$

Any smooth function f defined on a neighborhood of p in M defines an element of T^*M in the same way, but this element depends only on

$$\underline{f-f(p)}_p + F_p^2 \in F_p/F_p^2 .$$

Definition 3.8. *Let $h\colon M\longrightarrow N$ be a smooth map between smooth manifolds and $p\in M$.*

(1) The differential of h at p *is the map*

$$\mathrm{d}_ph\colon T_pM \longrightarrow T_{h(p)}N, \qquad \big\{\mathrm{d}_ph(v)\big\}\big(\underline{f}_{h(p)}\big) = v\big(\underline{f\circ h}_p\big) \quad \forall\, v\in T_pM,\ \underline{f}_{h(p)}\in\tilde{F}_{h(p)}\,. \tag{3.11}$$

(2) The pull-back map on the cotangent spaces *is the map*

$$h^* \equiv \big\{\mathrm{d}_ph\big\}^*\colon T^*_{h(p)}N \longrightarrow T_p^*M, \qquad \eta \longrightarrow \eta\circ \mathrm{d}_ph\,. \tag{3.12}$$

The map (3.11) is a vector-space homomorphism, and thus so is h^*. It is immediate from the definition that $\mathrm{d}_p\mathrm{id}_M=\mathrm{id}_{T_pM}$ and thus $\mathrm{id}_M^*=\mathrm{id}_{T_p^*M}$. If $N=\mathbb{R}$, $T_{h(p)}\mathbb{R}$ is canonically isomorphic to $\mathbb{R}$, via the map

$$T_{h(p)}\mathbb{R} \longrightarrow \mathbb{R}, \qquad w \longrightarrow w\big(\mathrm{id}_{\mathbb{R}}\big);$$

see (3.7). In particular, if $v\in T_pM$,

$$\mathrm{d}_ph(v) \longrightarrow \big\{\mathrm{d}_ph(v)\big\}\big(\mathrm{id}_{\mathbb{R}}\big) \equiv v\big(\mathrm{id}_{\mathbb{R}}\circ h\big) = v(h).$$

Thus, under the canonical identification $T_{h(p)}\mathbb{R}$ with $\mathbb{R}$, the differential d_ph of a smooth map $h\colon M\longrightarrow\mathbb{R}$ is given by

$$\boxed{\mathrm{d}_ph(v) = v(h) \qquad \forall\, v\in T_pM} \tag{3.13}$$

and so corresponds to the same element of T_p^*M as

$$\underline{h-h(p)}_p + F_p^2 \in F_p/F_p^2\,;$$

see (3.10).

Lemma 3.9. *If $g: M\longrightarrow N$ and $h: N\longrightarrow X$ are smooth maps between smooth manifolds and $p\in M$, then*

$$\mathrm{d}_p(h\circ g) = \mathrm{d}_{g(p)}h\circ \mathrm{d}_pg\colon T_pM \longrightarrow T_{h(g(p))}X. \tag{3.14}$$

Thus, $(h\circ g)^*=g^*\circ h^*: T^*_{h(g(p))}X\longrightarrow T_p^*M$ and

$$\boxed{g^*\mathrm{d}_{g(p)}f = \mathrm{d}_p(f\circ g)} \tag{3.15}$$

whenever f is a smooth function on a neighborhood of $g(p)$ in N.

If $v\in T_pM$ and f is a smooth function on a neighborhood of $h(g(p))$ in X, then by (3.11)

$$\begin{aligned}\big\{\{\mathrm{d}_p(h\circ g)\}(v)\big\}(f)=v\big(f\circ h\circ g\big)=\{\mathrm{d}_pg(v)\}(f\circ h)&=\big\{\mathrm{d}_{g(p)}h\big(\mathrm{d}_pg(v)\big)\big\}(f)\\&=\big\{\{\mathrm{d}_{g(p)}h\circ\mathrm{d}_pg\}(v)\big\}(f);\end{aligned}$$

thus, (3.14) holds. The second claim is the dual of the first. For the last claim, note that

$$g^*\mathrm{d}_{g(p)}f\equiv\mathrm{d}_{g(p)}f\circ\mathrm{d}_pg=\mathrm{d}_p(f\circ g) \tag{3.16}$$

by (3.12) and (3.14). For the purposes of applying (3.12) and (3.14), all expressions in (3.16) are viewed as maps to $T_{f(g(p))}\mathbb{R}$, before its canonical identification with $\mathbb{R}$. The identities of course continue to hold after this identification.

Let $\varphi=(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m$ be a chart on a neighborhood of a point p in M; so, $x_i=\pi_i\circ\varphi$, where $\pi_i\colon\mathbb{R}^m\longrightarrow\mathbb{R}$ is the projection to the i-th component as before. Since the map (3.8) induces the isomorphism (3.9) and $\{\underline{\pi_i-x_i(p)}_{\varphi(p)}\}_i$ is a basis for $F_{\varphi(p)}/F^2_{\varphi(p)}$,

$$\varphi^*\big(\{\underline{\pi_i-x_i(p)}_{\varphi(p)}\}_i\big)\equiv\big\{\underline{(\pi_i-x_i(p))\circ\varphi}_p\big\}_i=\big\{\underline{x_i-x_i(p)}_p\big\}_i$$

is a basis for F_p/F_p^2. Thus, $\{\mathrm{d}_px_i\}_i$ is a basis for T^*_pM, since d_px_i and $\underline{x_i-x_i(p)}_p$ act in the same way on all elements of T_pM; see the paragraph following Definition 3.8. For each $i=1,2,\ldots,m$, let

$$\frac{\partial}{\partial x_i}\bigg|_p=\mathrm{d}_{\varphi(p)}\varphi^{-1}\big(\partial_{e_i}|_{\varphi(p)}\big)\in T_pM.$$

By (3.11), for every smooth function f defined on a neighborhood of p in M

$$\begin{aligned}\frac{\partial}{\partial x_i}\bigg|_p(f)&=\big\{\mathrm{d}_{\varphi(p)}\varphi^{-1}\big(\partial_{e_i}|_{\varphi(p)}\big)\big\}(f)=\partial_{e_i}|_{\varphi(p)}\big(f\circ\varphi^{-1}\big)\\&=\partial_i\big(f\circ\varphi^{-1}\big)|_{\varphi(p)}\end{aligned} \tag{3.17}$$

is the i-th partial derivative of the function $f\circ\varphi^{-1}$ at $\varphi(p)$; this is a smooth function defined on a neighborhood of $\varphi(p)$ in $\mathbb{R}^m$. In particular, for all $i,j=1,2,\ldots,m$

$$\mathrm{d}_px_j\bigg(\frac{\partial}{\partial x_i}\bigg|_p\bigg)=\frac{\partial}{\partial x_i}\bigg|_p(x_j)=\partial_i\big(\pi_j\circ\varphi\circ\varphi^{-1}\big)=\delta_{ij}\,;$$

the first equality above is a special case of (3.13). Thus, $\{\frac{\partial}{\partial x_i}\big|_p\}_i$ is a basis for T_pM; it is dual to the basis $\{\mathrm{d}_px_i\}_i$ for T^*_pM. The coefficients of other elements of T_pM and T^*_pM with respect to these bases are given by

$$v=\sum_{i=1}^{i=m}\big(\mathrm{d}_px_i(v)\big)\frac{\partial}{\partial x_i}\bigg|_p=\sum_{i=1}^{i=m}v(x_i)\frac{\partial}{\partial x_i}\bigg|_p\qquad\forall\,v\in T_pM; \tag{3.18}$$

$$\eta=\sum_{i=1}^{i=m}\eta\bigg(\frac{\partial}{\partial x_i}\bigg|_p\bigg)\mathrm{d}_px_i\qquad\forall\,\eta\in T^*_pM. \tag{3.19}$$

The first identities in (3.18) and (3.19) are immediate from the two bases being dual to each other (each d_px_j gives the same values when evaluated on both sides of the first identity in (3.18); both sides of (3.19) evaluate to the same number on each $\frac{\partial}{\partial x_j}\big|_p$). The second equality in (3.18) follows from (3.13). If f is a smooth function on a neighborhood of p, by (3.19), (3.13), and (3.17)

$$\mathrm{d}_p f = \sum_{i=1}^{i=m} \mathrm{d}_p f\bigg(\frac{\partial}{\partial x_i}\bigg|_p\bigg)\mathrm{d}_p x_i = \sum_{i=1}^{i=m}\bigg(\frac{\partial}{\partial x_i}\bigg|_p(f)\bigg)\mathrm{d}_p x_i = \sum_{i=1}^{i=m}\big(\partial_i(f\circ\varphi^{-1})\big)_{\varphi(p)}\mathrm{d}_p x_i\,. \tag{3.20}$$

If $\psi=(y_1,\ldots,y_m)\colon V\longrightarrow\mathbb{R}^m$ is another chart around p, by (3.18) and (3.17)

$$\begin{aligned}\frac{\partial}{\partial x_j}\bigg|_p &= \sum_{i=1}^{i=m}\bigg(\frac{\partial}{\partial x_j}\bigg|_p(y_i)\bigg)\frac{\partial}{\partial y_i}\bigg|_p = \sum_{i=1}^{i=m}\Big(\partial_j(\pi_i\circ\psi\circ\varphi^{-1})_{\varphi(p)}\Big)\frac{\partial}{\partial y_i}\bigg|_p\\ \Longleftrightarrow\qquad \bigg(\frac{\partial}{\partial x_1}\bigg|_p,\ldots,\frac{\partial}{\partial x_n}\bigg|_p\bigg) &= \bigg(\frac{\partial}{\partial y_1}\bigg|_p,\ldots,\frac{\partial}{\partial y_n}\bigg|_p\bigg)\mathcal{J}\big(\psi\circ\varphi^{-1}\big)_{\varphi(p)},\end{aligned} \tag{3.21}$$

where $\mathcal{J}(\psi\circ\varphi^{-1})_{\varphi(p)}$ is the usual Jacobian (matrix of partial derivatives) of the smooth map $\psi\circ\varphi^{-1}$ between the open subsets $\varphi(U\cap V)$ and $\psi(U\cap V)$ of $\mathbb{R}^m$ at $\varphi(p)$; see Figure 1.2 with $\varphi_\alpha=\psi$ and $\varphi_\beta=\varphi$.

Suppose next that $f\colon M\longrightarrow N$ is a map between smooth manifolds and

$$\varphi=(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m \qquad\text{and}\qquad \psi=(y_1,\ldots,y_n)\colon V\longrightarrow\mathbb{R}^n$$

are coordinate charts around $p\in M$ and $f(p)\in N$, respectively; see Figure 1.5. By (3.18) and (3.11),

$$\begin{aligned}\mathrm{d}_p f\bigg(\frac{\partial}{\partial x_j}\bigg|_p\bigg) &= \sum_{i=1}^{i=n}\bigg\{\mathrm{d}_p f\bigg(\frac{\partial}{\partial x_j}\bigg|_p\bigg)\bigg\}(y_i)\frac{\partial}{\partial y_i}\bigg|_{f(p)} = \sum_{i=1}^{i=n}\bigg(\frac{\partial}{\partial x_j}\bigg|_p(y_i\circ f)\bigg)\frac{\partial}{\partial y_i}\bigg|_{f(p)}\\ &= \sum_{i=1}^{i=n}\big(\partial_j(\pi_i\circ\psi\circ f\circ\varphi^{-1})\big)_{\varphi(p)}\frac{\partial}{\partial y_i}\bigg|_{f(p)};\end{aligned} \tag{3.22}$$

so the matrix of the linear transformation $\mathrm{d}_p f\colon T_pM\longrightarrow T_{f(p)}N$ with respect to the bases $\{\frac{\partial}{\partial x_j}\big|_p\}_j$ and $\{\frac{\partial}{\partial y_i}\big|_{f(p)}\}_i$ is $\mathcal{J}(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}$, the Jacobian of the smooth map $\psi\circ f\circ\varphi^{-1}$ between the open subsets $\varphi(U\cap f^{-1}(V))$ and $\psi(V)$ of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, evaluated at $\varphi(p)$. In particular, $\mathrm{d}_p f$ is injective (surjective) if and only if $\mathcal{J}(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}$ is. The $f=\mathrm{id}$ case of (3.22) is the change-of-coordinates formula (3.21). If M and N are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, $\varphi=\mathrm{id}_M$, and $\psi=\mathrm{id}_N$, then under the canonical identifications $T_p\mathbb{R}^m=\mathbb{R}^m$ and $T_{f(p)}\mathbb{R}^n=\mathbb{R}^n$ the differential $\mathrm{d}_p f$ is simply the Jacobian $\mathcal{J}(f)_p$ of f at p. The chain-rule formula (3.14) states that the Jacobian of a composition of maps is the (matrix) product of the Jacobians of the maps; if M, N, and X are open subsets of Euclidean spaces, this yields the usual chain rule for smooth maps between open subsets of Euclidean spaces, for free (once it is checked that all definitions above make sense and correspond to the standard ones for Euclidean spaces).

By the above, if $\varphi=(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m$ is a chart around a point p in M, then $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M. A weak converse to this statement is true as well; see Corollary 3.12 below. The key tool in obtaining it is the Inverse Function Theorem for $\mathbb{R}^m$; see [4, Theorem 8.3], for example.

Theorem 3.10 (Inverse Function Theorem). *Let $U'\subset\mathbb{R}^m$ be an open subset and $f\colon U'\longrightarrow\mathbb{R}^m$ a smooth map. If the Jacobian $\mathcal{J}(f)_p$ of f is non-singular for some $p\in U'$, there exist neighborhoods U of p in U' and V of $f(p)$ in $\mathbb{R}^m$ such that $f\colon U\longrightarrow V$ is a diffeomorphism.*

Corollary 3.11 (Inverse Function Theorem for Manifolds)**.** *Let $f: M \longrightarrow N$ be a smooth map between smooth manifolds. If the differential $\mathrm{d}_p f: T_pM \longrightarrow T_{f(p)}N$ is an isomorphism for some $p \in M$, then there exist neighborhoods U of p in M and V of $f(p)$ in N such that $f: U \longrightarrow V$ is a diffeomorphism.*

Let $\varphi = (x_1, \ldots, x_m): U' \longrightarrow \mathbb{R}^m$ and $\psi = (y_1, \ldots, y_m): V' \longrightarrow \mathbb{R}^m$ be charts around p in M and $f(p)$ in N, respectively; see Figure 1.5. Then,

$$\psi \circ f \circ \varphi^{-1}: \varphi\big(U' \cap f^{-1}(V')\big) \longrightarrow \varphi(V') \subset \mathbb{R}^m$$

is a smooth map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^m$ such that $\mathcal{J}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$ is non-singular (since by (3.22) this is the matrix of the linear transformation $\mathrm{d}_p f$ with respect to bases $\{\frac{\partial}{\partial x_j|_p}\}_j$ and $\{\frac{\partial}{\partial y_i}|_{f(p)}\}_i$). Since φ and ψ are homeomorphisms onto the open subsets $\varphi(U')$ and $\psi(V')$ of $\mathbb{R}^m$, by Theorem 3.10 there exist open neighborhoods U of p in $U' \cap f^{-1}(V')$ and V of $f(p)$ in V' such that the restriction

$$\psi \circ f \circ \varphi^{-1}: \varphi(U) \longrightarrow \psi(V)$$

is a diffeomorphism. Since $\varphi: U \longrightarrow \varphi(U)$ and $\psi: V \longrightarrow \psi(V)$ are also diffeomorphisms, it follows that so is $f: U \longrightarrow V$ (being composition of $\psi \circ f \circ \varphi^{-1}$ with ψ^{-1} and φ).

Corollary 3.12. *Let M be a smooth m-manifold. If $x_1, \ldots, x_m: U' \longrightarrow \mathbb{R}$ are smooth functions such that $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M for some $p \in U'$, then there exists a neighborhood U of p in U' such that*

$$\varphi = (x_1, \ldots, x_m): U \longrightarrow \mathbb{R}$$

is a chart around p.

Let $f = (x_1, \ldots, x_m): U' \longrightarrow \mathbb{R}^m$. Since $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M, the differential

$$\mathrm{d}_p f = \begin{pmatrix} \mathrm{d}_p x_1 \\ \vdots \\ \mathrm{d}_p x_m \end{pmatrix}: T_pM \longrightarrow \mathbb{R}^m$$

is an isomorphism (for each $v \in T_pM - 0$, there exists i such that $\mathrm{d}_p x_i(v) \neq 0$). Thus, Corollary 3.12 follows immediately from Corollary 3.11 with $M = U'$ and $N = \mathbb{R}^m$.

Corollary 3.13. *Let M be a smooth m-manifold. If $x_1, \ldots, x_n: U' \longrightarrow \mathbb{R}$ are smooth functions such that the set $\{\mathrm{d}_p x_i\}_i$ spans T_p^*M for some $p \in U'$, then there exists a neighborhood U of p in U' such that an m-element subset of $\{x_i\}_i$ determines a chart around p on M.*

This claim follows from Corollary 3.12 by choosing a subset of $\{x_i\}_i$ so that the corresponding subset of $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M.

Corollary 3.14. *Let M be a smooth m-manifold. If $x_1, \ldots, x_k: U' \longrightarrow \mathbb{R}$ are smooth functions such that the set $\{\mathrm{d}_p x_i\}_i$ is linearly independent in T_p^*M for some $p \in U'$, then there exist a neighborhood U of p in U' and a set of smooth functions $x_{k+1}, \ldots, x_m: U \longrightarrow \mathbb{R}$ such that the map*

$$\varphi = \big(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\big): U \longrightarrow \mathbb{R}^m$$

is a chart around p on M.

This claim follows from Corollary 3.12 by choosing a chart $\psi = (y_1, \ldots, y_m) : U'' \longrightarrow \mathbb{R}^m$ on a neighborhood U'' of p in U' and adding some of the functions y_j to the set $\{x_i\}_i$ so that the corresponding set $\{\mathrm{d}_p x_i, \mathrm{d}_p y_j\}$ is a basis for T_p^*M.

Remark 3.15. The differential of a smooth map induces a functor from the category of pointed smooth manifolds (smooth manifolds with a choice of a point) and pointed smooth maps (smooth maps taking chosen points to each other) to the category of finite-dimensional vector spaces and vector-space homomorphisms:

$$(M,p) \longrightarrow T_pM, \qquad \big(h\colon (M,p) \longrightarrow (N,q)\big) \longrightarrow \big(\mathrm{d}_p h\colon T_pM \longrightarrow T_qN\big);$$

these mappings take a composition of morphisms to a composition of morphisms by (3.14) and id_M to id_{T_pM}. On the other hand, the pull-back map h^* on the cotangent spaces reverses compositions of morphisms by (3.15) and thus gives rise to a contravariant functor between the same two categories.

4 Immersions and Submanifolds

Definition 4.1. *Let M and N be smooth manifolds.*

(1) A smooth map $f\colon M \longrightarrow N$ is an immersion if the differential $\mathrm{d}_p f\colon T_pM \longrightarrow T_{f(p)}N$ is injective for every $p \in M$.

(2) The manifold M is a submanifold of N if $M \subset N$, M has the subspace topology, and the inclusion map $\iota\colon M \longrightarrow N$ is an immersion.

If $f\colon M \longrightarrow N$ is a diffeomorphism between smooth manifolds, then the differential

$$\mathrm{d}_p f\colon T_pM \longrightarrow T_{f(p)}N \tag{4.1}$$

is an isomorphism for every $p \in M$. Thus, a diffeomorphism between two smooth manifolds is a bijective immersion. On the other hand, if $f : M \longrightarrow N$ is an immersion, $\dim M \leq \dim N$. If $\dim M = \dim N$ and $f : M \longrightarrow N$ is an immersion, then the differential (4.1) is an isomorphism for every $p \in M$. Corollary 3.11 then implies that f is a local diffeomorphism. Thus, a bijective immersion $f : M \longrightarrow N$ between smooth manifolds of the same dimension is a diffeomorphism. The assumption that manifolds are second-countable topological spaces turns out to imply that a bijective immersion must be a map between manifolds of the same dimension; see Exercise 24. Thus, a bijective immersion is a diffeomorphism and vice versa.

A more interesting example of an immersion is the inclusion of $\mathbb{R}^m$ as the coordinate subspace $\mathbb{R}^m \times 0$ into $\mathbb{R}^n$, with $m \leq n$. By Proposition 4.3 below, every immersion $f\colon M \longrightarrow N$ locally (on M and N) looks like the inclusion of $\mathbb{R}^m$ as $\mathbb{R}^m \times 0$ into $\mathbb{R}^n$ and every submanifold $M \subset N$ locally (on N) looks like $\mathbb{R}^m \times 0 \subset \mathbb{R}^n$. We will use the following lemma in the proof of Proposition 4.3.

Lemma 4.2. *Let $f\colon M^m \longrightarrow N^n$ be a smooth map. If the differential $\mathrm{d}_p f$ is injective for $p \in M$, there exist a neighborhood U of p in M and a chart $\psi = (y_1, \ldots, y_n) : V \longrightarrow \mathbb{R}^n$ around $f(p) \in N$ such that*

$$\varphi = (y_1 \circ f, \ldots, y_m \circ f)\colon U \longrightarrow \mathbb{R}^m$$

is a chart around $p \in M$.

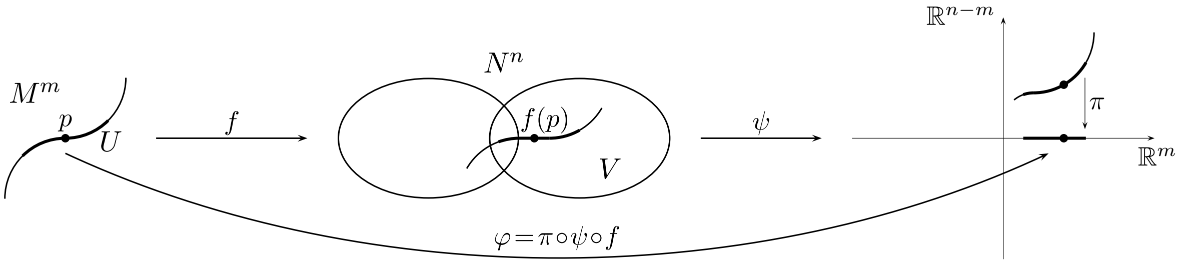


Figure 1.8: An immersion pull-backs a subset of the coordinates on the target to a chart on the domain

Since the differential $\mathrm{d}_p f\colon T_pM \longrightarrow T_{f(p)}N$ is injective, its dual

$$f^* = \{\mathrm{d}_p f\}^* \colon T^*_{f(p)}N \longrightarrow T^*_pM$$

is surjective. Thus, if $\psi = (y_1, \ldots, y_n) : V \longrightarrow \mathbb{R}^n$ is any chart around $f(p) \in N$, then the set

$$\big\{ f^* \mathrm{d}_{f(p)} y_i = \mathrm{d}_p(y_i \circ f) \big\}_i$$

spans T^*_pM (because the set $\{\mathrm{d}_{f(p)} y_i\}$ is a basis for $T^*_{f(p)}N$). By Corollary 3.13, a subset of $\{y_i \circ f\}_i$ determines a chart around p on M. If this subset is different from $\{y_1 \circ f, \ldots, y_m \circ f\}$, compose ψ with a diffeomorphism of $\mathbb{R}^n$ that switches the coordinates, sending the chosen coordinates (those in the subset) to the first m coordinates.

The statement of Lemma 4.2 is illustrated in Figure 1.8. In summary, if $\mathrm{d}_p f$ is injective, then m of the coordinates of a chart around $f(p)$ give rise to a chart around p. By re-ordering the coordinates around $f(p)$, it can be assumed that it is the first m coordinates that give rise to a chart around p, which is then $\varphi = \pi \circ \psi \circ f$, where $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the projection on the first m coordinates. In particular,

$$\pi \colon \psi(f(U)) \longrightarrow \varphi(U) \subset \mathbb{R}^m$$

is bijective; so the image of $f(U) \subset V \subset N$ under ψ is the graph of some function $g\colon \varphi(U) \longrightarrow \mathbb{R}^{n-m}$:

$$\psi(f(U)) = \big\{ (x, g(x)) \colon x \in \varphi(U) \big\}.$$

By construction,

$$\psi(f(p')) = \big(y_1(f(p')), \ldots, y_n(f(p'))\big) = \big(\varphi(p'), g(\varphi(p'))\big) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \qquad \forall\, p' \in U;$$

so $g = (y_{m+1}, \ldots, y_n) \circ f \circ \varphi^{-1}$. In the proof of the next proposition, we compose ψ with the diffeomorphism $(x, y) \longrightarrow (x, y - g(x))$ so that the image of $f(U)$ is shifted to $\mathbb{R}^m \times 0$.

Proposition 4.3 (Slice Lemma). *Let $f : M^m \longrightarrow N^n$ be a smooth map. If $\mathrm{d}_p f$ is injective for some $p \in M$, there exist charts*

$$\varphi\colon U \longrightarrow \mathbb{R}^m \qquad \textit{and} \qquad \psi\colon V \longrightarrow \mathbb{R}^n$$

around $p \in M$ and $f(p) \in N$, respectively, such that the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \mathbb{R}^m \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

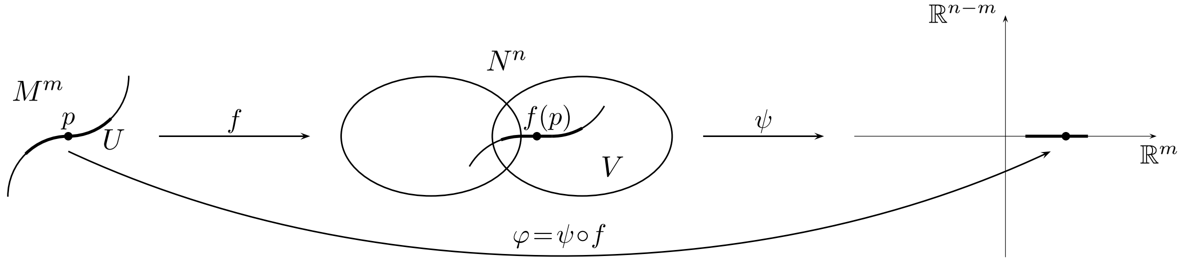


Figure 1.9: The local structure of immersions

commutes, where the bottom arrow is the natural inclusion of $\mathbb{R}^m$ *as* $\mathbb{R}^m{\times}0$*, and* $f(U)=\psi^{-1}(\mathbb{R}^m{\times}0)$*.*

By Lemma 4.2, there exist a neighborhood U of p in M and a chart $\psi'=(y_1,\ldots,y_n):V'\longrightarrow\mathbb{R}^n$ around $f(p)\in N$ such that

$$\varphi = \pi\circ\psi'\circ f : U \longrightarrow \mathbb{R}^m$$

is a chart around $p\in M$, where $\pi\colon\mathbb{R}^n\longrightarrow\mathbb{R}^m$ is the projection on the first m coordinates as before. In particular, $\varphi(U)\subset\mathbb{R}^m$ is an open subset and

$$\psi'\circ f = \big(\varphi, g\circ\varphi\big)\colon U \longrightarrow \mathbb{R}^m\times\mathbb{R}^{n-m},$$

where $g=(y_{m+1},\ldots,y_n)\circ f\circ\varphi^{-1}:\varphi(U)\longrightarrow\mathbb{R}^{n-m}$; this is a smooth function. Thus, the map

$$\Theta\colon \varphi(U)\times\mathbb{R}^{n-m} \longrightarrow \varphi(U)\times\mathbb{R}^{n-m}, \qquad (x,y)\longrightarrow \big(x, y-g(x)\big),$$

is smooth. It is clearly bijective, and

$$\mathcal{J}(\Theta)_{(x,y)} = \begin{pmatrix} \mathbb{I}_m & 0 \\ * & \mathbb{I}_{n-m} \end{pmatrix};$$

so Θ is a diffeomorphism. Let $V=\psi'^{-1}(\varphi(U)\times\mathbb{R}^{n-m})$ and

$$\psi=\Theta\circ\psi'\colon V \longrightarrow \mathbb{R}^n.$$

Since $\varphi(U)\times\mathbb{R}^{n-m}$ is open in $\mathbb{R}^n$, V is open in N. Since Θ is a diffeomorphism, ψ is a chart on N. Since $\psi'(V')$ and $\varphi(U)\times\mathbb{R}^{n-m}$ contain $\psi'(f(U))$, $f(U)$ is contained in V. By definition,

$$\begin{aligned}\psi\circ f(p') = \Theta\circ\psi'\circ f(p') = \Theta\big(\varphi(p'), g(\varphi(p'))\big) &= \big(\varphi(p'), g(\varphi(p')) - g(\varphi(p'))\big)\\ &= \big(\varphi(p'), 0\big)\in\varphi(U)\times 0 \quad \forall\, p'\in U.\end{aligned}$$

Since $\psi(f(U))=\varphi(U)=\psi(V)\cap\mathbb{R}^m\times 0$, $f(U)=\psi^{-1}(\mathbb{R}^m\times 0)$.

Corollary 4.4. *If* $M^m\subset N^n$ *is a submanifold, for every* $p\in M$ *there exists a chart* $\psi\colon V\longrightarrow\mathbb{R}^n$ *on* N *around* p *such that* $M\cap V=\psi^{-1}(\mathbb{R}^m\times 0)$ *and*

$$\psi\colon M\cap V \longrightarrow \mathbb{R}^m\times 0 = \mathbb{R}^m$$

is a chart on M*.*

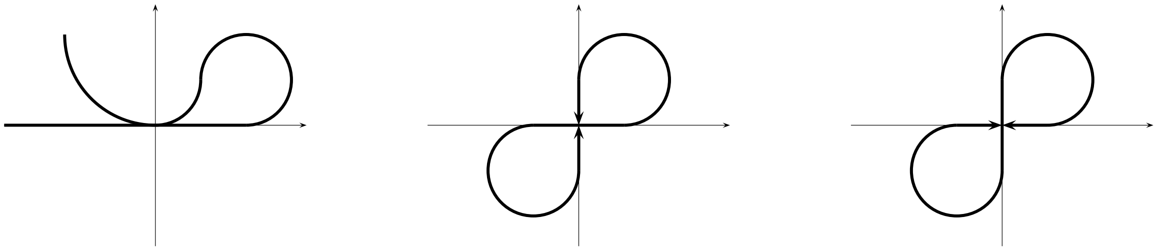


Figure 1.10: Images of some immersions $\mathbb{R} \longrightarrow \mathbb{R}^2$

Let U be an open neighborhood of p in M and (V,ψ) a chart on N around $p = f(p)$ provided by Proposition 4.4 for the inclusion map $f : M \longrightarrow N$. Since $M \subset N$ has subspace topology, there exists $W \subset V$ open so that $U = M \cap W$; the chart $(W, \psi|_W)$ then has the desired properties.

Proposition 4.3 completely describes the local structure of immersions, but says nothing about their global structure. Images of 3 different immersions of $\mathbb{R}$ into $\mathbb{R}^2$ are shown in Figure 1.10. Another type phenomena is illustrated by the injective immersion

$$\mathbb{R} \longrightarrow S^1 \times S^1, \qquad s \longrightarrow \big(\mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}\alpha s}\big), \tag{4.2}$$

where $\alpha \in \mathbb{R} - \mathbb{Q}$. The image of this immersion is a dense submanifold of $S^1 \times S^1$.

If $f\colon M \longrightarrow N$ is an injective map and $h\colon X \longrightarrow N$ is any map such that $h(X) \subset f(M)$, then there exists a unique map $h_0\colon X \longrightarrow M$ so that the diagram

$$\begin{tikzcd} & M \arrow[d, "f"] \\ X \arrow[ur, dashed, "h_0"] \arrow[r, "h"] & N \end{tikzcd}$$

commutes. If M, N, and X are topological spaces, f is an embedding, and h is continuous, then h_0 is also continuous [5, Theorem 7.2e]. An analogue of this property holds in the smooth category, as indicated by the next proposition.

Proposition 4.5. *Let $f\colon M \longrightarrow N$ be an injective immersion, $h\colon X \longrightarrow N$ a smooth map such that $h(X) \subset f(M)$, and $h_0\colon X \longrightarrow M$ the unique map such that $h = f \circ h_0$. If h_0 is continuous, then it is smooth; in particular, h_0 is smooth if f is an embedding (e.g. if M is a submanifold of N).*

It is sufficient to show that every point $q \in X$ has a neighborhood W on which h_0 is smooth. By Proposition 4.3, there exist charts

$$\varphi\colon U \longrightarrow \mathbb{R}^m \qquad \text{and} \qquad \psi\colon V \longrightarrow \mathbb{R}^n$$

around $h_0(q) \in M$ and $h(q) = f(h_0(q)) \in N$ such that the diagram

$$\begin{tikzcd} & U \arrow[r, "\varphi"] \arrow[d, "f"] & \mathbb{R}^m \arrow[d] \\ W \arrow[ur, dashed, "h_0"] \arrow[r, "h"] & V \arrow[r, "\psi"] & \mathbb{R}^n \end{tikzcd}$$

commutes, where $W=h_0^{-1}(U)$ and the right-most arrow is the standard inclusion of $\mathbb{R}^m$ as $\mathbb{R}^m\times 0$ in $\mathbb{R}^n$. Since h_0 is continuous, W is open in X. Since h is smooth and ψ is a chart on N, the map

$$\psi\circ h = \psi\circ f\circ h_0 = \big(\varphi\circ h_0, 0\big)\colon W \longrightarrow \mathbb{R}^m\times\mathbb{R}^{n-m}$$

is smooth. Thus, the map $\varphi\circ h_0\colon W\longrightarrow\mathbb{R}^m$ is also smooth. Since φ is a chart on M containing the image of $h_0|_W$, it follows that $h_0|_W$ is a smooth map.

It is possible for the map h_0 to be continuous even if $f\colon M\longrightarrow N$ is not an embedding (and even if the image of h is not contained in the image of any open subset of M on which f is an embedding). This is in particular the case for the immersion (4.2) and more generally for any integral immersion of a completely integral distribution. Such immersions may or may not be embeddings, but the map h_0 is necessarily continuous for them; see Proposition 5.11 below. On the other hand, h_0 need not be continuous in general. For example, it is not continuous at $h^{-1}(0)$ if f and h are immersions described by the middle and right-most diagrams, respectively, in Figure 1.10. A similar example can be obtained from the left diagram in Figure 1.10 if all branches of the curve have infinite contact with the x-axis at the origin (f and h can then differ by a "a branch switch" at the origin).

Corollary 4.6. *Let N be a smooth manifold, $M\subset N$, and $\iota\colon M\longrightarrow N$ the inclusion map.*

(1) If $\mathcal{T}_M$ is a topology on M, there exists at most one smooth structure $\mathcal{F}_M$ on $(M,\mathcal{T}_M)$ with respect to which ι is an immersion.

(2) If $\mathcal{T}_M$ is the subspace topology on M and $(M,\mathcal{T}_M)$ admits a smooth structure $\mathcal{F}_M$ with respect to which ι is an immersion, there exists no other topology $\mathcal{T}_M'$ admitting a smooth structure $\mathcal{F}_M'$ on M with respect to which ι is an immersion.

The first statement of this corollary follows easily from Proposition 4.5. The second statement depends on manifolds being second-countable; its proof makes use of Exercise 24.

Corollary 4.7. *A topological subspace $M\subset N$ admits a smooth structure with respect to which M is a submanifold of N if and only if for every $p\in M$ there exists a neighborhood U of p in N such that the topological subspace $M\cap U$ of N admits a smooth structure with respect to which $M\cap U$ is a submanifold of N.*

By Corollary 4.6, the smooth structures on the overlaps of such open subsets must agree.

The middle and right-most diagrams in Figure 1.10 are examples of a subset M of a smooth manifold N that admits two different manifold structures $(M,\mathcal{T}_M,\mathcal{F}_M)$, in different topologies, with respect to which the inclusion map $\iota\colon M\longrightarrow N$ is an embedding. In light of the second statement of Proposition 4.6, this is only possible because M does not admit such a smooth structure in the subspace topology. On the other hand, if manifolds were not required to be second-countable, the discrete topology on $\mathbb{R}$ would provide a second manifold structure with respect to which the identity map $\mathbb{R}\longrightarrow\mathbb{R}$, with the target $\mathbb{R}$ having the standard manifold structure, would be an immersion.

5 Implicit Function Theorems

This section is in a sense dual to Section 4. It describes ways of constructing new immersions and submanifolds by studying properties of submersions (smooth maps with surjective differentials),

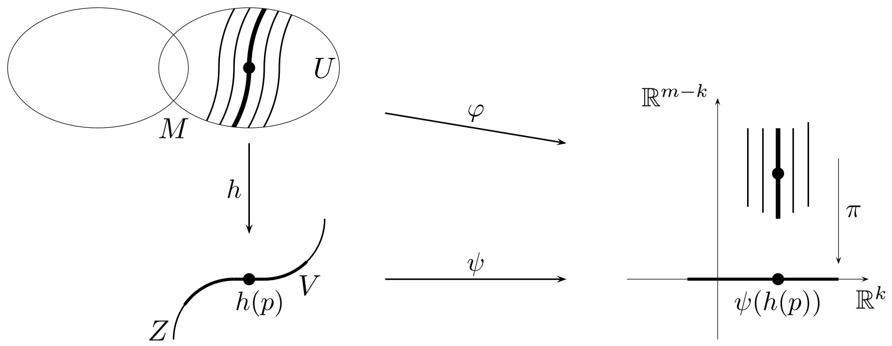


Figure 1.11: The local structure of submersions

rather than studying properties of immersions and submanifolds. While Section 4 exploits Corollary 3.13, this section makes use of Corollary 3.14, as well as of the Slice Lemma. We will use the following lemma in the proof of the Implicit Function Theorem for Manifolds, Theorem 5.3, below.

Lemma 5.1. *Let $h\colon M^m \longrightarrow Z^k$ be a smooth map. If the differential $\mathrm{d}_p h$ is surjective for some $p\in M$, there exist charts*

$$\varphi\colon U \longrightarrow \mathbb{R}^m \qquad \text{and} \qquad \psi\colon V \longrightarrow \mathbb{R}^k$$

around $p\in M$ and $h(p)\in Z$, respectively, such that the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "h"'] & \mathbb{R}^m \arrow[d] \\ V \arrow[r, "\psi"] & \mathbb{R}^k \end{tikzcd}$$

commutes, where the right arrow is the natural projection map from $\mathbb{R}^m$ to $\mathbb{R}^k\times 0\subset\mathbb{R}^m$.

Let $\psi=(y_1,\ldots,y_k)\colon V\longrightarrow\mathbb{R}^k$ be a chart on Z around $f(p)$. Since the differential $\mathrm{d}_p h$ is surjective, its dual map

$$h^*=\{\mathrm{d}_p h\}^*\colon T^*_{h(p)}N \longrightarrow T^*_p M$$

is injective. Since $\{\mathrm{d}_{h(p)} y_i\}$ is a basis for $T^*_{h(p)}N$, it follows that the set

$$\{h^*\mathrm{d}_{h(p)} y_i = \mathrm{d}_p(y_i\circ h)\}$$

is linearly independent in $T^*_p M$. By Corollary 3.14, it can be extended to a chart

$$\varphi\colon \big(y_1\circ f,\ldots,y_k\circ h, x_{k+1},\ldots,x_m\big)\colon U \longrightarrow \mathbb{R}^k\times\mathbb{R}^{m-k}$$

on M, where U is a neighborhood of p in $h^{-1}(V)$.

Lemma 5.1 can be seen as a counter-part of the Slice Lemma (Proposition 4.3). While an immersion locally looks like the inclusion

$$\mathbb{R}^m \longrightarrow \mathbb{R}^m\times 0 \subset \mathbb{R}^n, \qquad m\le n,$$

a submersion locally looks like the projection

$$\mathbb{R}^m \longrightarrow \mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^m, \qquad k \le m.$$

Thus, an immersion can locally be represented by a horizontal slice in a chart, while the pre-image of a point in the target of a submersion is locally a vertical slice (it is customary to present projections vertically, as in Figure 1.11).

Corollary 5.2. *Let $h\colon M \longrightarrow Z$ be a smooth map. If the differential $\mathrm{d}_p h$ is surjective for some $p \in M$, there exist a neighborhood U of p in M and a smooth structure on the subspace $h^{-1}(h(p)) \cap U$ of M so that $h^{-1}(h(p)) \cap U$ is a submanifold of M and*

$$\mathrm{codim}_M\big(h^{-1}(h(p)) \cap U\big) \equiv \dim M - \dim\big(h^{-1}(h(p)) \cap U\big) = \dim Z.$$

If $\psi\colon V \longrightarrow \mathbb{R}^k$ and $\varphi = (\psi \circ h, \phi) : U \longrightarrow \mathbb{R}^k \times \mathbb{R}^{m-k}$ are charts on Z around $h(p)$ and on M around p, respectively, provided by Lemma 5.1,

$$h^{-1}(h(p)) \cap U = \{\psi \circ h\}^{-1}\big(\psi(h(p))\big) \cap U = \{\pi \circ \varphi\}^{-1}\big(\psi(h(p))\big) = \varphi^{-1}\big(\psi(h(p)) \times \mathbb{R}^{m-k}\big).$$

Since $\varphi\colon U \longrightarrow \varphi(U)$ is a homeomorphism, so is the map

$$\varphi\colon h^{-1}(h(p)) \cap U \longrightarrow \psi(h(p)) \times \mathbb{R}^{m-k} \cap \varphi(U)$$

in the subspace topologies. Thus,

$$\phi\colon h^{-1}(h(p)) \cap U \longrightarrow \mathbb{R}^{m-k}$$

induces a smooth structure on $h^{-1}(h(p)) \cap U \subset M$ in the subspace topology with respect to which the inclusion $h^{-1}(h(p)) \cap U \longrightarrow M$ is an immersion because so is the inclusion

$$\psi(h(p)) \times \mathbb{R}^{m-k} \longrightarrow \mathbb{R}^k \times \mathbb{R}^{m-k}.$$

Theorem 5.3 (Implicit Function Theorem for Manifolds)**.** *Let $f\colon M \longrightarrow N$ be a smooth map and $Y \subset N$ an embedded submanifold. If*

$$T_{f(p)}N = \mathrm{Im}\,\mathrm{d}_p f + T_{f(p)}Y \qquad \forall\, p \in f^{-1}(Y), \tag{5.1}$$

then $f^{-1}(Y)$ has a structure of an embedded submanifold of M and $\mathrm{codim}_M f^{-1}(Y) = \mathrm{codim}_N Y$.

By Corollary 4.7, it is sufficient to show that for every $p \in f^{-1}(Y)$ there exists a neighborhood U of p in M such that $f^{-1}(Y) \cap U$ has a structure of an embedded submanifold of M. As provided by Corollary 4.4, let $\psi\colon V \longrightarrow \mathbb{R}^n$ be a chart on N around $f(p) \in Y$ such that $Y \cap V = \psi^{-1}(\mathbb{R}^l \times 0)$, where $l = \dim Y$. Let $\tilde\pi\colon \mathbb{R}^n \longrightarrow 0 \times \mathbb{R}^{n-l}$ be the projection map and

$$h = \tilde\pi \circ \psi \circ f : f^{-1}(V) \longrightarrow V \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^{n-l}.$$

Since $\mathbb{R}^l \times 0 = \tilde\pi^{-1}(0)$, $Y \cap V = \psi^{-1}(\tilde\pi^{-1}(0))$ and

$$f^{-1}(Y) \cap f^{-1}(V) = f^{-1}(Y \cap V) = f^{-1}\big(\psi^{-1}(\tilde\pi^{-1}(0))\big) = h^{-1}(0). \tag{5.2}$$

On the other hand, by the chain rule (3.14)

$$\mathrm{d}_p h = \mathrm{d}_{\psi(f(p))}\tilde\pi \circ \mathrm{d}_{f(p)}\psi \circ \mathrm{d}_p f \colon T_pM \longrightarrow T_{f(p)}N \longrightarrow T_{\psi(f(p))}\mathbb{R}^n \longrightarrow T_0(0 \times \mathbb{R}^{n-l}). \tag{5.3}$$

The homomorphism $\mathrm{d}_{\psi(f(p))}\tilde{\pi}$ is onto, as is the homomorphism $\mathrm{d}_{f(p)}\psi$. On the other hand,

$$\mathrm{d}_{\psi(f(p))}\tilde{\pi}\circ\mathrm{d}_{f(p)}\psi = \mathrm{d}_{f(p)}(\tilde{\pi}\circ\psi)\colon T_{f(p)}N \longrightarrow T_{\psi(f(p))}\mathbb{R}^n \longrightarrow T_0(0\times\mathbb{R}^{n-l})$$

by the chain rule (3.14) and thus vanishes on $T_{f(p)}Y$ (since $\tilde{\pi}\circ\psi$ maps Y to 0 in $0\times\mathbb{R}^{n-l}$). So, by (5.1), the restriction

$$\mathrm{d}_{\psi(f(p))}\tilde{\pi}\circ\mathrm{d}_{f(p)}\psi\colon \mathrm{Im}\,\mathrm{d}_p f \longrightarrow T_0(0\times\mathbb{R}^{n-l})$$

is onto, i.e. the homomorphism (5.3) is surjective. By Corollary 5.2 and (5.2), there exists a neighborhood U of p in $f^{-1}(V)$ such that

$$f^{-1}(Y)\cap U = f^{-1}(Y)\cap f^{-1}(V)\cap U = h^{-1}(0)\cap U$$

admits a structure of an embedded submanifold of M, as required.

Corollary 5.4. *Let $f\colon M\longrightarrow N$ be a smooth map and $q\in N$. If*

$$\mathrm{d}_p f\colon T_pM \longrightarrow T_qN \quad \textit{is onto} \quad \forall\; p\in f^{-1}(q), \tag{5.4}$$

then $f^{-1}(q)$ has a structure of an embedded submanifold of M and $\mathrm{codim}_M f^{-1}(q) = \dim N$.

This is just the $Y=\{q\}$ case of Theorem 5.3.

Example 5.5. Let $f:\mathbb{R}^{m+1}\longrightarrow\mathbb{R}$ be given by $f(x)=|x|^2$. This is a smooth map, and its differential at $x\in\mathbb{R}^{m+1}$ with respect to the standard bases for $T_x\mathbb{R}^{m+1}$ and $T_{f(x)}\mathbb{R}$ is

$$\mathcal{J}(f)_x = \begin{pmatrix}2x_1 & 2x_2 & \ldots & 2x_{m+1}\end{pmatrix}\colon \mathbb{R}^{m+1}\longrightarrow\mathbb{R}.$$

Thus, $\mathrm{d}_x f$ is surjective if and only if $x\neq 0$, i.e. $f(x)\neq 0$. By Corollary 5.4, $f^{-1}(q)$ with $q\neq 0$ then has a structure of an embedded submanifold of $\mathbb{R}^{m+1}$ and its codimension is 1 (so the dimension is m). This is indeed the case, since $f^{-1}(q)$ is the sphere of radius $\sqrt{q}$ centered at the origin if $q>0$ and the empty set (which is a smooth manifold of any dimension) if $q<0$. If $q=0$, $f^{-1}(q)=\{0\}$; this happens to be a smooth submanifold of $\mathbb{R}^{m+1}$, but of the wrong dimension.

Example 5.6. Corollary 5.4 can be used to show that the group SO_n is a smooth submanifold of $\mathrm{Mat}_{n\times n}\mathbb{R}$, while U_n and SU_n are smooth submanifolds of $\mathrm{Mat}_{n\times n}\mathbb{C}$. For example, with $\mathrm{Symm}_{n\times n}\mathbb{R}$ denoting the space of symmetric $n\times n$ real matrices, define

$$f\colon \mathrm{Mat}_{n\times n}\mathbb{R}\longrightarrow \mathrm{Symm}_{n\times n}\mathbb{R}, \qquad \text{by} \quad f(A)=AA^{\mathrm{tr}}.$$

Then, $O(n)=f^{-1}(\mathbb{I}_n)$. It is then sufficient to show that the differential $\mathrm{d}_A f$ is onto for all $A\in O(n)$. Since $f=f\circ R_A$ for every $A\in O(n)$, where the diffeomorphism

$$R_A\colon \mathrm{Mat}_{n\times n}\mathbb{R}\longrightarrow \mathrm{Mat}_{n\times n}\mathbb{R} \qquad \text{is given by} \quad R_A(B)=BA,$$

it is sufficient to establish that $\mathrm{d}_{\mathbb{I}} f$ is surjective. This is a direct check.

Corollary 5.7 (Implicit Function Theorem for Maps)**.** *Let $f\colon X\longrightarrow M$ and $g\colon Y\longrightarrow M$ be smooth maps. If*

$$T_{f(x)}M = \mathrm{Im}\,\mathrm{d}_x f + \mathrm{Im}\,\mathrm{d}_y g \qquad \forall\; (x,y)\in X\times Y \;\; \textit{s.t}\; f(x)=g(y), \tag{5.5}$$

then the space

$$X\times_M Y \equiv \big\{(x,y)\in X\times Y\colon\; f(x)=g(y)\big\}$$

has a structure of an embedded submanifold of $X\times Y$ and its codimension equals to the dimension of M. Furthermore, the projection map $\pi_1=\pi_X: X\times_M Y\longrightarrow X$ is injective (immersion) if $g\colon Y\longrightarrow M$ is injective (immersion).

This corollary is obtained by applying Theorem 5.3 to the smooth map

$$h=(f,g)\colon X\times Y\longrightarrow M\times M.$$

Its last statement immediately implies Warner's Theorem 1.39. The commutative diagram

$$\begin{tikzcd}
X\times_M Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & M
\end{tikzcd}$$

is known as a fibered square.

Corollary 5.8 (Implicit Function Theorem for Intersections). *Let $X,Y\subset M$ be embedded submanifolds. If*

$$T_pM=T_pX+T_pY \qquad \forall\, p\in X\cap Y, \tag{5.6}$$

then $X\cap Y$ is a smooth submanifold of X, Y, and M and

$$\dim X\cap Y=\dim X+\dim Y-\dim M.$$

This corollary is a special case of Corollary 5.7.

Remark 5.9. Submanifolds $X,Y\subset M$ satisfying (5.6) are said to be transverse (in M); this is written as $X\pitchfork Y$ or $X\pitchfork_M Y$ to be specific. For example, two distinct lines in the plane are transverse, but two intersecting lines in $\mathbb{R}^3$ are not. Similarly, smooth maps $f: X\longrightarrow M$ and $g\colon Y\longrightarrow M$ satisfying (5.5) are called transverse; this is written as $f\pitchfork g$ or $f\pitchfork_M g$. If $f: M\longrightarrow N$ satisfies (5.1) with respect to a submanifold $Y\subset N$, f is said to transverse to Y; this is written as $f\pitchfork Y$ or $f\pitchfork_N Y$. Finally, if $f: M\longrightarrow N$ satisfies (5.4) with respect to $q\in N$, q is said to be a regular value of f. By Corollary 5.4, the pre-image of a regular value is a smooth submanifold in the domain of codimension equal to the dimension of the target. By Sard's Theorem [2, §2], the set of a regular values is dense in the target (in fact, its complement is a set of measure 0); so the pre-images of most points in the target of a smooth map are smooth submanifolds of the domain, though in some cases they may all be empty (e.g. if the dimension of the domain is lower than the dimension of the target).

The standard version of the Implicit Function Theorem for $\mathbb{R}^m$, Corollary 5.10 below, says that under certain conditions a system of k equations in m variables has a locally smooth $(m-k)$-dimensional space of solutions which can be described as a graph of a function $g\colon\mathbb{R}^{m-k}\longrightarrow\mathbb{R}^k$. It is normally obtained as an application of the Inverse Function Theorem for $\mathbb{R}^m$, Theorem 3.10 above. It can also be deduced from the proof of Lemma 5.1 and by itself implies Corollary 5.2.

Corollary 5.10 (Implicit Function Theorem for $\mathbb{R}^m$). *Let $U\subset\mathbb{R}^{m-k}\times\mathbb{R}^k$ be an open subset and $f: U\longrightarrow\mathbb{R}^k$ a smooth function. If $(x_0,y_0)\in f^{-1}(0)$ is such that the right $k\times k$ submatrix of $\mathcal{J}(f)_{(x_0,y_0)}$, $\frac{\partial f}{\partial y}|_{(x_0,y_0)}$, is non-singular, then there exist open neighborhoods V of x_0 in $\mathbb{R}^{m-k}$ and W of y_0 in $\mathbb{R}^k$ and a smooth function $g\colon V\longrightarrow W$ such that*

$$f^{-1}(0)\cap V\times W=\big\{(x,g(x))\colon x\in V\big\}.$$

Exercises

1. Show that every Hausdorff locally Euclidean space is regular.

2. Show that every regular second-countable space is normal.

3. Show that the collection (1.1) is indeed a smooth structure on M, according to Definition 1.3.

4. Show that the two smooth structures $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{R}^1$ in Example 1.6 are not the same, but $(\mathbb{R}^1, \mathcal{F})$ and $(\mathbb{R}^1, \mathcal{F}')$ are diffeomorphic smooth manifolds.

5. Show that the maps $\varphi_\pm\colon U_\pm \longrightarrow \mathbb{R}^m$ described after Example 1.7 are indeed charts on S^m and the overlap map between them is

$$\varphi_+ \circ \varphi_-^{-1}\colon \varphi_-(U_+ \cap U_-) = \mathbb{R}^m - 0 \longrightarrow \varphi_+(U_+ \cap U_-) = \mathbb{R}^m - 0, \qquad x \longrightarrow \frac{x}{|x|^2}\,.$$

6. Show that the map $\varphi_{1/2}$ in Example 1.8 is well-defined and is indeed a homeomorphism.

7. With notation as in Example 1.10, show that

(a) the map $S^{2n+1}/S^1 \longrightarrow (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$ induced by inclusions $S^{2n+1} \longrightarrow \mathbb{C}^{2n+1}$ and $S^1 \longrightarrow \mathbb{C}^*$ is a homeomorphism with respect to the quotient topologies;

(b) the quotient topological space, $\mathbb{C}P^n$, is a compact topological $2n$-manifold which admits a structure of a *complex* (in fact, *algebraic*) n-manifold, i.e. it can be covered by charts whose overlap maps, $\varphi_\alpha \circ \varphi_\beta^{-1}$, are holomorphic maps between open subsets of $\mathbb{C}^n$ (and rational functions on $\mathbb{C}^n$);

(c) $\mathbb{C}P^n$ contains $\mathbb{C}^n$ (with its complex structure) as a dense open subset.

8. Let V and W be finite-dimensional vector spaces with the canonical smooth structures of Example 1.5. Show that the canonical smooth structure on the vector space $V \oplus W = V \times W$ is the same as the product smooth structure.

9. Let $(M, \mathcal{F}_M)$ and $(N, \mathcal{F}_N)$ be smooth manifolds and $\mathcal{F}_{M;0}$ and $\mathcal{F}_{N;0}$ collections of charts on M and N, respectively, generating $\mathcal{F}_M$ and $\mathcal{F}_N$ in the sense of (1.1). Show that a continuous map $f\colon M \longrightarrow N$ is smooth (or a local diffeomorphism) if and only if the map (2.1) is smooth (or a local diffeomorphism) for every $(U, \varphi) \in \mathcal{F}_{M;0}$ and all $(V, \psi) \in \mathcal{F}_{N;0}$ covering $f(U)$.

10. Show that a composition of two smooth maps (local diffeomorphisms, diffeomorphisms) is again smooth (a local diffeomorphism, a diffeomorphism).

11. Let $S^1 \subset \mathbb{C}$ and MB be the unit circle and the infinite Mobius band with the smooth structures of Examples 1.7 and 1.8, respectively. Show that the map

$$\mathrm{MB} = \big([0,1] \times \mathbb{R}\big)/\!\sim\, \longrightarrow S^1, \qquad [s,t] \longrightarrow \mathrm{e}^{2\pi \mathrm{i} s}\,,$$

is well-defined and smooth.

12. Let $(M, \mathcal{F})$ be a smooth m-manifold and $U \subset M$ an open subset. Show that $\mathcal{F}|_U$ is the unique smooth structure on the topological subspace U of M satisfying either of the following two properties:

(SSM1) the inclusion map $\iota\colon U \longrightarrow M$ is a local diffeomorphism;

(SSM2) if N is a smooth manifold, a continuous map $f\colon N \longrightarrow U$ is smooth if and only if the map $\iota\circ f\colon N \longrightarrow M$ is smooth.

13. Let $(M, \mathcal{F}_M)$ and $(N, \mathcal{F}_N)$ be smooth manifolds and $\mathcal{F}_{M\times N}$ the product smooth structure on $M\times N$ of Proposition 1.11. Show that $\mathcal{F}_{M\times N}$ is the unique smooth structure on the product topological space $M\times N$ satisfying either of the following two properties:

(PSM1) the slice inclusion maps $\iota_q\colon M \longrightarrow M\times N$, with $q\in N$, and $\iota_p\colon M \longrightarrow M\times N$, with $p\in M$, and the projection maps $\pi_M, \pi_N\colon M\times N \longrightarrow M, N$ are smooth;

(PSM2) if X is a smooth manifold, continuous maps $f\colon X \longrightarrow M$ and $g\colon X \longrightarrow N$ are smooth if and only if the map $f\times g\colon X \longrightarrow M\times N$ is smooth.

14. Verify Lemmas 2.3 and 2.4.

15. Verify Proposition 2.5.

16. Show that the actions (2.4), (2.5), and (1.3) satisfy the assumptions of Proposition 2.5 and that the quotient smooth structures on

$$S^1 = \mathbb{R}/\mathbb{Z}, \qquad \mathrm{MB} = (\mathbb{R}\times\mathbb{R})/\mathbb{Z}, \quad \text{and} \quad \mathbb{R}P^n = S^n/\mathbb{Z}_2,$$

are the same as the smooth structures of Examples 1.7, 1.8, and 1.9, respectively.

17. Verify that the addition and product operations on $\tilde{F}_p$ described after Definition 3.1 are well-defined and make $\tilde{F}_p$ an $\mathbb{R}$-algebra.

18. Deduce (3.6) from Lemma 3.5.

19. Verify that the map (3.8) is well-defined and is indeed an $\mathbb{R}$-algebra homomorphism.

20. Verify that the differential $\mathrm{d}_p h$ of a smooth map $h\colon M \longrightarrow N$, as defined in (3.11), is indeed well-defined. In other words, $\mathrm{d}_p h(v)$ is a derivation on $\tilde{F}_{h(p)}$ for all $v \in T_pM$. Show that $\mathrm{d}_p h\colon T_pM \longrightarrow T_{h(p)}N$ is a vector-space homomorphism.

21. Let M be a non-empty compact m-manifold. Show that there exists no immersion $f\colon M \longrightarrow \mathbb{R}^m$.

22. Let M be an embedded submanifold in $\mathbb{R}^n$ and $\iota\colon M \longrightarrow \mathbb{R}^n$ the inclusion map. Show that for every $p\in M$ the image of the differential

$$\mathrm{d}_p\iota\colon T_pM \longrightarrow T_p\mathbb{R}^n = \mathbb{R}^n$$

is the subspace of $\mathbb{R}^n$ consisting of the vectors $\alpha'(0)$, where $\alpha\colon (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^n$ is a smooth map such that $\operatorname{Im}\alpha \subset M$ and $\alpha(0) = p$.

23. Show that the map (4.2) is an injective immersion and that its image is dense in $S^1\times S^1$.

24. Show that a bijective immersion $f\colon M \longrightarrow N$ between two smooth manifolds is a diffeomorphism. *Hint:* you'll need to use that M is second-countable, along with either

(i) if $f\colon U \longrightarrow \mathbb{R}^n$ is a smooth map from an open subset of $\mathbb{R}^m$ with $m<n$, the measure of $f(U) \subset \mathbb{R}^n$ is 0;

(ii) Proposition 4.3 (Slice Lemma) and Baire Category Theorem [5, Theorem 7.2].

25. Verify Corollary 4.6.

26. Show that the smooth structures on S^m of Example 5.5 and Exercise 5 are the same.

27. (a) For what values of $t\in\mathbb{R}$, is the subspace

$$\big\{(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}\colon x_1^2+\ldots+x_n^2-x_{n+1}^2=t\big\}$$

a smooth embedded submanifold of $\mathbb{R}^{n+1}$?

(b) For such values of t, determine the diffeomorphism type of this submanifold (i.e. show that it is diffeomorphic to something rather standard). *Hint:* Draw some pictures.

28. Show that the special unitary group

$$SU_n=\big\{A\in\mathrm{Mat}_n\mathbb{C}\colon \bar{A}^tA=\mathbb{I}_n,\ \det A=1\big\}$$

is a smooth compact manifold. What is its dimension?

29. Verify Corollary 5.7.

30. With notation as in Corollary 5.7, show that every pair of continuous maps $p\colon Z\longrightarrow X$ and $q\colon Z\longrightarrow Y$ such that $f\circ p=g\circ q$ factors through a unique continuous map $r\colon Z\longrightarrow X\times_M Y$,

$$\begin{tikzcd}
Z \arrow[dr, dashed, "r"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& X\times_M Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & M
\end{tikzcd}$$

and that $X\times_M Y$ is the unique (up to homeomorphism) topological space possessing this property for all (p,q) as above. If in addition the assumption (5.5) holds and p and q are smooth, then r is also smooth, and $X\times_M Y$ is the unique (up to diffeomorphism) smooth manifold possessing this property for all (p,q) as above.

31. Verify Corollary 5.8.

32. Deduce Corollary 5.10 from the proof of Lemma 5.1 and Corollary 5.2 from Corollary 5.10.

Proposition 5.11. *Let $f: M \longrightarrow N$ be an injective immersion, $h: X \longrightarrow N$ a smooth map such that $h(X) \subset f(M)$, and $h_0: X \longrightarrow M$ the unique map so that the diagram*

$$\begin{tikzcd} & M \arrow[d, "f"] \\ X \arrow[ur, dashed, "h_0"] \arrow[r, "h"] & N \end{tikzcd}$$

commutes. If there exists a completely integrable distribution $\mathcal{D} \subset TN$ such that $\mathrm{Im}\, \mathrm{d}_p f = \mathcal{D}_{f(p)}$ for all $p \in M$, then the map h_0 is continuous (and thus smooth by Proposition 4.5).

Definition 5.12. *A smooth partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of a smooth manifold M is a collection $\{\eta_\alpha\}_{\alpha \in \mathcal{A}}$ of smooth functions on M with values in $[0,1]$ such that*

(PU1) *the collection $\{\mathrm{supp}\, \eta_\alpha\}_{\alpha \in \mathcal{A}}$ is locally finite;*

(PU2) $\mathrm{supp}\, \eta_\alpha \subset U_\alpha$ *for every $\alpha \in \mathcal{A}$;*

(PU3) $\sum_{\alpha \in \mathcal{A}} \eta_\alpha \equiv 1$.

Chapter 2

Smooth Vector Bundles

6 Definitions and Examples

A *(smooth) real vector bundle* V of rank k over a smooth manifold M is a smoothly varying family of k-dimensional real vector spaces which is locally trivial. Formally, it is a triple (M,V,π), where M and V are smooth manifolds and

$$\pi\colon V \longrightarrow M$$

is a smooth map. For each $p\!\in\! M$, the fiber $V_p\!\equiv\!\pi^{-1}(p)$ of V over p is a real k-dimensional vector space; see Figure 2.1. The vector-space structures in V_p vary smoothly with $p\in M$. This means that the scalar multiplication map

$$\mathbb{R}\times V \longrightarrow V, \qquad (c,v) \longrightarrow c\cdot v, \tag{6.1}$$

and the addition map

$$V\times_M V \equiv \big\{(v_1,v_2)\!\in\! V\!\times\! V\colon \pi(v_1)\!=\!\pi(v_2)\in M\big\} \longrightarrow V, \qquad (v_1,v_2)\longrightarrow v_1\!+\!v_2, \tag{6.2}$$

are smooth. Note that we can add $v_1,v_2\!\in\! V$ only if they lie in the same fiber over M, i.e.

$$\pi(v_1)\!=\!\pi(v_2) \qquad \Longleftrightarrow \qquad (v_1,v_2)\in V\times_M V.$$

The space $V\!\times_M\! V$ is a smooth submanifold of $V\!\times\! V$, by Corollary 5.7. The local triviality condition means that for every point $p\!\in\! M$ there exist a neighborhood U of p in M and a diffeomorphism

$$h\colon V|_U \equiv \pi^{-1}(U) \longrightarrow U\times\mathbb{R}^k,$$

such that h takes every fiber of π to the corresponding fiber of the projection map $\pi_1\colon U\!\times\!\mathbb{R}^k \longrightarrow U$, i.e. $\pi_1\!\circ\! h\!=\!\pi$ on $V|_U$ so that the diagram

$$\begin{tikzcd} V|_U\equiv\pi^{-1}(U) \arrow[rr, "h", "\approx"'] \arrow[dr, "\pi"'] & & U\times\mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

commutes, and the restriction of h to each fiber is linear:

$$h(c_1v_1\!+\!c_2v_2) = c_1h(v_1) + c_2h(v_2) \in x\times\mathbb{R}^k \qquad \forall\ c_1,c_2\!\in\!\mathbb{R},\ v_1,v_2\!\in\! V_x,\ x\!\in\! U.$$

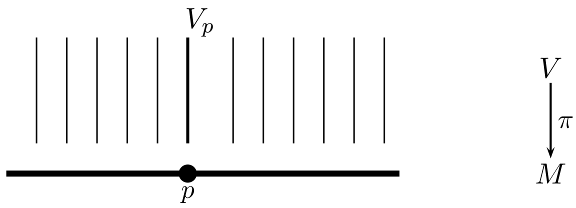


Figure 2.1: Fibers of a vector-bundle projection map are vector spaces of the same rank.

These conditions imply that the restriction of h to each fiber V_x of π is an isomorphism of vector spaces. In summary, V locally (and not just pointwise) looks like bundles of $\mathbb{R}^k$'s over open sets in M glued together. This is in a sense analogous to an m-manifold being open subsets of $\mathbb{R}^m$ glued together in a nice way. Here is a formal definition.

Definition 6.1. *A real vector bundle of rank k is a tuple $(M, V, \pi, \cdot, +)$ such that*

($\mathbb{R}$VB1) *M and V are smooth manifolds and $\pi\colon V \longrightarrow M$ is a smooth map;*

($\mathbb{R}$VB2) *$\cdot\colon \mathbb{R}\times V \longrightarrow V$ is a map s.t. $\pi(c\cdot v) = \pi(v)$ for all $(c, v) \in \mathbb{R}\times V$;*

($\mathbb{R}$VB3) *$+\colon V\times_M V \longrightarrow V$ is a map s.t. $\pi(v_1+v_2) = \pi(v_1) = \pi(v_2)$ for all $(v_1, v_2) \in V\times_M V$;*

($\mathbb{R}$VB4) *for every point $p \in M$ there exist a neighborhood U of p in M and a diffeomorphism $h\colon V|_U \longrightarrow U\times\mathbb{R}^k$ such that*

($\mathbb{R}$VB4-a) *$\pi_1\circ h = \pi$ on $V|_U$ and*

($\mathbb{R}$VB4-b) *the map $h|_{V_x}\colon V_x \longrightarrow x\times\mathbb{R}^k$ is an isomorphism of vector spaces for all $x \in U$.*

The spaces M and V are called the base and the total space of the vector bundle (M, V, π). It is customary to call $\pi : V \longrightarrow M$ a vector bundle and V a vector bundle over M. If M is an m-manifold and $V \longrightarrow M$ is a real vector bundle of rank k, then V is an $(m+k)$-manifold. Its local coordinate charts are obtained by restricting the trivialization maps h for V, as above, to small coordinate charts in M.

Example 6.2. If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1\colon M\times\mathbb{R}^k \longrightarrow M$$

is a real vector bundle of rank k over M. It is called the trivial rank k real vector bundle over M.

Example 6.3. Let $M = S^1$ be the unit circle and $V = \mathrm{MB}$ the infinite Mobius band of Example 1.8. With notation as in Example 1.8, the map

$$\pi\colon V \longrightarrow M, \qquad [s,t] \longrightarrow \mathrm{e}^{2\pi \mathrm{i} s},$$

defines a real line bundle (i.e. rank one bundle) over S^1. Trivializations of this vector bundle can be constructed as follows. With $U_\pm = S^1 - \{\pm 1\}$, let

$$h_+\colon V|_{U_+} \longrightarrow U_+\times\mathbb{R}, \qquad [s,t] \longrightarrow \big(\mathrm{e}^{2\pi \mathrm{i} s}, t\big);$$
$$h_-\colon V|_{U_-} \longrightarrow U_-\times\mathbb{R}, \qquad [s,t] \longrightarrow \begin{cases} (\mathrm{e}^{2\pi \mathrm{i} s}, t), & \text{if } s \in (1/2, 1]; \\ (\mathrm{e}^{2\pi \mathrm{i} s}, -t), & \text{if } s \in [0, 1/2). \end{cases}$$

Both maps are diffeomorphisms, with respect to the smooth structures of Example 1.8 on MB and of Example 1.7 on S^1. Furthermore, $\pi_1 \circ h_\pm = \pi$ and the restriction of $h_\pm$ to each fiber of π is a linear map to $\mathbb{R}$.

Example 6.4. Let $\mathbb{R}P^n$ be the real projective space of dimension n described in Example 1.9 and

$$\gamma_n = \big\{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in \ell\big\}.$$

If $U_i \subset \mathbb{R}P^n$ is as in Example 1.9, the map

$$h_i : \gamma_n \cap U_i \times \mathbb{R}^{n+1} \longrightarrow U_i \times \mathbb{R}, \qquad \big(\ell, (v_0, \ldots, v_n)\big) \longrightarrow (\ell, v_i),$$

is a homeomorphism. The overlap maps,

$$h_i \circ h_j^{-1} : U_i \cap U_j \times \mathbb{R} \longrightarrow U_i \cap U_j \times \mathbb{R}, \qquad (\ell, c) \longrightarrow \big(\ell, (X_i/X_j)c\big),$$

are smooth. By Lemma 2.3, the collection $\{(\gamma_n \cap U_i \times \mathbb{R}^{n+1}, h_i)\}$ of generalized charts then induces a smooth structure on the topological subspace $\gamma_n \subset \mathbb{R}P^n \times \mathbb{R}^{n+1}$. With this smooth structure, γ_n is an embedded submanifold of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ and the projection on the first component,

$$\pi = \pi_1 : \gamma_n \longrightarrow \mathbb{R}P^n\,,$$

defines a smooth real line bundle. The fiber over a point $\ell \in \mathbb{R}P^n$ is the *one-dimensional subspace* ℓ *of* $\mathbb{R}^{n+1}$! For this reason, γ_n is called the tautological line bundle over $\mathbb{R}P^n$. Note that $\gamma_1 \longrightarrow S^1$ is the infinite Mobius band of Example 6.3.

Example 6.5. If M is a smooth m-manifold, let

$$TM = \bigsqcup_{p \in M} T_pM, \qquad \pi : TM \longrightarrow M, \quad \pi(v) = p \;\text{ if } v \in T_pM.$$

If $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^m$ is a smooth chart on M, let

$$\tilde{\varphi}_\alpha : TM|_{U_\alpha} \equiv \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^m, \qquad \tilde{\varphi}_\alpha(v) = \big(\pi(v), \mathrm{d}_{\pi(v)}\varphi_\alpha v\big).$$

If $\varphi_\beta : U_\beta \longrightarrow \mathbb{R}^m$ is another smooth chart, the overlap map

$$\tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^m \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^m$$

is a smooth map between open subsets of $\mathbb{R}^{2m}$. By Lemma 2.4, the collection of generalized charts

$$\big\{(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha) : (U_\alpha, \varphi_\alpha) \in \mathcal{F}_M\big\},$$

where $\mathcal{F}_M$ is the smooth structure of M, then induces a manifold structure on the set TM. With this smooth structure on TM, the projection $\pi : TM \longrightarrow M$ defines a smooth real vector bundle of rank m, called the tangent bundle of M.

Definition 6.6. *A complex vector bundle of rank k is a tuple $(M, V, \pi, \cdot, +)$ such that*

(ℂVB1) *M and V are smooth manifolds and $\pi : V \longrightarrow M$ is a smooth map;*

(ℂVB2) *$\cdot : \mathbb{C} \times V \longrightarrow V$ is a map s.t. $\pi(c \cdot v) = \pi(v)$ for all $(c, v) \in \mathbb{C} \times V$;*

(ℂVB3) $+ : V \times_M V \longrightarrow V$ *is a map s.t.* $\pi(v_1+v_2)=\pi(v_1)=\pi(v_2)$ *for all* $(v_1,v_2)\in V\times_M V$;

(ℂVB4) *for every point* $p\in M$ *there exists a neighborhood* U *of* p *in* M *and a diffeomorphism* $h : V|_U \longrightarrow U\times\mathbb{C}^k$ *such that*

(ℂVB4-a) $\pi_1\circ h=\pi$ *on* $V|_U$ *and*

(ℂVB4-b) *the map* $h|_{V_x} : V_x \longrightarrow x\times\mathbb{C}^k$ *is an isomorphism of complex vector spaces for all* $x\in U$.

Similarly to a real vector bundle, a complex vector bundle over M locally looks like bundles of $\mathbb{C}^k$'s over open sets in M glued together. If M is an m-manifold and $V\longrightarrow M$ is a complex vector bundle of rank k, then V is an $(m+2k)$-manifold. A complex vector bundle of rank k is also a real vector bundle of rank $2k$, but a real vector bundle of rank $2k$ need not in general admit a complex structure.

Example 6.7. If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1 : M\times\mathbb{C}^k \longrightarrow M$$

is a complex vector bundle of rank k over M. It is called the trivial rank-k complex vector bundle over M.

Example 6.8. Let $\mathbb{C}P^n$ be the complex projective space of dimension n described in Example 1.10 and

$$\gamma_n = \big\{(\ell,v)\in\mathbb{C}P^n\times\mathbb{C}^{n+1} : v\in\ell\big\}.$$

The projection $\pi : \gamma_n \longrightarrow \mathbb{C}P^n$ defines a smooth complex line bundle. The fiber over a point $\ell\in\mathbb{C}P^n$ is the *one-dimensional complex subspace* ℓ *of* $\mathbb{C}^{n+1}$. For this reason, γ_n is called the tautological line bundle over $\mathbb{C}P^n$.

Example 6.9. If M is a complex m-manifold, the tangent bundle TM of M is a complex vector bundle of rank m over M.

7 Sections and Homomorphisms

Definition 7.1. (1) *A (smooth)* section *of a (real or complex) vector bundle* $\pi : V \longrightarrow M$ *is a (smooth) map* $s : M\longrightarrow V$ *such that* $\pi\circ s=\mathrm{id}_M$, *i.e.* $s(x)\in V_x$ *for all* $x\in M$.

(2) *A* vector field *on a smooth manifold is a section of the tangent bundle* $TM\longrightarrow M$.

If s is a smooth section, then $s(M)$ is an embedded submanifold of V: the injectivity of s and $\mathrm{d}s$ is immediate from $\pi\circ s=\mathrm{id}_M$, while the embedding property follows from the continuity of π. Every fiber V_x of V is a vector space and thus has a distinguished element, the zero-vector in V_x, which we denote by 0_x. It follows that every vector bundle admits a canonical section, called the zero section,

$$s_0(x) = (x, 0_x) \in V_x.$$

This section is smooth, since on a trivialization of V over an open subset U of M it is given by the inclusion of U as $U\times 0$ into $U\times\mathbb{R}^k$ or $U\times\mathbb{C}^k$. Thus, M can be thought of as sitting inside of V as the zero section, which is a deformation retract of V; see Figure 2.2. The set of all smooth sections of a vector bundle $\pi : V \longrightarrow M$ is denoted by $\Gamma(M;V)$. This is naturally a module over the ring $C^\infty(M)$ of smooth functions on M, since $fs\in\Gamma(M;V)$ whenever $f\in C^\infty(M)$ and $s\in\Gamma(M;V)$.

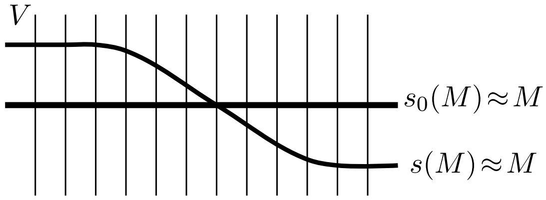


Figure 2.2: The image of a vector-bundle section is an embedded submanifold of the total space.

Definition 7.2. (1) *Suppose $\pi\colon V \longrightarrow M$ and $\pi'\colon V' \longrightarrow N$ are real (or complex) vector bundles. A smooth map $\tilde{f}\colon V \longrightarrow V'$ is a* vector bundle homomorphism *if $\tilde{f}$ descends to a map $f\colon M \longrightarrow N$, i.e. the diagram*

$$\begin{tikzcd} V \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & V' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes, and the restriction $\tilde{f}\colon V_x \longrightarrow V_{f(x)}$ is linear (or $\mathbb{C}$-linear, respectively) for all $x \in M$.

(2) *If $\pi\colon V \longrightarrow M$ and $\pi'\colon V' \longrightarrow M$ are vector bundles, a vector bundle homomorphism $\tilde{f}\colon V \longrightarrow V'$ is an* isomorphism of vector bundles *if $\pi' \circ \tilde{f} = \pi$, i.e. the diagram*

$$\begin{tikzcd} V \arrow[rr, "\tilde{f}"] \arrow[dr, "\pi"'] & & V' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes, and $\tilde{f}$ is a diffeomorphism (or equivalently, its restriction to each fiber is an isomorphism of vector spaces). If such an isomorphism exists, then V and V' are said to be isomorphic vector bundles.

Lemma 7.3. *The real line bundle $V \longrightarrow S^1$ given by the infinite Mobius band of Example 6.3 is not isomorphic to the trivial line bundle $S^1 \times \mathbb{R} \longrightarrow S^1$.*

Proof: In fact, (V, S^1) is not even homeomorphic to $(S^1 \times \mathbb{R}, S^1)$. Since

$$S^1 \times \mathbb{R} - s_0(S^1) \equiv S^1 \times \mathbb{R} - S^1 \times 0 = S^1 \times \mathbb{R}^- \sqcup S^1 \times \mathbb{R}^+,$$

the space $S^1 \times \mathbb{R} - S^1$ is not connected. On the other hand, $V - s_0(S^1)$ is connected. If MB is the standard Mobius Band and $S^1 \subset MB$ is the central circle, $MB - S^1$ is a deformation retract of $V - S^1$. On the other hand, the boundary of MB has only one connected component (this is the primary feature of MB) and is a deformation retract of $MB - S^1$. Thus, $V - S^1$ is connected as well.

Lemma 7.4. *If $\pi : V \longrightarrow M$ is a real (or complex) vector bundle of rank k, V is isomorphic to the trivial real (or complex) vector bundle of rank k over M if and only if V admits k sections $s_1, \ldots, s_k$ such that the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent (over $\mathbb{C}$, respectively) in V_x for all $x \in M$.*

Proof: We consider the real case; the proof in the complex case is nearly identical.
(1) Suppose $h\colon M\times\mathbb{R}^k \longrightarrow V$ is an isomorphism of vector bundles over M. Let $e_1,\ldots,e_k$ be the standard coordinate vectors in $\mathbb{R}^k$. Define sections $s_1,\ldots,s_k$ of V over M by

$$s_l(x) = h\big(x,e_l\big) \qquad \forall\, l=1,\ldots,k,\ x\in M.$$

Since the maps $x\longrightarrow(x,e_l)$ are sections of $M\times\mathbb{R}^k$ over M and h is a bundle homomorphism, the maps s_l are sections of V. Since the vectors (x,e_l) are linearly independent in $x\times\mathbb{R}^k$ and h is an isomorphism on every fiber, the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x for all $x\in M$, as needed.
(2) Suppose $s_1,\ldots,s_k$ are sections of V such that the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x for all $x\in M$. Define the map

$$h\colon M\times\mathbb{R}^k \longrightarrow V \qquad \text{by} \qquad h(x,c_1,\ldots,c_k) = c_1s_1(x)+\ldots+c_ks_k(x)\in V_x.$$

Since the sections $s_1,\ldots,s_k$ and the vector space operations on V are smooth, the map h is smooth. It is immediate that $\pi(h(x,c))=x$ and the restriction of h to $x\times\mathbb{R}^k$ is linear; thus, h is a vector bundle homomorphism. Since the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x, the homomorphism h is injective and thus an isomorphism on every fiber. We conclude that h is an isomorphism between vector bundles over M.

8 Transition Data

Suppose $\pi\colon V\longrightarrow M$ is a real vector bundle of rank k. By Definition 6.1, there exists a collection $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for V such that $\bigcup_{\alpha\in\mathcal{A}}U_\alpha=M$. Since (U_α,h_α) is a trivialization for V,

$$h_\alpha\colon V|_{U_\alpha}\longrightarrow U_\alpha\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_\alpha=\pi$ and the restriction $h_\alpha\colon V_x\longrightarrow x\times\mathbb{R}^k$ is linear for all $x\in U_\alpha$. Thus, for all $\alpha,\beta\in\mathcal{A}$,

$$h_{\alpha\beta}\equiv h_\alpha\circ h_\beta^{-1}\colon \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k \longrightarrow \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_{\alpha\beta}=\pi_1$, i.e. $h_{\alpha\beta}$ maps $x\times\mathbb{R}^k$ to $x\times\mathbb{R}^k$, and the restriction of $h_{\alpha\beta}$ to $x\times\mathbb{R}^k$ defines an isomorphism of $x\times\mathbb{R}^k$ with itself. Such a diffeomorphism must be given by

$$(x,v)\longrightarrow\big(x,g_{\alpha\beta}(x)v\big) \qquad \forall\, v\in\mathbb{R}^k,$$

for a unique element $g_{\alpha\beta}(x)\in\mathrm{GL}_k\mathbb{R}$ (the general linear group of $\mathbb{R}^k$). The map $h_{\alpha\beta}$ is then given by

$$h_{\alpha\beta}(x,v)=\big(x,g_{\alpha\beta}(x)v\big) \qquad \forall\, x\in U_\alpha\cap U_\beta,\ v\in\mathbb{R}^k,$$

and is completely determined by the map $g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}$ (and $g_{\alpha\beta}$ is determined by $h_{\alpha\beta}$). Since $h_{\alpha\beta}$ is smooth, so is $g_{\alpha\beta}$.

Example 8.1. Let $\pi\colon V\longrightarrow S^1$ be the Mobius band line bundle of Example 6.3. If $\{(U_\pm,h_\pm)\}$ is the pair of trivializations described in Example 6.3, then

$$h_-\circ h_+^{-1}\colon U_+\cap U_-\times\mathbb{R}\longrightarrow U_+\cap U_-\times\mathbb{R}, \quad (p,v)\longrightarrow\begin{cases}(p,v), & \text{if } \operatorname{Im}p<0;\\ (p,-v), & \text{if } \operatorname{Im}p>0;\end{cases} = \big(p,g_{-+}(p)v\big),$$

$$\text{where} \qquad g_{-+}\colon U_+\cap U_-=S^1-\{\pm1\}\longrightarrow\mathrm{GL}_1\mathbb{R}=\mathbb{R}^*, \quad g_{-+}(p)=\begin{cases}-1, & \text{if } \operatorname{Im}p>0;\\ 1, & \text{if } \operatorname{Im}p<0.\end{cases}$$

In this case, the transition maps $g_{\alpha\beta}$ are locally constant, which is rarely the case.

By the above, starting with a real rank k vector bundle $\pi\colon V \longrightarrow M$, we can obtain an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth transition maps

$$\big\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}.$$

These transition maps satisfy:

(VBT1) $g_{\alpha\alpha} \equiv \mathbb{I}_k$, since $h_{\alpha\alpha} \equiv h_\alpha\circ h_\alpha^{-1} = \mathrm{id}$;

(VBT2) $g_{\alpha\beta}g_{\beta\alpha} \equiv \mathbb{I}_k$, since $h_{\alpha\beta}h_{\beta\alpha} = \mathrm{id}$;

(VBT3) $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} \equiv \mathbb{I}_k$, since $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} = \mathrm{id}$.

The last condition is known as the (Čech) cocycle condition (more details in Chapter 5 of Warner). It is sometimes written as

$$g_{\alpha_1\alpha_2}g_{\alpha_0\alpha_2}^{-1}g_{\alpha_0\alpha_1} \equiv \mathbb{I}_k \qquad \forall\, \alpha_0,\alpha_1,\alpha_2 \in \mathcal{A}.$$

In light of (VBT2), the two versions of the cocycle condition are equivalent.

Conversely, given an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\big\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfy (VBT1)-(VBT3), we can assemble a rank k vector bundle $\pi'\colon V' \longrightarrow M$ as follows. Let

$$V' = \Big(\bigsqcup_{\alpha\in\mathcal{A}} \alpha\times U_\alpha\times\mathbb{R}^k\Big)\Big/\sim, \qquad \text{where}$$
$$(\beta,x,v) \sim \big(\alpha,x,g_{\alpha\beta}(x)v\big) \quad \forall\; \alpha,\beta\in\mathcal{A},\; x\in U_\alpha\cap U_\beta,\; v\in\mathbb{R}^k.$$

The relation $\sim$ is reflexive by (VBT1), symmetric by (VBT2), and transitive by (VBT3) and (VBT2). Thus, $\sim$ is an equivalence relation, and V' carries the quotient topology. Let

$$q\colon \bigsqcup_{\alpha\in\mathcal{A}} \alpha\times U_\alpha\times\mathbb{R}^k \longrightarrow V' \qquad \text{and} \qquad \pi'\colon V' \longrightarrow M, \quad [\alpha,x,v] \longrightarrow x,$$

be the quotient map and the natural projection map (which is well-defined). If $\beta\in\mathcal{A}$ and W is a subset of $U_\beta\times\mathbb{R}^k$, then

$$q^{-1}\big(q(\beta\times W)\big) = \bigsqcup_{\alpha\in\mathcal{A}} \alpha\times h_{\alpha\beta}(W), \qquad \text{where}$$
$$h_{\alpha\beta}\colon \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k \longrightarrow \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k, \qquad h_{\alpha\beta}(x,v) = \big(x,g_{\alpha\beta}(x)v\big).$$

In particular, if $\beta\times W$ is an open subset of $\beta\times U_\beta\times\mathbb{R}^k$, then $q^{-1}\big(q(\beta\times W)\big)$ is an open subset of $\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times U_\alpha\times\mathbb{R}^k$. Thus, q is an open continuous map. Since its restriction

$$q_\alpha \equiv q|_{\alpha\times U_\alpha\times\mathbb{R}^k}$$

is injective, $(q_\alpha(\alpha\times U_\alpha\times\mathbb{R}^k), q_\alpha^{-1})$ is a chart on V' in the sense of Lemma 2.3. The overlap maps between these charts are the maps $h_{\alpha\beta}$ and thus smooth.[1] Thus, by Lemma 2.3, these charts induce

[1]Formally, the overlap map is $(\beta\longrightarrow\alpha)\times h_{\alpha\beta}$.

a smooth structure on V'. The projection map $\pi'\colon V' \longrightarrow M$ is smooth with respect to this smooth structure, since it induces projection maps on the charts. Since

$$\pi_1 = \pi' \circ q_\alpha \colon \alpha \times U_\alpha \times \mathbb{R}^k \longrightarrow U_\alpha \subset M,$$

the diffeomorphism q_α induces a vector space structure in V'_x for each $x \in U_\alpha$ such that the restriction of q_α to each fiber is a vector space isomorphism. Since the restriction of the overlap map $h_{\alpha\beta}$ to $x \times \mathbb{R}^k$, with $x \in U_\alpha \cap U_\beta$, is a vector space isomorphism, the vector space structures defined on V'_x via the maps q_α and q_β are the same. We conclude that $\pi'\colon V' \longrightarrow M$ is a real vector bundle of rank k.

If $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\big\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$ are transition data arising from a vector bundle $\pi\colon V \longrightarrow M$, then the vector bundle V' constructed in the previous paragraph is isomorphic to V. Let $\{(U_\alpha, h_\alpha)\}$ be the trivializations as above, giving rise to the transition functions $g_{\alpha\beta}$. We define

$$\tilde{f}\colon V \longrightarrow V' \qquad \text{by} \qquad \tilde{f}(v) = \big[\alpha, h_\alpha(v)\big] \quad \text{if } \ \pi(v) \in U_\alpha.$$

If $\pi(v) \in U_\alpha \cap U_\beta$, then

$$\big[\beta, h_\beta(v)\big] = \big[\alpha, h_{\alpha\beta}(h_\beta(v))\big] = \big[\alpha, h_\alpha(v)\big] \in V',$$

i.e. the map $\tilde{f}$ is well-defined (depends only on v and not on α). It is immediate that $\pi' \circ \tilde{f} = \pi$. Since the map

$$q_\alpha^{-1} \circ \tilde{f} \circ h_\alpha^{-1} \colon U_\alpha \times \mathbb{R}^k \longrightarrow \alpha \times U_\alpha \times \mathbb{R}^k$$

is the identity (and thus smooth), $\tilde{f}$ is a smooth map. Since the restrictions of q_α and h_α to every fiber are vector space isomorphisms, it follows that so is $\tilde{f}$. We conclude that $\tilde{f}$ is a vector-bundle isomorphism.

In summary, a real rank k vector bundle over M determines a set of transition data with values in $\mathrm{GL}_k\mathbb{R}$ satisfying (VBT1)-(VBT3) above (many such sets, of course) and a set of transition data satisfying (VBT1)-(VBT3) determines a real rank-k vector bundle over M. These two processes are well-defined and are inverses of each other when applied to the set of equivalence classes of vector bundles and the set of equivalence classes of transition data satisfying (VBT1)-(VBT3). Two vector bundles over M are defined to be equivalent if they are isomorphic as vector bundles over M. Two sets of transition data

$$\big\{g_{\alpha\beta}\big\}_{\alpha,\beta\in\mathcal{A}} \qquad \text{and} \qquad \big\{g'_{\alpha\beta}\big\}_{\alpha,\beta\in\mathcal{A}},$$

with $\mathcal{A}$ consisting of *all* sufficiently small open subsets of M, are said to be equivalent if there exists a collection of smooth functions $\{f_\alpha\colon U_\alpha \longrightarrow \mathrm{GL}_k\mathbb{R}\}_{\alpha\in\mathcal{A}}$ such that

$$g'_{\alpha\beta} = f_\alpha g_{\alpha\beta} f_\beta^{-1}, \qquad \forall \alpha, \beta \in \mathcal{A}, [2]$$

i.e. the two sets of transition data differ by the action of a Čech 0-chain (more in Chapter 5 of Warner). Along with the cocycle condition on the gluing data, this means that isomorphism classes of real rank k vector bundles over M can be identified with $\check{H}^1(M; \mathrm{GL}_k\mathbb{R})$, the quotient of the space of Čech cocycles of degree one by the subspace of Čech boundaries.

[2]Such a collection $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ corresponds, via trivializations, to an isomorphism between the vector bundles determined by $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ and $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$.

Remark 8.2. In Chapter 5 of Warner, Čech cohomology groups, $\check{H}^m$, are defined for (sheafs of) abelian groups. However, the first two groups, $\check{H}^0$ and $\check{H}^1$, easily generalize to non-abelian groups as well.

If $\pi\colon V \longrightarrow M$ is a complex rank k vector bundle over M, we can similarly obtain transition data for V consisting of an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\big\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{C}\big\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfies (VBT1)-(VBT3). Conversely, given such transition data, we can construct a complex rank k vector bundle over M. The set of isomorphism classes of complex rank k vector bundles over M can be identified with $\check{H}^1(M;\mathrm{GL}_k\mathbb{C})$.

9 Operations on Vector Bundles

Vector bundles can be restricted to smooth submanifolds and pulled back by smooth maps. All natural operations on vector spaces, such as taking quotient vector space, dual vector space, direct sum of vector spaces, tensor product of vector spaces, and exterior powers also carry over to vector bundles via transition functions.

Restrictions and pullbacks

If N is a smooth manifold, $M\subset N$ is an embedded submanifold, and $\pi\colon V \longrightarrow N$ is a vector bundle of rank k (real or complex) over N, then its restriction to M,

$$\pi\colon V|_M \equiv \pi^{-1}(M) \longrightarrow M,$$

is a vector bundle of rank k over N. It inherits smooth structure from V by the Slice Lemma or the Implicit Function Theorem, Theorem 5.3. If $\{(U_\alpha, h_\alpha)\}$ is a collection of trivializations for $V \longrightarrow N$, then $\{(M\cap U_\alpha, h_\alpha|_{\pi^{-1}(M\cap U_\alpha)})\}$ is a collection of trivializations for $V|_M \longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V \longrightarrow N$, then $\{g_{\alpha\beta}|_{M\cap U_\alpha\cap U_\beta}\}$ is transition data for $V|_M \longrightarrow M$.

If $f\colon M \longrightarrow N$ is a smooth map and $\pi\colon V \longrightarrow N$ is a vector bundle of rank k, there is a pullback bundle over M:

$$f^*V \equiv M\times_N V \equiv \big\{(p,v)\in M\times V\colon f(p)=\pi(v)\big\} \xrightarrow{\pi_1} M. \tag{9.1}$$

Note that f^*V is the maximal subspace of $M\times V$ so that the diagram

$$\begin{tikzcd} f^*V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & V \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. By Corollary 5.7, f^*V is a smooth submanifold of $M\times V$. By construction, the fiber of π_1 over $p\in M$ is $p\times V_{f(p)}\subset M\times V$, i.e. the fiber of π over $f(p)\in N$:

$$\big(f^*V\big)_p = p\times V_{f(p)} \qquad \forall\, p\in M. \tag{9.2}$$

If $\{(U_\alpha, h_\alpha)\}$ is a collection of trivializations for $V \longrightarrow N$, then $\{(f^{-1}(U_\alpha), h_\alpha\circ f)\}$ is a collection of trivializations for $f^*V \longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V \longrightarrow N$, then $\{g_{\alpha\beta}\circ f\}$

is transition data for $f^*V \longrightarrow M$. The case discussed in the previous paragraph corresponds to f being the inclusion map.

The above pullback operation on vector bundles extends to homomorphisms. Let $f: M \longrightarrow N$ be a smooth map and $\pi_V: V \longrightarrow N$ and $\pi_W: W \longrightarrow N$ be vector bundles. Any vector-bundle homomorphism $\varphi\colon V \longrightarrow W$ over N induces a vector-bundle homomorphism $f^*\varphi\colon f^*V \longrightarrow f^*W$ over M so that the diagram

$$\begin{tikzcd}
f^*V \arrow[rr, "\pi_2"] \arrow[dr, "f^*\varphi"] \arrow[ddr, "\pi_1"'] & & V \arrow[dr, "\varphi"] \arrow[dd, "\pi_V"] & \\
& f^*W \arrow[rr, "\pi_2"] \arrow[d, "\pi_1"] & & W \arrow[dl, "\pi_W"] \\
& M \arrow[r, "f"] & N &
\end{tikzcd} \tag{9.3}$$

commutes. The vector-bundle homomorphism $f^*\varphi$ is given by

$$(f^*\varphi)_p = \mathrm{id}\times\varphi_{f(p)}\colon (f^*V)_p = p\times V_{f(p)} \longrightarrow (f^*W)_p = p\times W_{f(p)}, \qquad (p,v) \longrightarrow \big(p,\varphi(v)\big),$$

where φ_p is the restriction of φ to the fiber $V_{f(p)} = \pi_V^{-1}(f(p))$ over $f(p)\in N$.

If $f\colon M \longrightarrow N$ is a smooth map, then $\mathrm{d}_p f\colon T_pM \longrightarrow T_{(p)}N$ is a linear map which varies smoothly with p. It thus gives rises to a smooth map,

$$\mathrm{d}f\colon TM \longrightarrow TN, \qquad v \longrightarrow \mathrm{d}_{\pi(v)}f(v). \tag{9.4}$$

However, this description of $\mathrm{d}f$ gives no indication that $\mathrm{d}f$ maps $v\in T_pM$ to $T_{f(p)}N$ or that this map is linear on each T_pM. One way to fix this defect is to state that (9.4) is a bundle homomorphism covering the map $f\colon M \longrightarrow N$, i.e. that the diagram

$$\begin{tikzcd}
TM \arrow[r, "\mathrm{d}f"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi'"] \\
M \arrow[r, "f"] & N
\end{tikzcd} \tag{9.5}$$

commutes. Since $(f^*TN)_p = p\times T_{f(p)}N$ by (9.2), another way to fix this is to state that $\mathrm{d}f$ is a bundle homomorphism from TM to f^*TN, i.e. that the diagram

$$\begin{tikzcd}
TM \arrow[rr, "\mathrm{d}f"] \arrow[dr, "\pi"'] & & f^*TN \arrow[r, dashed, "\pi_2"] \arrow[dl, "\pi_1"] & TN \arrow[d, dashed, "\pi'"] \\
& M \arrow[rr, dashed, "f"] & & N
\end{tikzcd} \tag{9.6}$$

commutes. The triangular part of (9.6) is generally the preferred way of describing $\mathrm{d}f$. The description (9.5) factors through the triangular part of (9.6), as indicated by the dashed arrows. The triangular part of (9.6) also leads to a more precise statement of the Implicit Function Theorem, which is rather useful in topology of manifolds; see Theorem 9.2 below.

Quotient Bundles

If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and $V' \subset V$ is a linear subspace, then we can form the quotient vector space, V/V'. If W is another vector space, $W' \subset W$ is a linear subspace, and $g\colon V \longrightarrow W$ is a linear map such that $g(V') \subset W'$, then g descends to a linear map between the quotient spaces:

$$\bar{g}\colon V/V' \longrightarrow W/W'.$$

If we choose bases for V and W such that the first few vectors in each basis form bases for V' and W', then the matrix for g with respect to these bases is of the form:

$$g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

The matrix for $\bar{g}$ is then D. If g is an isomorphism from V to W that restricts to an isomorphism from V' to W', then $\bar{g}$ is an isomorphism from V/V' to W/W'. Any vector-space homomorphism $\varphi\colon V \longrightarrow W$ such that $V' \subset \ker\varphi$ descends to a homomorphism $\bar{\varphi}$, so that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "q"'] & W \\ V/V' \arrow[ur, dashed, "\bar{\varphi}"'] & \end{tikzcd}$$

commutes.

Definition 9.1. *Let $\pi\colon V \longrightarrow M$ be a smooth vector bundle of rank k. A subbundle of V of rank k' is a smooth submanifold V' of V such that $\pi|_{V'}\colon V' \longrightarrow M$ is a vector bundle of rank k'.*

A subbundle of course cannot have a larger rank than the ambient bundle; so $k' \le k$ in Definition 9.1 and the equality holds if and only if $V' = V$.

If $V' \subset V$ is a subbundle, we can form a quotient bundle, $V/V' \longrightarrow M$, such that

$$(V/V')_p = V_p/V'_p \qquad \forall\, p \in M.$$

The topology on V/V' is the quotient topology for the natural surjective map $q : V \longrightarrow V/V'$. The vector-bundle structure on V/V' is determined from those of V and V' by requiring that q be a smooth vector-bundle homomorphism; so if s is a smooth section of V, then $q \circ s$ is a smooth section of V/V'. This also gives a short exact sequence[3] of vector bundles over M,

$$0 \longrightarrow V' \longrightarrow V \xrightarrow{\;q\;} V/V' \longrightarrow 0,$$

where the zeros denote the zero vector bundle $M \times 0 \longrightarrow M$. We can choose a system of trivializations $\{(U_\alpha, h_\alpha)\}_{\alpha \in \mathcal{A}}$ such that

$$h_\alpha\big(V'|_{U_\alpha}\big) = U_\alpha \times (\mathbb{R}^{k'} \times 0) \subset U_\alpha \times \mathbb{R}^k \qquad \forall\, \alpha \in \mathcal{A}. \tag{9.7}$$

Let $q_{k'} : \mathbb{R}^k \longrightarrow \mathbb{R}^{k-k'}$ be the projection onto the last $(k-k')$ coordinates. Then, the trivializations for V/V' are given by $\{(U_\alpha, \{\mathrm{id} \times q_{k'}\} \circ h_\alpha)\}$. Alternatively, if $\{g_{\alpha\beta}\}$ is transition data for V such

[3] exact means that at each position the kernel of the outgoing vector-bundle homomorphism equals the image of the incoming one; short means that it consists of five terms with zeros at the ends

that the upper-left $k' \times k'$-submatrices of $g_{\alpha\beta}$ correspond to V' (as is the case for the above trivializations h_α) and $\bar{g}_{\alpha\beta}$ is the lower-right $(k-k') \times (k-k')$ matrix of $g_{\alpha\beta}$, then $\{\bar{g}_{\alpha\beta}\}$ is transition data for V/V'. Any vector-bundle homomorphism $\varphi\colon V \longrightarrow W$ over M such that $\varphi(v)=0$ for all $v \in V'$ descends to a vector-bundle homomorphism $\bar{\varphi}$ so that $\varphi = \bar{\varphi} \circ q$.

For example, if $\iota\colon Y \longrightarrow N$ is an immersion, the bundle homomorphism $\mathrm{d}\iota$ as in (9.6) is injective and the image of $\mathrm{d}\iota$ in ι^*TN is a subbundle of ι^*TN. In this case, the quotient bundle,

$$\mathcal{N}_N\iota \equiv \iota^*TN/\mathrm{Im}\,\mathrm{d}\iota \longrightarrow Y,$$

is called the normal bundle for the immersion ι. If Y is an embedded submanifold and ι is the inclusion map, TY is a subbundle of $\iota^*TN = TN|_Y$ and the quotient subbundle,

$$\mathcal{N}_N Y \equiv \mathcal{N}_N\iota = \iota^*TN/\mathrm{Im}\,\mathrm{d}\iota = TN|_Y/TY \longrightarrow Y,$$

is called the normal bundle of Y in N; its rank is the codimension of Y in N. If $f: M \longrightarrow N$ is a smooth map and $X \subset M$ is an embedded submanifold, the vector-bundle homomorphism $\mathrm{d}f$ in (9.6) restricts (pulls back by the inclusion map) to a vector-bundle homomorphism

$$\mathrm{d}f|_X\colon TM|_X \longrightarrow (f^*TN)\big|_Y$$

over X, which can be composed with the inclusion homomorphism $TX \longrightarrow TM|_X$,

$$TX \longrightarrow TM|_X \xrightarrow{\mathrm{d}f|_X} (f^*TN)\big|_Y\,.$$

If in addition $f(X) \subset Y$, then the above sequence can be composed with the f^*-pullback of the projection homomorphism $q\colon TN|_Y \longrightarrow \mathcal{N}_N Y$,

$$TX \longrightarrow TM|_X \xrightarrow{\mathrm{d}f|_X} (f^*TN)\big|_Y \xrightarrow{f^*q} f^*\mathcal{N}_N Y\,. \tag{9.8}$$

This composite vector-bundle homomorphism is 0, since $\mathrm{d}_x f(v) \in T_{f(x)}Y$ for all $x \in X$. Thus, it descends to a vector-bundle homomorphism

$$\mathrm{d}f\colon \mathcal{N}_M X \longrightarrow f^*\mathcal{N}_N Y \tag{9.9}$$

over X. If $f \pitchfork_N Y$ as in (5.1), then the map $TM|_X \longrightarrow f^*\mathcal{N}_N Y$ in (9.8) is onto and thus the vector-bundle homomorphism (9.9) is surjective on every fiber. Finally, if $X = f^{-1}(Y)$, the ranks of the two bundles in (9.9) are the same by the last statement in Theorem 5.3, and so (9.9) is an isomorphism of vector bundles over X. Combining this observation with Theorem 5.3, we obtain a more precise statement of the latter.

Theorem 9.2. *Let $f\colon M \longrightarrow N$ be a smooth map and $Y \subset N$ an embedded submanifold. If $f \pitchfork_N Y$ as in (5.1), then $X \equiv f^{-1}(Y)$ is an embedded submanifold of M and the differential $\mathrm{d}f$ induces a vector-bundle isomorphism*

$$\begin{tikzcd}
\mathcal{N}_M X \arrow[rr, "\mathrm{d}f"] \arrow[dr, "\pi"'] & & f^*(\mathcal{N}_N Y) \arrow[dl, "\pi_1"] \\
& M &
\end{tikzcd} \tag{9.10}$$

Since the ranks of $\mathcal{N}_M X$ and $f^*(\mathcal{N}_N Y)$ are the codimensions of X in M and Y in N, respectively, this theorem implies Theorem 9.2. If $Y=\{q\}$ for some point $q\in Y$, then $\mathcal{N}_N Y$ is a trivial vector bundle and thus so is $\mathcal{N}_M X \approx f^*(\mathcal{N}_N Y)$. For example, the unit sphere $S^m \subset \mathbb{R}^{m+1}$ has trivial normal bundle, because

$$S^m = f^{-1}(1), \qquad \text{where} \quad f\colon \mathbb{R}^{m+1} \longrightarrow \mathbb{R}, \quad f(x) = |x|^2.$$

A trivialization of the normal bundle to S^m is given by

$$T\mathbb{R}^{m+1}/TS^m \longrightarrow S^m \times \mathbb{R}, \qquad (x,v) \longrightarrow (x, x\cdot v).$$

Corollary 9.3. *Let $f: X \longrightarrow M$ and $g: Y \longrightarrow M$ be smooth maps. If $f\pitchfork_M g$ as in (5.5), then the space*

$$X\times_M Y \equiv \big\{(x,y)\in X\times Y:\ f(x)=g(y)\big\}$$

is an embedded submanifold of $X\times Y$ and the differential $\mathrm{d}f$ *induces a vector-bundle isomorphism*

$$\begin{tikzcd} \mathcal{N}_{X\times Y}(X\times_M Y) \arrow[rr, "\mathrm{d}(\pi_X\circ f)+\mathrm{d}(\pi_Y\circ g)"] \arrow[dr, "\pi"'] & & \pi_X^* f^*TM = \pi_Y^* g^*TM \arrow[dl] \\ & X\times_M Y & \end{tikzcd} \tag{9.11}$$

Furthermore, the projection map $\pi_1=\pi_X: X\times_M Y \longrightarrow X$ is injective (immersion) if $g: Y\longrightarrow M$ is injective (immersion).

This corollary is obtained by applying Theorem 9.2 to the smooth map

$$f\times g\colon X\times Y \longrightarrow M\times M.$$

All other versions of the Implicit Function Theorem stated in these notes are special cases of this corollary.

Direct Sums

If V and W are two vector spaces, we can form a new vector space, $V\oplus W = V\times W$, the direct sum of V and W. There are natural inclusions $V, W \longrightarrow V\oplus W$ and projections $V\oplus W \longrightarrow V, W$. If $f\colon V\longrightarrow V'$ and $g\colon W\longrightarrow W'$ are linear maps, they induce a linear map

$$f\oplus g\colon V\oplus W \longrightarrow V'\oplus W'.$$

If we choose bases for V, W, V', and W' so that f and g correspond to some matrices A and D, then in the induced bases for $V\oplus W$ and $V'\oplus W'$,

$$f\oplus g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

If $\pi_V: V \longrightarrow M$ and $\pi_W: W \longrightarrow M$ are smooth vector bundles, we can form their direct sum, $V\oplus W$, so that

$$(V\oplus W)_p = V_p\oplus W_p \qquad \forall\, p\in M.$$

The vector-bundle structure on $V \oplus W$ is determined from those of V and W by requiring that either the natural inclusion maps $V, W \longrightarrow V \oplus W$ or the projections $V \oplus W \longrightarrow V, W$ be smooth vector-bundle homomorphisms over M. Thus, if s_V and s_W are sections of V and W, then $s_V \oplus s_W$ is a smooth section of $V \oplus W$ if and only if s_V and s_W are smooth. If $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and W, transition data for $V \oplus W$ is given by $\{g_{\alpha\beta} \oplus g'_{\alpha\beta}\}$, i.e. we put the first matrix in the top left corner and the second matrix in the bottom right corner. Alternatively, let

$$d\colon M \longrightarrow M \times M, \qquad d(p) = (p, p),$$

be the diagonal embedding. Then,

$$\pi_V \times \pi_W \colon V \times W \longrightarrow M \times M$$

is a smooth vector bundle (with the product structure), and

$$V \oplus W = d^*(V \times W).$$

If $V, W \longrightarrow M$ are vector bundles, then V and W are vector subbundles of $V \oplus W$. It is immediate from Section 9 that

$$\big(V \oplus W\big)/V = W \qquad \text{and} \qquad \big(V \oplus W\big)/W = V.$$

These equalities hold in the holomorphic category as well (i.e. when the bundles and the base manifold carry complex structures and all trivializations and transition maps are holomorphic). Conversely, if V' is a subbundle of V, by Section 10 below

$$V \approx (V/V') \oplus V'$$

as smooth vector bundles, real or complex. However, if V and V' are holomorphic bundles, V may not have the same holomorphic structure as $(V/V') \oplus V'$ (i.e. the two bundles are isomorphic as smooth vector bundles, but not as holomorphic ones).

Dual Bundles

If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$), the dual vector space is the space of the linear homomorphisms to the field ($\mathbb{R}$ or $\mathbb{C}$, respectively):

$$V^* = \mathrm{Hom}_{\mathbb{R}}\big(V, \mathbb{R}\big) \qquad \text{or} \qquad V^* = \mathrm{Hom}_{\mathbb{C}}\big(V, \mathbb{C}\big).$$

A linear map $g : V \longrightarrow W$ between two vector spaces, induces a dual map in the "opposite" direction:

$$g^* \colon W^* \longrightarrow V^*, \qquad \big\{g^*(L)\big\}(v) = L\big(g(v)\big) \quad \forall\, L \in W^*,\ v \in V.$$

If $V = \mathbb{R}^k$ and $W = \mathbb{R}^n$, then g is given by an $n \times k$-matrix, and its dual is given by the transposed $k \times n$-matrix.

If $\pi \colon V \longrightarrow M$ is a smooth vector bundle of rank k (say, over $\mathbb{R}$), the dual bundle of V is a vector bundle $V^* \longrightarrow M$ such that

$$(V^*)_p = V_p^* \qquad \forall\, p \in M.$$

The vector-bundle structure on V^* is determined from that of V by requiring that the natural map

$$V \oplus V^* = V \times_M V^* \longrightarrow \mathbb{R} \text{ (or } \mathbb{C}), \qquad (v, L) \longrightarrow L(v), \tag{9.12}$$

be smooth. Thus, if s and ψ are smooth sections of V and V^*, $\psi(s)$ is a smooth function on M. If $\{g_{\alpha\beta}\}$ is transition data for V, i.e. the transitions between charts are given by

$$h_\alpha \circ h_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p, v) \longrightarrow \big(p, g_{\alpha\beta}(p)v\big),$$

the dual transition maps are then given by

$$U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p, v) \longrightarrow \big(p, g_{\alpha\beta}(p)^{\mathrm{tr}} v\big).$$

However, these maps reverse the direction, i.e. they go from the α-side to the β-side. To fix this problem, we simply take the inverse of $g_{\alpha\beta}(p)^{\mathrm{tr}}$:

$$U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p, v) \longrightarrow \big(p, \{g_{\alpha\beta}(p)^{\mathrm{tr}}\}^{-1} v\big).$$

So, transition data for V^* is $\{(g_{\alpha\beta}^{\mathrm{tr}})^{-1}\}$. As an example, if V is a line bundle, then $g_{\alpha\beta}$ is a smooth nowhere-zero function on $U_\alpha \cap U_\beta$ and $(g^*)_{\alpha\beta}$ is the smooth function given by $1/g_{\alpha\beta}$.

Tensor Products

If V and V' are two vector spaces, we can form a new vector space, $V \otimes V'$, the tensor product of V and V'. If $g\colon V \longrightarrow W$ and $g'\colon V' \longrightarrow W'$ are linear maps, they induce a linear map

$$g \otimes g' : V \otimes V' \longrightarrow W \otimes W'.$$

If we choose bases $\{e_j\}$, $\{e'_n\}$, $\{f_i\}$, and $\{f'_m\}$ for V, V', W, and W', respectively, then $\{e_j \otimes e'_n\}_{(j,n)}$ and $\{f_i \otimes f'_m\}_{(i,m)}$ are bases for $V \otimes V'$ and $W \otimes W'$. If the matrices for g and g' with respect to the chosen bases for V, V', W, and W' are $(g_{ij})_{i,j}$ and $(g'_{mn})_{m,n}$, then the matrix for $g \otimes g'$ with respect to the induced bases for $V \otimes V'$ and $W \otimes W'$ is $(g_{ij} g'_{mn})_{(i,m),(j,n)}$. The rows of this matrix are indexed by the pairs (i, m) and the columns by the pairs (j, n). In order to actually write down the matrix, we need to order all pairs (i, m) and (j, n). If all four vector spaces are one-dimensional, g and g' correspond to single numbers g_{ij} and g'_{mn}, while $g \otimes g'$ corresponds to the single number $g_{ij} g'_{mn}$.

If $\pi : V \longrightarrow M$ and $\pi' : V' \longrightarrow M$ are smooth vector bundles, we can form their tensor product, $V \otimes V'$, so that

$$(V \otimes V')_m = V_m \otimes V'_m \qquad \forall\, m \in M.$$

The topology and smooth structure on $V \otimes V'$ are determined from those of V and V' by requiring that if s and s' are smooth sections of V and V', then $s \otimes s'$ is a smooth section of $V \otimes V'$. More explicitly, suppose $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and V'. Then, transition data for $V \otimes V'$ is given by $\{g_{\alpha\beta} \otimes g'_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $g_{\alpha\beta} \otimes g'_{\alpha\beta}$ from $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ as in the previous paragraph. As an example, if V and V' are line bundles, then $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are smooth nowhere-zero functions on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $(g \otimes g')_{\alpha\beta}$ is the smooth function given by $g_{\alpha\beta} g'_{\alpha\beta}$.

Exterior Products

If V is a vector space and k is a nonnegative integer, we can form the k-th exterior power, $\Lambda^k V$, of V. A linear map $g\colon V \longrightarrow W$ induces a linear map

$$\Lambda^k g\colon \Lambda^k V \longrightarrow \Lambda^k W.$$

If n is a nonnegative integer, let $S_k(n)$ be the set of increasing k-tuples of integers between 1 and n:

$$S_k(n) = \big\{(i_1,\ldots,i_k)\in\mathbb{Z}^k\colon 1\le i_1 < i_2 < \ldots < i_k \le n\big\}.$$

If $\{e_j\}_{j=1,\ldots,n}$ and $\{f_i\}_{i=1,\ldots,m}$ are bases for V and W, then $\{e_\eta\}_{\eta\in S_k(n)}$ and $\{f_\mu\}_{\mu\in S_k(m)}$ are bases for $\Lambda^k V$ and $\Lambda^k W$, where

$$e_{(\eta_1,\ldots,\eta_k)} = e_{\eta_1}\wedge\ldots\wedge e_{\eta_k} \qquad \text{and} \qquad f_{(\mu_1,\ldots,\mu_k)} = f_{\mu_1}\wedge\ldots\wedge f_{\mu_k}.$$

If $(g_{ij})_{i=1,\ldots,m,j=1,\ldots,n}$ is the matrix for g with respect to the chosen bases for V and W, then

$$\big(\det\big((g_{\mu_r\eta_s})_{r,s=1,\ldots,k}\big)\big)_{(\mu,\eta)\in I_k(m)\times I_k(n)}$$

is the matrix for $\Lambda^k g$ with respect to the induced bases for $\Lambda^k V$ and $\Lambda^k W$. The rows and columns of this matrix are indexed by the sets $S_k(m)$ and $S_k(n)$, respectively. The (μ,η)-entry of the matrix is the determinant of the $k\times k$-submatrix of $(g_{ij})_{i,j}$ with the rows and columns indexed by the entries of μ and η, respectively. In order to actually write down the matrix, we need to order the sets $S_k(m)$ and $S_k(n)$. If $k=m=n$, then $\Lambda^k V$ and $\Lambda^k W$ are one-dimensional vector spaces, called the top exterior power of V and W, with bases

$$\big\{e_1\wedge\ldots\wedge e_k\big\} \qquad \text{and} \qquad \big\{f_1\wedge\ldots\wedge f_k\big\}.$$

With respect to these bases, the homomorphism $\Lambda^k g$ corresponds to the number $\det(g_{ij})_{i,j}$. If $k>n$ (or $k>m$), then $\Lambda^k V$ (or $\Lambda^k W$) is the zero vector space and the corresponding matrix is empty.

If $\pi\colon V\longrightarrow M$ is a smooth vector bundle, we can form its k-th exterior power, $\Lambda^k V$, so that

$$(\Lambda^k V)_m = \Lambda^k V_m \qquad \forall\, m\in M.$$

The topology and smooth structure on $\Lambda^k V$ are determined from those of $\Lambda^k V$ by requiring that if $s_1,\ldots,s_k$ are smooth sections of V, then $s_1\wedge\ldots\wedge s_k$ is a smooth section of $\Lambda^k V$. More explicitly, suppose $\{g_{\alpha\beta}\}$ is transition data for V. Then, transition data for $\Lambda^k V$ is given by $\{\Lambda^k g_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $\Lambda^k g_{\alpha\beta}$ from each matrix $g_{\alpha\beta}$ as in the previous paragraph. As an example, if the rank of V is k, then the transition data for the line bundle $\Lambda^k V$, called the top exterior power of V, is $\{\det g_{\alpha\beta}\}$.

It follows directly from the definitions that if $V\longrightarrow M$ is a vector bundle of rank k and $L\longrightarrow M$ is a line bundle (vector bundle of rank one), then

$$\Lambda^{\mathrm{top}}(V\oplus L) \equiv \Lambda^{k+1}(V\oplus L) = \Lambda^k V\otimes L \equiv \Lambda^{\mathrm{top}} V\otimes L.$$

More generally, if $V,W\longrightarrow M$ are any two vector bundles, then

$$\Lambda^{\mathrm{top}}(V\oplus W) = (\Lambda^{\mathrm{top}} V)\otimes(\Lambda^{\mathrm{top}} W) \qquad \text{and} \qquad \Lambda^k(V\oplus W) = \bigoplus_{i+j=k}(\Lambda^i V)\otimes(\Lambda^j W).$$

Remark: For complex vector bundles, the above constructions (exterior power, tensor product, direct sum, etc.) are always done over $\mathbb{C}$, unless specified otherwise. So if V is a complex vector bundle of rank k over M, the top exterior power of V is the complex line bundle $\Lambda^k V$ over M (could also be denoted as $\Lambda^k_{\mathbb{C}} V$). In contrast, if we forget the complex structure of V (so that it becomes a real vector bundle of rank $2k$), then its top exterior power is the real line bundle $\Lambda^{2k} V$ (could also be denoted as $\Lambda^{2k}_{\mathbb{R}} V$).

10 Metrics on Fibers

If V is a vector space over $\mathbb{R}$, a positive-definite inner-product on V is a symmetric bilinear map

$$\langle\cdot,\cdot\rangle\colon V\times V \longrightarrow \mathbb{R}, \quad (v,w)\longrightarrow \langle v,w\rangle, \qquad \text{s.t.} \qquad \langle v,v\rangle>0 \ \ \forall\, v\in V-0.$$

If $\langle,\rangle$ and $\langle,\rangle'$ are positive-definite inner-products on V and $a,a'\in\bar{\mathbb{R}}^+$ are not both zero, then

$$a\langle,\rangle+a'\langle,\rangle'\colon V\times V\longrightarrow\mathbb{R}, \qquad \big\{a\langle,\rangle+a'\langle,\rangle'\big\}(v,w)=a\langle v,w\rangle+a'\langle v,w\rangle',$$

is also a positive-definite inner-product. If W is a subspace of V and $\langle,\rangle$ is a positive-definite inner-product on V, let

$$W^\perp=\big\{v\in V\colon \langle v,w\rangle=0\ \forall\, w\in W\big\}$$

be the orthogonal complement of W in V. In particular,

$$V=W\oplus W^\perp.$$

Furthermore, the quotient projection map

$$\pi\colon V\longrightarrow V/W$$

induces an isomorphism from $W^\perp$ to V/W so that

$$V\approx W\oplus(V/W).$$

If M is a smooth manifold and $V\longrightarrow M$ is a smooth real vector bundle of rank k, a Riemannian metric on V is a positive-definite inner-product in each fiber $V_x\approx\mathbb{R}^k$ of V that varies smoothly with $x\in M$. More formally, the smoothness requirement is one of the following equivalent conditions:

(a) the map $\langle,\rangle\colon V\times_M V\longrightarrow\mathbb{R}$ is smooth;

(b) the section $\langle,\rangle$ of the vector bundle $(V\otimes V)^*\longrightarrow M$ is smooth;

(c) if s_1,s_2 are smooth sections of the vector bundle $V\longrightarrow M$, then the map

$$\big\langle s_1,s_2\big\rangle\colon M\longrightarrow\mathbb{R}, \qquad m\longrightarrow\big\langle s_1(m),s_2(m)\big\rangle,$$

is smooth;

(d) if $h\colon V|_{\mathcal{U}}\longrightarrow\mathcal{U}\times\mathbb{R}^k$ is a trivialization of V, then the matrix-valued function,

$$B\colon\mathcal{U}\longrightarrow \mathrm{Mat}_k\mathbb{R} \quad \text{s.t.} \quad \big\langle h^{-1}(m,v),h^{-1}(m,w)\big\rangle=v^tB(m)w \quad \forall\, m\in\mathcal{U},\ v,w\in\mathbb{R}^k,$$

is smooth.

Every real vector bundle admits a Riemannian metric. Such a metric can be constructed by covering M by a locally finite collection of trivializations for V and patching together positive-definite inner-products on each trivialization using a partition of unity. If W is a subspace of V and $\langle , \rangle$ is a Riemannian metric on V, let

$$W^{\perp} = \big\{ v \in V : \langle v, w \rangle = 0 \ \forall\, w \in W \big\}$$

be the orthogonal complement of W in V. Then $W^{\perp} \longrightarrow M$ is a vector subbundle of V and

$$V = W \oplus W^{\perp}.$$

Furthermore, the quotient projection map

$$\pi : V \longrightarrow V/W$$

induces a vector bundle isomorphism from $W^{\perp}$ to V/W so that

$$V \approx W \oplus (V/W).$$

If V is a vector space over $\mathbb{C}$, a nondegenerate Hermitian inner-product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}, \quad (v, w) \longrightarrow \langle v, w \rangle,$$

which is $\mathbb{C}$-antilinear in the first input, $\mathbb{C}$-linear in the second input,

$$\langle w, v \rangle = \overline{\langle v, w \rangle} \qquad \text{and} \qquad \langle v, v \rangle > 0 \ \ \forall\, v \in V - 0.$$

If $\langle , \rangle$ and $\langle , \rangle'$ are nondegenerate Hermitian inner-products on V and $a, a' \in \bar{\mathbb{R}}^+$ are not both zero, then $a\langle , \rangle + a'\langle , \rangle'$ is also a nondegenerate Hermitian inner-product on V. If W is a complex subspace of V and $\langle , \rangle$ is a nondegenerate Hermitian inner-product on V, let

$$W^{\perp} = \big\{ v \in V : \langle v, w \rangle = 0 \ \forall\, w \in W \big\}$$

be the orthogonal complement of W in V. In particular,

$$V = W \oplus W^{\perp}.$$

Furthermore, the quotient projection map

$$\pi : V \longrightarrow V/W$$

induces an isomorphism from $W^{\perp}$ to V/W so that

$$V \approx W \oplus (V/W).$$

If M is a smooth manifold and $V \longrightarrow M$ is a smooth complex vector bundle of rank k, a Hermitian metric on V is a nondegenerate Hermitian inner-product in each fiber $V_x \approx \mathbb{C}^k$ of V that varies smoothly with $x \in M$. More formally, the smoothness requirement is one of the following equivalent conditions:

(a) the map $\langle , \rangle \colon V \times_M V \longrightarrow \mathbb{C}$ is smooth;

(b) the section $\langle , \rangle$ of the vector bundle $(V \otimes_{\mathbb{R}} V)^* \longrightarrow M$ is smooth;

(c) if s_1, s_2 are smooth sections of the vector bundle $V \longrightarrow M$, then the function $\langle s_1, s_2 \rangle$ on M is smooth;

(d) if $h \colon V|_{\mathcal{U}} \longrightarrow \mathcal{U} \times \mathbb{C}^k$ is a trivialization of V, then the matrix-valued function,

$$B \colon \mathcal{U} \longrightarrow \mathrm{Mat}_k \mathbb{C} \quad \text{s.t.} \quad \langle h^{-1}(m,v), h^{-1}(m,w) \rangle = \bar{v}^t B(m) w \quad \forall\, m \in M,\ v, w \in \mathbb{C}^k,$$

is smooth.

Similarly to the real case, every complex vector bundle admits a Hermitian metric. If W is a subspace of V and $\langle , \rangle$ is a Hermitian metric on V, let

$$W^{\perp} = \big\{ v \in V \colon \langle v, w \rangle = 0 \ \forall\, w \in W \big\}$$

be the orthogonal complement of W in V. Then $W^{\perp} \longrightarrow M$ is a complex vector subbundle of V and

$$V = W \oplus W^{\perp}.$$

Furthermore, the quotient projection map

$$\pi \colon V \longrightarrow V/W$$

induces an isomorphism of complex vector bundles over M so that

$$V \approx W \oplus (V/W).$$

If $V \longrightarrow M$ is a real vector bundle of rank k with a Riemannian metric $\langle , \rangle$ or a complex vector bundle of rank k with a Hermitian metric $\langle , \rangle$, let

$$SV \equiv \big\{ v \in V \colon \langle v, v \rangle = 1 \big\} \longrightarrow M$$

be the sphere bundle of V. In the real case, the fiber of SV over every point of M is S^{k-1}. Furthermore, if $\mathcal{U}$ is a small open subset of M, then $SV|_{\mathcal{U}} \approx \mathcal{U} \times S^{k-1}$ as bundles over $\mathcal{U}$, i.e. SV is an S^{k-1}-fiber bundle over M. In the complex case, SV is an S^{2k-1}-fiber bundle over M. If $V \longrightarrow M$ is a real line bundle (vector bundle of rank one) with a Riemannian metric $\langle , \rangle$, then $SV \longrightarrow M$ is an S^0-fiber bundle. In particular, if $\mathcal{U}$ is a small open subset of M, $SV|_{\mathcal{U}}$ is diffeomorphic to $\mathcal{U} \times \{\pm 1\}$. Thus, $SV \longrightarrow M$ is a $2:1$-covering map. If M is connected, the covering space SV is connected if and only if V is *not* orientable; see Section 11 below.

11 Orientations

If V is a real vector space of dimension k, the top exterior power of V, i.e.

$$\Lambda^{\mathrm{top}} V \equiv \Lambda^k V$$

is a one-dimensional vector space. Thus, $\Lambda^{\mathrm{top}}V-0$ has exactly two connected components. An orientation on V is a component $\mathcal{C}$ of V. If $\mathcal{C}$ is an orientation on V, then a basis $\{e_i\}$ for V is called oriented (with respect to $\mathcal{C}$) if

$$e_1\wedge\ldots\wedge e_k\in\mathcal{C}.$$

If $\{f_j\}$ is another basis for V and A is the change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, i.e.

$$\big(f_1,\ldots,f_k\big)=\big(e_1,\ldots,e_k\big)A \qquad\Longleftrightarrow\qquad f_j=\sum_{i=1}^{i=k}A_{ij}e_i,$$

then

$$f_1\wedge\ldots\wedge f_k=(\det A)e_1\wedge\ldots\wedge e_k.$$

Thus, two different bases for V belong to the same orientation on V if and only of the determinant of the corresponding change-of-basis matrix is positive.

Suppose $V\longrightarrow M$ is a real vector bundle of rank k. An orientation for V is an orientation for each fiber $V_x\approx\mathbb{R}^k$, which varies smoothly (or continuously, or is locally constant) with $x\in M$. This means that if

$$h\colon V|_{\mathcal{U}}\longrightarrow\mathcal{U}\times\mathbb{R}^k$$

is a trivialization of V and $\mathcal{U}$ is connected, then h is either orientation-preserving or orientation-reversing (with respect to the standard orientation of $\mathbb{R}^k$) on every fiber. If V admits an orientation, V is called orientable.

Lemma 11.1. *Suppose $V\longrightarrow M$ is a smooth real vector bundle.*
(1) V is orientable if and only if there exists a collection $\{\mathcal{U}_\alpha,h_\alpha\}$ of trivializations that covers M such that

$$\det g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathbb{R}^+,$$

where $\{g_{\alpha\beta}\}$ is the corresponding transition data.
(2) V is orientable if and only if the line bundle $\Lambda^{\mathrm{top}}V\longrightarrow M$ is orientable.
(3) If V is a line bundle, V is orientable if and only if V is (isomorphic to) the trivial line bundle $M\times\mathbb{R}$.
(4) If V is a line bundle with a Riemannian metric $\langle,\rangle$, V is orientable if and only if SV is not connected.

Proof: (1) If V has an orientation, we can choose a collection $\{\mathcal{U}_\alpha,h_\alpha\}$ of trivializations that covers M such that the restriction of h_α to each fiber is orientation-preserving (if it is orientation-preserving, simply multiply its first component by -1). Then, the corresponding transition data $\{g_{\alpha\beta}\}$ is orientation-preserving, i.e.

$$\det g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathbb{R}^+.$$

Conversely, suppose $\{\mathcal{U}_\alpha,h_\alpha\}$ is a collection of trivializations that covers M such that

$$\det g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathbb{R}^+.$$

Then, if $x\in\mathcal{U}_\alpha$ for some α, define an orientation on V_x by requiring that

$$h_\alpha\colon V_x\longrightarrow x\times\mathbb{R}^k$$

is orientation-preserving. Since $\det g_{\alpha\beta}$ is $\mathbb{R}^+$-valued, the orientation on V_x is independent of α such that $x \in \mathcal{U}_\alpha$. Each of the trivializations h_α is then orientation-preserving on each fiber.
(2) An orientation for V is the same as an orientation for Λ^{top}, since

$$\Lambda^{\mathrm{top}} V = \Lambda^{\mathrm{top}}\big(\Lambda^{\mathrm{top}} V\big).$$

Furthermore, if $\{(\mathcal{U}_\alpha, h_\alpha)\}$ is a collection of trivializations for V such that the corresponding transition functions $g_{\alpha\beta}$ have positive determinant, then $\{(\mathcal{U}_\alpha, \Lambda^{\mathrm{top}} h_\alpha)\}$ is a collection of trivializations for $\Lambda^{\mathrm{top}} V$ such that the corresponding transition functions $\Lambda^{\mathrm{top}} g_{\alpha\beta} = \det(g_{\alpha\beta})$ have positive determinant as well.
(3) The trivial line bundle $M \times \mathbb{R}$ is orientable, with an orientation determined by the standard orientation on $\mathbb{R}$. Thus, if V is isomorphic to the trivial line bundle, then V is orientable. Conversely, suppose V is an oriented line bundle. For each $x \in M$, let

$$\mathcal{C}_x \subset \Lambda^{\mathrm{top}} V = V$$

be the chosen orientation of the fiber. Choose a Riemannian metric on V and define a section s of V by requiring that for all $x \in M$

$$\big\langle s(x), s(x) \big\rangle = 1 \qquad \text{and} \qquad s(x) \in \mathcal{C}_x.$$

This section is well-defined and smooth (as can be seen by looking on a trivialization). Since it does not vanish, the line bundle V is trivial by Lemma 7.4.
(4) If V is orientable, then V is isomorphic to $M \times \mathbb{R}$, and thus

$$SV = S(M \times \mathbb{R}) = M \times S^0 = M \sqcup M$$

is not connected. Conversely, if M is connected and SV is not connected, let SV^+ be one of the components of V. Since $SV \longrightarrow M$ is a covering projection, so is $SV^+ \longrightarrow M$. Since the latter is one-to-one, it is a diffeomorphism, and its inverse determines a nowhere-zero section of V. Thus, V is isomorphic to the trivial line bundle by Lemma 7.4.

If V is a complex vector space of dimension k, V has a canonical orientation as a real vector space of dimension $2k$. If $\{e_i\}$ is a basis for V over $\mathbb{C}$, then

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\}$$

is a basis for V over $\mathbb{R}$. The orientation determined by such a basis is the canonical orientation for the underlying real vector space V. If $\{f_j\}$ is another basis for V over $\mathbb{C}$, B is the complex change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, A is the real change-of-basis matrix from

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\} \qquad \text{to} \qquad \big\{f_1, \mathrm{i}f_1, \ldots, f_k, \mathrm{i}f_k\big\},$$

then

$$\det A = (\det B)\overline{\det B} \in \mathbb{R}^+.$$

Thus, the two bases over $\mathbb{R}$ induced by complex bases for V determine the same orientation for V. This implies that every complex vector bundle $V \longrightarrow M$ is orientable as a real vector bundle.

Exercises

1 Let $\pi: V \longrightarrow M$ be a vector bundle. Show that

(a) the scalar-multiplication map (6.1) is smooth;

(b) the space $V \times_M V$ is a smooth submanifold of $V \times V$ and the addition map (6.2) is smooth.

2 Verify all claims made in Example 6.5, thus establishing that the tangent bundle TM of a smooth manifold is indeed a vector bundle. What is its transition data?

3 Show that the tangent bundle TS^1 of S^1 is isomorphic to the trivial real line bundle over S^1.

4 Show that the complex tautological line bundle $\gamma_n \longrightarrow \mathbb{C}P^n$ is indeed a complex line bundle as claimed in Example 6.8. What is its transition data? Why is it non-trivial for $n \geq 1$?

5 Let $\pi: V \longrightarrow M$ be a smooth vector bundle of rank k and $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ a collection of trivializations covering M. Show that a section s of π is continuous (smooth) if and only if the map

$$s_\alpha \equiv \pi_2 \circ h_\alpha \circ s: U_\alpha \longrightarrow \mathbb{R}^k\,,$$

where $\pi_2: U_\alpha \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is the projection on the second component, is continuous (smooth) for every $\alpha \in \mathcal{A}$.

6 Let M be a smooth m-manifold. Show that

(TM1) the topology on TM constructed in Example 6.5 is the unique one so that $\pi: TM \longrightarrow M$ is a topological vector bundle with the canonical vector-space structure on the fibers and so that for every vector field X on TM and smooth function $f: U \longrightarrow \mathbb{R}$, where U is an open subset of $\mathbb{R}$, the function $X(f): U \longrightarrow \mathbb{R}$ is continuous if and only if X is continuous;

(TM2) the smooth structure on TM constructed in Example 6.5 is the unique one so that $\pi: TM \longrightarrow M$ is a smooth vector bundle with the canonical vector-space structure on the fibers and so that for every vector field X on TM and smooth function $f: U \longrightarrow \mathbb{R}$, where U is an open subset of $\mathbb{R}$, the function $X(f): U \longrightarrow \mathbb{R}$ is smooth if and only if X is smooth.

7 Show that the two versions of the last condition on $\tilde{f}$ in (2) in Definition 7.2 are indeed equivalent.

8 Suppose that $f: M \longrightarrow N$ is a smooth map and $\pi: V \longrightarrow N$ is a smooth vector bundle of rank k with transition data $\{g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_n\mathbb{R}\}_{\alpha,\beta\in\mathcal{A}}$. Show that

(a) the space f^*V defined by (9.1) is a smooth submanifold of $M \times V$ and the projection $\pi_1: f^*V \longrightarrow M$ is a vector bundle of rank k with transition data

$$\big\{f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f: f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \longrightarrow \mathrm{GL}_n\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}\,;$$

(b) if M is an embedded submanifold of N and f is the inclusion map, then the projection $\pi_2: f^*V \longrightarrow V$ induces an isomorphism $f^*V \longrightarrow V|_M$ of vector bundles over M.

9 Let $f: M \longrightarrow N$ be a smooth map and $\varphi: V \longrightarrow W$ a smooth vector-bundle homomorphism over N. Show that the pullback vector-bundle homomorphism $f^*\varphi: f^*V \longrightarrow f^*W$ is also smooth.

10 Let $\varphi\colon V \longrightarrow W$ be a smooth surjective vector-bundle homomorphism over a smooth manifold M. Show that

$$\ker\varphi \equiv \{v\in V\colon\ \varphi(v)=0\} \longrightarrow M$$

is a subbundle of V.

11 Let $V \longrightarrow M$ be a vector bundle of rank k and $V'\subset V$ a smooth subbundle of rank k'. Show that

(a) there exists a collection $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for V covering M so that (9.7) holds and thus the corresponding transition data has the form

$$g_{\alpha\beta} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\colon U_\alpha\cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{R},$$

where the top left block is $k'\times k'$;

(b) the vector-bundle structure on V/V' described in Section 9 is the unique one so that the natural projection map $V \longrightarrow V/V'$ is a smooth vector-bundle homomorphism;

(c) if $\varphi\colon V \longrightarrow W$ is a vector-bundle homomorphism over M such that $\varphi(v)=0$ for all $v\in V'$, then the induced vector-bundle homomorphism $\bar{\varphi}\colon V/V' \longrightarrow W$ is smooth.

12 Let $f=(f_1,\ldots,f_k) : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ be a smooth map, $q\in\mathbb{R}^k$ a regular value of f, and $X=f^{-1}(q)$. Denote by ∇f_i the gradient of f_i. Show that

$$TX = \{(p,v)\in X\times\mathbb{R}^m\colon\ \nabla f_i|_p\cdot v=0\ \forall\, i=1,2,\ldots,k\}$$

under the canonical identifications $TX\subset T\mathbb{R}^m|_X$ and $T\mathbb{R}^m=\mathbb{R}^m\times\mathbb{R}^m$. Use this description of TX to give a trivialization of $\mathcal{N}_{\mathbb{R}^m}X$.

13 Obtain Corollary 9.3 from Theorem 9.2.

14 Let $V, W \longrightarrow M$ be smooth vector bundles. Show that the two constructions of $V\oplus W$ in Section 9 produce the same vector bundle and that this is the unique vector-bundle structure on the total space of

$$V\oplus W = \bigsqcup_{p\in M} V_p\oplus W_p$$

so that

(VB$\bigoplus$1) the projection maps $V\oplus W \longrightarrow V, W$ are smooth bundle homomorphisms over M;

(VB$\bigoplus$2) the inclusion maps $V, W \longrightarrow V\oplus W$ are smooth bundle homomorphisms over M.

15 Let $\pi_V : V \longrightarrow M$ and $\pi_W : W \longrightarrow N$ be smooth vector bundles and $\pi_M, \pi_N : M\times N \longrightarrow M, N$ the component projection maps. Show that the total of the vector bundle

$$\pi\colon \pi_M^*V \oplus \pi_N^*W \longrightarrow M\times N$$

is $V\times W$ (with the product smooth structure) and $\pi=\pi_V\times\pi_W$.

16 Let M and N be smooth manifolds and $\pi_M, \pi_N : M\times N \longrightarrow M, N$ the projection maps. Show that $\mathrm{d}\pi_M$ and $\mathrm{d}\pi_N$ viewed as maps from $T(M\times N)$ to

(a) TM and TN, respectively, induce a diffeomorphism $T(M\times N)\longrightarrow TM\times TN$ that commutes with the projections from the tangent bundles to the manifolds and is linear on the fibers of these projections;

(b) π_M^*TM and π_N^*TN, respectively, induce a vector-bundle isomorphism

$$T(M\times N)\longrightarrow \pi_M^*TM\oplus\pi_N^*TN.$$

Why are the above two statements the same?

17 Show that the vector-bundle structure on the total space of V^* constructed in Section 9 is the unique one so that the map (9.12) is smooth.

18 Suppose $k<n$. Show that the map

$$\iota\colon \mathbb{C}P^k\longrightarrow \mathbb{C}P^n, \qquad [X_0,\ldots,X_k]\longrightarrow [X_0,\ldots,X_k,\underbrace{0,\ldots,0}_{n-k}],$$

is a complex embedding (i.e. a smooth embedding that induces holomorphic maps between the charts that determine the complex structures on $\mathbb{C}P^k$ and $\mathbb{C}P^n$) and that the normal bundle to this immersion, $\mathcal{N}_\iota$, is isomorphic to

$$(n-k)\gamma_k^*\equiv \underbrace{\gamma_k^*\oplus\ldots\oplus\gamma_k^*}_{n-k},$$

where $\gamma_k\longrightarrow\mathbb{C}P^k$ is the tautological line bundle (isomorphic as complex line bundles). *Hint:* there are a number of ways of doing this, including:

(i) construct an isomorphism between the two vector bundles;

(ii) determine transition data for $\mathcal{N}_\iota$ and $(n-k)\gamma_k^*$;

(iii) show that there exists a holomorphic diffeomorphism between $(n-k)\gamma_k^*$ and a neighborhood of $\iota(\mathbb{C}P^k)$ in $\mathbb{C}P^n$, fixing $\iota(\mathbb{C}P^k)$, and that this implies that $\mathcal{N}_\iota=(n-k)\gamma_k^*$.

19 Let $\pi\colon V\longrightarrow M$ be a real vector bundle.

(a) Show that the vector-bundle homomorphism $\mathrm{d}\pi: TV\longrightarrow\pi^*TM$ is surjective and thus $\ker\mathrm{d}\pi\longrightarrow V$ is a vector bundle; it is called the vertical tangent bundle of V and denoted TV^{vrt}.

(b) Show that there is a canonical isomorphism $TV^{\mathrm{vrt}}\longrightarrow\pi^*V$ of vector bundles over V. Conclude that there is a short exact sequence of vector bundles

$$0\longrightarrow TV^{\mathrm{vrt}}\stackrel{\iota}{\longrightarrow}TV\stackrel{\mathrm{d}\pi}{\longrightarrow}\pi^*M\longrightarrow 0$$

over V, where ι is the inclusion map.

20 Let $\gamma_n\longrightarrow\mathbb{C}P^n$ be the tautological line bundle as in Example 6.8 and $P:\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ a homogeneous polynomial of degree one. Show that

$$s_P\colon\mathbb{C}P^n\longrightarrow\gamma_n^*, \qquad \big\{s_P(\ell)\big\}(\ell,v)=P(v)\quad\forall\,(\ell,v)\in\gamma_n,$$

is a well-defined holomorphic section of γ_n^*, while the line bundle $\gamma_n\longrightarrow\mathbb{C}P^n$ admits no nonzero holomorphic section.

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