# MAT 531: Topology&Geometry, II
# Spring 2010

## Problem Set 5
**Due on Thursday, 3/4, in class**

*Note:* This problem set has two pages.

1. Let $\alpha$ be a 1-form on a smooth manifold $M$.

   (a) Show directly from the definitions that for any pair of vector fields $X_0$ and $X_1$ on $M$
   $$\mathrm{d}\alpha(X_0, X_1) = X_0\alpha(X_1) - X_1\alpha(X_0) - \alpha([X_0, X_1]).$$
   *Hint:* first show that the value of RHS at any $p \in M$ depends only on the values of $X_0$ and $X_1$ at $p$.

   (b) Use the same approach to obtain Warner's 2.25f directly.

2. Let $V$ be a vector space of dimension $n$ and $\Omega \in \Lambda^n V^*$ a nonzero element. Show that the homomorphism
   $$V \longrightarrow \Lambda^{n-1}V^*, \qquad v \longrightarrow i_v\Omega,$$
   where $i_v$ is the contraction map, is an isomorphism.

3. Suppose $M$ is a smooth $n$-manifold.
   (a) Let $\Omega$ be a nowhere-zero $n$-form on $M$. Show that for every $p \in M$ there exists a chart $(x_1, \ldots, x_n)\colon U \longrightarrow \mathbb{R}^n$ around $p$ such that
   $$\Omega|_U = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$
   (b) Let $\alpha$ be a nowhere-zero closed $(n-1)$-form on $M$. Show that for every $p \in M$ there exists a chart $(x_1, \ldots, x_n)\colon U \longrightarrow \mathbb{R}^n$ around $p$ such that
   $$\alpha|_U = \mathrm{d}x_2 \wedge dx_3 \wedge \ldots \wedge \mathrm{d}x_n.$$

4. Let $V \longrightarrow M$ be a smooth vector bundle of rank $k$ and $W \subset V$ a smooth subbundle of $V$ of rank $k'$. Show that
   $$\mathrm{Ann}(W) \equiv \big\{\alpha \in V_p^*\colon\ \alpha(w) = 0\ \forall\, w \in W,\ p \in M\big\}$$
   is a smooth subbundle of $V^*$ of rank $k - k'$.

5. Suppose $M$ is a 3-manifold, $\alpha$ is a nowhere-zero one-form on $M$, and $p \in M$. Show that

   (a) if there exists an embedded 2-dimensional submanifold $P \subset M$ such that $p \in P$ and $\alpha|_{TP} = 0$, then $(\alpha \wedge \mathrm{d}\alpha)|_p = 0$.

   (b) if there exists a neighborhood $U$ of $p$ in $M$ such that $(\alpha \wedge \mathrm{d}\alpha)|_U = 0$, then there exists an embedded 2-dimensional submanifold $P \subset M$ such that $p \in P$ and $\alpha|_{TP} = 0$.

   *Note:* If the top form $\alpha \wedge \mathrm{d}\alpha$ on $M$ is nowhere-zero, $\alpha$ is called a contact form. In this case, it has no integrable submanifolds at all.

6. A two-form $\omega$ on a smooth manifold $M$ is called symplectic if $\omega$ is closed (i.e. $\mathrm{d}\omega = 0$) and everywhere nondegenerate[1]. Suppose $\omega$ is a symplectic form on $M$.

   (a) Show that the dimension of $M$ is even and the map

   $$TM \longrightarrow T^*M, \qquad X \longrightarrow i_X\omega,$$

   is a vector bundle isomorphism ($i_X$ is the contraction w.r.t. $X$, i.e. the dual of $X\wedge$).

   (b) If $H: M \longrightarrow \mathbb{R}$ is a smooth map, let $X_H \in \Gamma(M; TM)$ be the preimage of $\mathrm{d}H$ under this isomorphism. Assume that $X_H$ is a complete vector field, so that the flow

   $$\varphi: \mathbb{R}\times M \longrightarrow M, \qquad (t,p) \longrightarrow \varphi_t(p),$$

   is globally defined. Show that for every $t\in\mathbb{R}$, the time-$t$ flow $\varphi_t: M \longrightarrow M$ is a symplectomorphism, i.e. $\varphi_t^*\omega = \omega$.

   *Note:* In such a situation, $H$ is called a Hamiltonian and $\varphi_t$ a Hamiltonian symplectomorphism.

[1] This means that $\omega_p \in \Lambda^2 T_p^* M$ is nondegenerate for every $p\in M$, i.e. for every $v\in T_pM - 0$ there exists $v' \in T_pM$ such that $\omega_p(v,v') \neq 0$.