

MAT 531: Topology & Geometry, II Spring 2010

Problem Set 10

Due on Thursday, 4/29, in class

- Chapter 6, #6 (p252)
- Suppose M is a compact Riemannian manifold. If M is not orientable, the Hodge $*$ -operator is defined only up to sign. However, as the definition of δ in Section 6.1 involves two $*$'s, the linear operator δ is well-defined. Show that δ is the adjoint of d whether or not M is oriented.
Hint: It is sufficient to assume that M is connected and non-orientable. Deduce the claim from the orientable case proved in the book. The proof is a couple of lines.
- Suppose M is a compact connected *non*-orientable n -manifold. Show that $H_{\text{deR}}^n(M) = 0$.
Remark: In fact, if M is a *non*-compact connected n -manifold, orientable or not, $H_{\text{deR}}^n(M) = 0$.
Hint: Deduce from the orientable case. The proof is a couple of lines.
- Chapter 6, #16 (p254); you may want to do (d) before (b) and (c).
Remark: In (e)-(h), it suffices to write up only proofs of claims omitted in the statement.
- Suppose M and N are smooth compact Riemannian manifolds. A differential form γ on $M \times N$ is called **decomposable** if

$$\gamma = \pi_M^* \alpha \wedge \pi_N^* \beta \quad \text{for some } \alpha \in E^*(M), \beta \in E^*(N).$$

(a) Show that

$$\Delta_{M \times N}(\pi_M^* \alpha \wedge \pi_N^* \beta) = \pi_M^* \Delta_M \alpha \wedge \pi_N^* \beta + \pi_M^* \alpha \wedge \pi_N^* \Delta_N \beta \quad \forall \alpha \in E^*(M), \beta \in E^*(N).$$

(b) Show that the \mathbb{R} -span of the decomposable forms on $M \times N$ is L^2 -dense in $E^*(M \times N)$.

(c) Conclude that

$$\mathcal{H}^*(M \times N) \approx \mathcal{H}^*(M) \otimes \mathcal{H}^*(N), \quad \pi_M^* \alpha \wedge \pi_N^* \beta \longleftrightarrow \alpha \otimes \beta.$$

(d) Conclude that

$$H_{\text{deR}}^p(M \times N) \approx \bigoplus_{q+r=p} H_{\text{deR}}^q(M) \otimes H_{\text{deR}}^r(N), \quad \pi_M^* \alpha \wedge \pi_N^* \beta \longleftrightarrow \alpha \otimes \beta.$$

This is the *Kunneth Formula* for compact manifolds.