MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 8

Solution to Problem p335, #4

Suppose $A \subset X$ *and* $r: X \longrightarrow A$ *is a retraction, i.e.* $r|_A = \mathrm{id}_A$. *Show that for any* $a_0 \in A$ *the homomorphism*

$$r_*\colon \pi_1(X, a_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

The condition on r means that

$$r \circ \iota = \mathrm{id}_A\colon A \longrightarrow A,$$

where $\iota\colon A \longrightarrow X$ is the inclusion map. Thus,

$$r_* \circ \iota_* = (r \circ \iota)_* = \mathrm{id}_{A*} = \mathrm{Id}\colon \pi_1(A, a_0) \longrightarrow \pi_1(A, a_0).$$

Since the composition $r_* \circ \iota_*$ is surjective, so is r_*.

Solution to Problem p335, #5

Suppose $A \subset \mathbb{R}^n$ *and* $h\colon (A, a_0) \longrightarrow (Y, y_0)$ *is a continuous map that extends to a continuous map from* $\mathbb{R}^n$ *to* Y. *Show that*

$$r_*\colon \pi_1(A, a_0) \longrightarrow \pi_1(X, x_0)$$

is the trivial homomorphism.

Suppose $k\colon \mathbb{R}^n \longrightarrow Y$ is a continuous map such that

$$k|_A = h \qquad \Longleftrightarrow \qquad h = k \circ \iota,$$

where $\iota\colon A \longrightarrow \mathbb{R}^n$ is the inclusion map. Then,

$$h_* = (k \circ \iota)_* = k_* \circ \iota_*\colon \pi_1(A, a_0) \longrightarrow \pi_1(\mathbb{R}^n, a_0) \longrightarrow \pi_1(X, x_0).$$

Since $\pi_1(\mathbb{R}^n, a_0)$ is trivial (consists just of the identity), the homomorphism h_* is trivial (i.e. it image is the identity element in $\pi_1(X, x_0)$.

Solution to Problem p341, #3

Let $p\colon E \longrightarrow B$ be a covering map. Suppose B is connected and $f^{-1}(b_0)$ has k-elements for some $b_0 \in B$. Show that $p^{-1}(b)$ has k elements for every $b \in B$.

For each $n \in \mathbb{Z}^+ \cup \{\infty\}$, let

$$\mathcal{U}_n = \big\{ b \in B \colon |p^{-1}(b)| = n \big\}.$$

Since p is surjective,

$$B = \bigcup_{n \in \mathbb{Z}^+ \cup \{\infty\}} \mathcal{U}_n.$$

If $b \in \mathcal{U}_n$ and V is an evenly covered neighborhood of b, then

$$|p^{-1}(b')| = n \quad \forall\, b' \in V \qquad \Longrightarrow \qquad V \subset \mathcal{U}_n.$$

Thus, $\mathcal{U}_n$ is an open subset of B. Let

$$W = \bigcup_{n \in \mathbb{Z}^+ \cup \{\infty\}, n \neq k} \mathcal{U}_n.$$

Then, $B = \mathcal{U}_k \sqcup W$. Since B is connected, $\mathcal{U}_n$ and W are open, $\mathcal{U}_k$ is non-empty (it contains b_0), W must be empty. Thus, $B = \mathcal{U}_k$, i.e. $p^{-1}(b)$ has k elements for every $b \in B$.

Solution to Problem p341, #3

Let $g, h\colon S^1 \longrightarrow S^1$ be the given maps by

$$g(z) = z^n \qquad \textit{and} \qquad h(z) = 1/z^n.$$

Determine the homomorphisms

$$g_*, h_* \colon \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1).$$

Let $f\colon I \longrightarrow S^1$ be the loop based at 1 given by

$$f(s) = e^{2\pi i s},$$

i.e. f goes around the circle counterclockwise once. Let

$$\alpha = [f] \in \pi_1(S^1, 1).$$

By the proof of Lemma 54.4,

$$\pi_1(S^1,1)=\mathbb{Z}[\alpha],$$

i.e. α generates $\pi_1(S^1,1)$. On the other hand,

$$g_*\alpha = g_*[f] = [g\circ f] = [\underbrace{f * \ldots * f}_{n \text{ times}}] = \underbrace{[f] * \ldots * [f]}_{n \text{ times}} = n[f] = n\cdot\alpha.$$

The third equality holds because $g\circ f$ "does f" on each of the n intervals $[(k-1)/n, k/n]$ with $k=1,\ldots,n$. Thus, the homomorphism

$$g_*\colon \pi_1(S^1,1)\longrightarrow \pi_1(S^1,1)$$

is the multiplication by n in the infinite cyclic group $\mathbb{Z}[\alpha]$.

Let $\eta\colon S^1\longrightarrow S^1$ be the map given by $\eta(z)=1/z$. Then,

$$\eta\big(f(s)\big) = 1/e^{2\pi i s} = e^{-2\pi i s} = e^{2\pi i(1-s)} = f(1-s) \quad\Longrightarrow\quad \eta\circ f = \bar{f} \quad\Longrightarrow\quad f_*\alpha = \bar{\alpha} = -\alpha.$$

Thus, the homomorphism

$$\eta_*\colon \pi_1(S^1,1)\longrightarrow \pi_1(S^1,1)$$

is the multiplication by -1. Since $h=\eta\circ g$, the homomorphism

$$h_* = (\eta\circ g)_* = \eta_*\circ g_*\colon \pi_1(S^1,1)\longrightarrow \pi_1(S^1,1)$$

is the multiplication by $(-1)n=-n$ in the infinite cyclic group $\mathbb{Z}[\alpha]$.