# MAT 530: Topology&Geometry, I
# Fall 2005

## Problem Set 6

### Solution to Problem p260, #2

(a) *Show that the product of a paracompact space and a compact space is paracompact.*
(b) *Conclude that $S_\Omega$ is not paracompact.*

(a) Suppose $X$ is a paracompact topological space and $Y$ is a compact topological space. Let $\mathcal{A}$ be an open cover of $X\times Y$. We will show that $\mathcal{A}$ has a locally finite open refinement that covers $X$.

For every $x\in X$, $\mathcal{A}$ is an open cover of the slice

$$Y_x \equiv \{x\}\times Y \subset X\times Y.$$

Since $\{x\}\times Y$ is homeomorphic to $Y$ and $Y$ is compact, there exists a finite open subcollection $\mathcal{B}_x$ of $\mathcal{A}$ that covers $\{x\}\times Y$. By the Tube Lemma, Lemma 26.8, there exists an open subset $\mathcal{U}_x$ of $X$ such that

$$x \in \mathcal{U}_x \qquad \text{and} \qquad \mathcal{U}_x\times Y \subset \bigcup_{W\in\mathcal{B}_x} W. \tag{1}$$

Since $X$ is paracompact and

$$\mathcal{C} \equiv \big\{\mathcal{U}_x : x\in X\big\}$$

is an open cover of $X$, there exists a locally finite open refinement $\mathcal{D}$ of $\mathcal{C}$ that covers $X$. For each $V\in\mathcal{D}$, choose $x(V)\in X$ such that $V\subset\mathcal{U}_{x(V)}$. Let

$$\mathcal{E} = \big\{(V\times Y)\cap W : V\in\mathcal{D},\, W\in\mathcal{B}_{x(V)}\big\}.$$

Since $\mathcal{B}_{x(V)}\subset\mathcal{A}$ for all $V\in\mathcal{D}$, $\mathcal{E}$ is an open refinement of $\mathcal{A}$. Since $V\subset\mathcal{U}_{x(V)}$,

$$V\times Y \subset \bigcup_{W\in\mathcal{B}_{x(V)}} W \qquad \forall\, V\in\mathcal{C}$$

by (1). Since $\mathcal{C}$ covers $X$, we find

$$\bigcup_{V\in\mathcal{C}} \bigcup_{W\in\mathcal{B}_{x(V)}} (V\times Y)\cap W = \bigcup_{V\in\mathcal{C}}\Big((V\times Y)\cap \bigcup_{W\in\mathcal{B}_{x(V)}} W\Big) = \bigcup_{V\in\mathcal{C}}(V\times Y) = \Big(\bigcup_{V\in\mathcal{C}} V\Big)\times Y = X\times Y.$$

Thus, $\mathcal{E}$ covers $X\times Y$.

It remains to show that $\mathcal{E}$ is locally finite. If $x\times y\in X\times Y$, choose an open subset $\mathcal{U}$ of $X$ such that $x\in\mathcal{U}$ and $\mathcal{U}$ intersects only finitely many elements of $\mathcal{C}$. Then,

$$\big\{(V\times Y)\cap W\in\mathcal{E} : (\mathcal{U}\times Y)\cap((V\times Y)\cap W)\neq\emptyset\big\} \subset \big\{(V\times Y)\cap W : \mathcal{U}\cap V\neq\emptyset,\; W\in\mathcal{B}_{x(V)}\big\}.$$

Since $\mathcal{B}_{x(V)}$ is finite for all $V\in\mathcal{C}$, $\mathcal{U}\times Y$ intersects only finitely many elements of $\mathcal{E}$.

(b) By Lemma 41.1, a paracompact Hausdorff space is normal. Since any space in the order topology is Hausdorff, $S_\Omega$ and $\bar{S}_\Omega$ are Hausdorff. Since any product of Hausdorff spaces is Hausdorff, $S_\Omega\times\bar{S}_\Omega$ is Hausdorff. On the other hand, by Example 2 in Section 32, $S_\Omega\times\bar{S}_\Omega$ is not normal and thus not paracompact. Since $\bar{S}_\Omega$ is well-ordered and thus has the least-upper property, every closed interval in $\bar{S}_\Omega$, including $\bar{S}_\Omega$, is compact by Theorem 27.1. Since $S_\Omega\times\bar{S}_\Omega$ is not paracompact, $S_\Omega$ is not paracompact by part (a).

## Solution to Problem p260, #7

*Let $X$ be a regular space.*
(a) *If $X$ is a finite union of closed paracompact subspaces of $X$, then $X$ is paracompact.*
(b) *If $X$ is a countable union of closed paracompact subspaces of $X$ whose interiors cover $X$, then $X$ is paracompact.*

(a) We will show that if $\mathcal{A}$ is an open cover of $X$, there exists a locally finite refinement $\mathcal{D}$ of $\mathcal{A}$ that covers $X$. Since $X$ is regular, the equivalence of (2) and (4) in Lemma 41.3 then implies that $X$ is paracompact.

Since a finite union of closed subsets is closed, it is sufficient to consider the case when $X=B\cup C$, where $B,C\subset X$ are closed subsets of $X$ that are paracompact in the subspace topologies. Since $\mathcal{A}$ is an open cover of $X$,

$$\mathcal{A}_B\equiv\big\{\mathcal{U}\cap B:\mathcal{U}\in\mathcal{A}\big\}\qquad\text{and}\qquad\mathcal{A}_C\equiv\big\{\mathcal{U}\cap C:\mathcal{U}\in\mathcal{A}\big\}$$

are open covers of $B$ and $C$, respectively. Since $B$ and $C$ are paracompact, there exist (open) refinements $\mathcal{B}$ of $\mathcal{A}_B$ and $\mathcal{C}$ of $\mathcal{A}_C$ that cover $B$ and $C$, respectively, and are locally finite in $B$ and $C$, respectively. Let

$$\mathcal{D}=\mathcal{B}\cup\mathcal{C}.$$

Since $\mathcal{B}$ and $\mathcal{C}$ cover $B$ and $C$, $\mathcal{D}$ covers $X$. Since $\mathcal{B}$ and $\mathcal{C}$ refine $\mathcal{A}_B$ and $\mathcal{A}_C$, which in turn refine $\mathcal{A}$, $\mathcal{D}$ also refines $\mathcal{A}$. Below we show that the collections $\mathcal{B}$ and $\mathcal{C}$ are locally finite in $X$. Thus, so is their union $\mathcal{D}$.

We show that $\mathcal{B}$ is locally finite in $X$; by symmetry, so is $\mathcal{C}$. Let $x\in X$ be any point. If $x\in X-B$, the set $W\equiv X-B$ is open in $X$, since $B$ is closed, and contains $x$. Since every element $D\in\mathcal{B}$ is a subset of $B$, $W$ intersects no element of $\mathcal{B}$. Suppose next that $x\in B$. Since $\mathcal{B}$ is locally finite in $B$, there exists an open subset $W$ of $X$ such that the set $W\cap B$ contains $x$ and intersects only finitely many elements of $\mathcal{B}$. Since every element of $D\in\mathcal{B}$ is a subset of $B$,

$$\big(W\cap B\big)\cap D=W\cap D\quad\forall\,D\in\mathcal{B}\qquad\Longrightarrow\qquad\big\{D\in\mathcal{B}\colon W\cap D\neq\emptyset\big\}=\big\{D\in\mathcal{B}\colon (W\cap B)\cap D\neq\emptyset\big\}.$$

Thus, the open subset $W$ of $X$ contains $x$ and intersects only finitely many elements of $\mathcal{B}$.

(b) We will show that if $\mathcal{A}$ is an open cover of $X$, there exists a $\sigma$-locally finite open refinement $\mathcal{D}$ of $\mathcal{A}$ that covers $X$. Since $X$ is regular, the equivalence of (1) and (4) in Lemma 41.3 then implies that $X$ is paracompact.

Suppose $X=\bigcup_{n\in\mathbb{Z}^+}\text{Int}\, X_n$, where $X_n$ is a closed subset of $X$ which is paracompact in the subspace topology, for every $n\in Z^+$. Since $\mathcal{A}$ is an open cover of $X$, for every $n\in\mathbb{Z}^+$

$$\mathcal{A}_n \equiv \{\mathcal{U}\cap X_n : \mathcal{U}\in\mathcal{A}\}$$

is an open cover of $X_n$. Since $X_n$ is paracompact, there exists a refinement $\mathcal{B}_n$ of $\mathcal{A}_n$ that covers $X_n$, is locally finite in $X_n$, and is open in $X_n$. For each $D\in\mathcal{B}_n$ in $X_n$, choose an open subset $V(D)\subset X$ such that

$$D = V(D)\cap X_n.$$

Let

$$\mathcal{D}_n = \{V(D)\cap\text{Int}\, X_n : D\in\mathcal{B}_n\} \quad \forall\, n\in\mathbb{Z}^+ \qquad \text{and} \qquad \mathcal{D} = \bigcup_{n\in\mathbb{Z}^+}\mathcal{D}_n.$$

All elements of $\mathcal{D}$ are open in $X$. Since $\mathcal{B}_n$ refines $\mathcal{A}_n$ (and thus $\mathcal{A}$) and

$$V(D)\cap\text{Int}\, X_n \subset V(D)\cap X_n = D \qquad \forall\,\mathcal{D}\in\mathcal{B}_n,$$

$\mathcal{D}_n$ refines $\mathcal{A}$. Since $\mathcal{B}_n$ covers $X_n$ and $X=\bigcup_{n\in\mathbb{Z}^+}\text{Int}\, X_n$,

$$\begin{aligned}\bigcup_{n\in\mathbb{Z}^+}\bigcup_{D\in\mathcal{B}_n}\big(V(D)\cap\text{Int}\, X_n\big) &= \bigcup_{n\in\mathbb{Z}^+}\bigg(\Big(\bigcup_{D\in\mathcal{B}_n}V(D)\Big)\cap\text{Int}\, X_n\bigg) \supset \bigcup_{n\in\mathbb{Z}^+}\bigg(\Big(\bigcup_{D\in\mathcal{B}_n}D\Big)\cap\text{Int}\, X_n\bigg)\\ &= \bigcup_{n\in\mathbb{Z}^+}\big(X_n\cap\text{Int}\, X_n\big) = \bigcup_{n\in\mathbb{Z}^+}X_n = X.\end{aligned}$$

Thus, $\mathcal{D}$ is an open refinement of $\mathcal{A}$ that covers $X$. By the last paragraph of part (a), the collection $\mathcal{B}_n$ is locally finite in $X$. Since

$$V(D)\cap\text{Int}\, X_n \subset V(D)\cap X_n = D \qquad \forall\,\mathcal{D}\in\mathcal{B}_n,$$

the collection $\mathcal{D}_n$ is also locally finite in $X$. Thus, $\mathcal{D}$ is $\sigma$-locally finite.