MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 5

Solution to Problem p200, #9

Let

$$A = \{x\times(-x) : x\in\mathbb{Q}\} \subset \mathbb{R}_l^2 \qquad \text{and} \qquad B = \{x\times(-x) : x\in\mathbb{R}-\mathbb{Q}\} \subset \mathbb{R}_l^2.$$

If $\mathcal{U}$ *and* V *are open subsets of* $\mathbb{R}_l^2$ *containing* A *and* B*, respectively, show that* $\mathcal{U}\cap V\neq\emptyset$*. Thus,* $\mathbb{R}_l^2$ *is not normal.*

(a) *Let*

$$K_n = \{x\in[0,1]-\mathbb{Q} : [x,x+1/n)\times[-x,-x+1/n)\subset V\}.$$

Show that $[0,1]$ *is the union of the sets* K_n *and countably many one-point sets.*

(b) *Show that some set* $\bar{K}_n$ *contains a nonempty open interval* (a,b) *of* $\mathbb{R}$*.*

(c) *Show that* V *contains the open parallelogram*

$$\{x\times(-x+\epsilon) : x\in(a,b),\ \epsilon\in(0,1/n)\}.$$

(d) *Conclude that if* $q\in(a,b)\cap\mathbb{Q}$*, then* $q\times(-q)\in\mathbb{R}_l^2$ *is a limit point of* V*. Thus, any open subset* $\mathcal{U}$ *of* $\mathbb{R}_l^2$ *containing* $q\times(-q)$ *intersects* V*.*

(a) We will show that

$$[0,1] = \bigcup_{n\in\mathbb{Z}^+} K_n \ \cup \bigcup_{q\in[0,1]\cap\mathbb{Q}} \{q\}.$$

This is equivalent to saying that for every $x\in[0,1]-\mathbb{Q}$, there exists $n\in\mathbb{Z}^+$ such that $x\in K_n$, i.e.

$$[x,x+1/n)\times[-x,-x+1/n) \subset V.$$

The set V is open in $\mathbb{R}_l^2$ and contains $x\times(-x)$, if $x\in[0,1]-\mathbb{Q}$. Since

$$\{[x,x+1/n)\times[-x,-x+1/n) : n\in\mathbb{R}_l^2\}$$

is a basis for $\mathbb{R}_l^2$ at $x\times(-x)$, it follows that

$$[x,x+1/n)\times[-x,-x+1/n) \subset V$$

for some $n\in\mathbb{Z}^+$, as needed.

(b) In the standard, i.e. order, topology, $[0,1]$ is a compact Hausdorff space. By part (a),

$$[0,1] = \bigcup_{n\in\mathbb{Z}^+} \bar{K}_n \ \cup \bigcup_{q\in[0,1]\cap\mathbb{Q}} \{q\},$$

where $\bar{K}_n$ is the closure of K_n in the standard topology on $[0,1]$. The sets $\bar{K}_n$, with $n\in\mathbb{Z}^+$, and $\{q\}$, with $q\in[0,1]\cap\mathbb{Q}$, are closed in $[0,1]$, and there are countably many of them. Since the interior of their union is $[0,1]$, and thus nonempty, the interior of one of these countably many sets is nonempty by Exercise 5 on p178, from PS4. The interior of $\{q\}$ is of course empty. Thus, for some $n \in \mathbb{Z}^+$, the interior of $\bar{K}_n$ is nonempty, i.e. $\bar{K}_n$ contains a nonempty open subset of $[0,1]$. Thus, $\bar{K}_n$ contains a nonempty open interval (a,b).

(c) Let n, a, and b be as in part (b). Suppose $s \times t$ belongs to the open parallelogram corresponding to a, b, and n, i.e.

$$a < s < b \qquad \text{and} \qquad -s < t < -s + 1/n.$$

Let $\delta = s+t \in (0, 1/n)$. Since $\bar{K}_n$ contains (a,b), there exists $x \in K_n$ such that

$$\begin{aligned} x \in (s-\delta, s) \quad &\Longrightarrow \quad -t = s-\delta < x < s \quad \Longrightarrow \quad x < s < x+\delta, \quad -x < t < -x + 1/n \\ \Longrightarrow \quad s \in [x, x+1/n), \quad t \in [-x, -x+1/n) \quad &\Longrightarrow \quad s \times t \in [x, x+1/n) \times [-x, -x+1/n) \subset V. \end{aligned}$$

Thus, entire open parallelogram is contained in V.

(d) Let n, a, and b be as in parts (b) and (c). If $q \in (a,b)$, a basis element for $\mathbb{R}_l^2$ at $q \times (-q)$ is given by

$$\mathcal{U}_\delta = [q, q+\delta) \times [-q, -q+\delta)$$

for some $\delta > 0$. Any such basis element intersects the above parallelogram. For example, let $s \in (a,b)$ be such that

$$s \in \big(q, q + \min(\delta, 1/n)\big) \cap (a,b).$$

Then, $(s, -q)$ belongs to $\mathcal{U}_\delta$ and the parallelogram. Since every basis for $\mathbb{R}_l^2$ at $q \times (-q)$ intersects V, so does the open set $\mathcal{U}$.

Solution to Problem p213, #5

Theorem (Strong Form of the Urysohn Lemma): Suppose X is a normal topological space and A and B are subsets of X. There exists a continuous function $f \colon X \longrightarrow [0,1]$ such that $f(A) = \{0\}$ $f(B) = \{1\}$, and $f(X-A-B) \subset (0,1)$ if and only if A and B are disjoint closed G_δ-sets in X.

Suppose $f \colon X \longrightarrow [0,1]$ is a continuous function such that

$$f(A) = \{0\}, \qquad f(B) = \{1\}, \quad \text{and} \quad f(X-A-B) \subset (0,1).$$

By the first two assumptions on f, the sets A and B are disjoint. By the three assumptions on f,

$$A = f^{-1}\big(\{0\}\big) = f^{-1}\bigg(\bigcap_{n \in \mathbb{Z}^+} [0, 1/n)\bigg) = \bigcap_{n \in \mathbb{Z}^+} f^{-1}\big([0, 1/n)\big).$$

Since f is continuous and $\{0\}$ is closed in $[0,1]$, A is a closed subset of X. Since f is continuous and $[0, 1/n)$ is an open subset of $[0,1]$, $f^{-1}([0,1/n))$ is open in X. Thus, A is a G_δ-set in X. Similarly,

$$B = f^{-1}\big(\{1\}\big) = f^{-1}\bigg(\bigcap_{n \in \mathbb{Z}^+} (1-1/n, 1]\bigg) = \bigcap_{n \in \mathbb{Z}^+} f^{-1}\big((1-1/n, 1]\big),$$

and B is a closed G_δ-set in X.

Suppose A and B are disjoint closed G_δ-sets in X. We will show that there exists a continuous function

$$g\colon X \longrightarrow [0,1] \qquad \text{s.t.} \qquad g(A)=\{0\}, \quad g(B)=\{1\}, \quad \text{and} \quad g(X-A-B)\subset(0,1].$$

By symmetry, there exists a continuous function

$$h\colon X \longrightarrow [0,1] \qquad \text{s.t.} \qquad h(A)=\{0\}, \quad h(B)=\{1\}, \quad \text{and} \quad h(X-A-B)\subset[0,1).$$

The function $f=(g+h)/2\colon X\longrightarrow[0,1]$ is continuous and

$$f(A)=\{0\}, \qquad f(B)=\{1\}, \quad \text{and} \quad f(X-A-B)\subset(0,1),$$

as needed.

Since A is a G_δ-set in X, there exist open subsets $\mathcal{U}_1,\mathcal{U}_2,\ldots$ in X such that

$$A=\bigcap_{n\in\mathbb{Z}^+}\mathcal{U}_n.$$

Since B is closed and disjoint from A, we can assume that for every $n\in\mathbb{Z}^+$

$$\mathcal{U}_n\cap B=\emptyset \qquad \Longleftrightarrow \qquad B\subset X-\mathcal{U}_n;$$

otherwise, we simply replace $\mathcal{U}_n$ by $\mathcal{U}_n-B$. Since X is normal and the closed sets A and $X-\mathcal{U}_n$ are disjoint, by the Urysohn Lemma there exists a continuous function

$$g_n\colon X\longrightarrow[0,1] \qquad \text{s.t.} \qquad g_n(A)=\{0\} \quad \text{and} \quad g_n(X-\mathcal{U}_n)=\{1\}.$$

For each $x\in X$, let

$$g(x)=\sum_{n=1}^{\infty}2^{-n}g_n(x).$$

Since

$$\sum_{n=1}^{\infty}2^{-n}|g_n(x)|\le\sum_{n=1}^{\infty}2^{-n}=1,$$

the first sum converges absolutely and uniformly. Thus, $g(x)\in\mathbb{R}$ is well-defined and $g\colon X\longrightarrow\mathbb{R}$ is continuous. Furthermore, $|g(x)|\le 1$ for all $x\in X$. Since $g_n(x)$ is nonnegative for all n, $g(x)\ge 0$ for all $x\in X$. Thus, $g\colon X\longrightarrow[0,1]$ is a continuous map. Since $g_n(x)=0$ for all $x\in A$,

$$g(x)=0 \ \ \forall\, x\in A \qquad \Longrightarrow \qquad g(A)=\{0\}.$$

Since $B\subset X-\mathcal{U}_n$ for all n,

$$g_n(x)=1 \ \ \forall\, x\in B \qquad \Longrightarrow \qquad g(x)=\sum_{n=1}^{\infty}2^{-n}=1 \ \ \forall\, x\in B \qquad \Longrightarrow \qquad g(B)=\{1\}.$$

Finally, since $g_n(x)\ge 0$ for all $x\in X$,

$$g^{-1}(0)=\bigcap_{n\in\mathbb{Z}^+}g_n^{-1}(0)\subset\bigcap_{n\in\mathbb{Z}^+}\mathcal{U}_n=A \qquad\Longrightarrow\qquad g(X-A-B)\subset g(X-A)\subset(0,1],$$

as needed.

Solution to Problem p223, #3

Suppose X is a metrizable topological space. Show that the following conditions on X are equivalent:
(i) *X is bounded under every metric that gives the topology of X;*
(ii) *every continuous function $f\colon X\longrightarrow\mathbb{R}$ is bounded;*
(iii) *X is limit point compact.*

(iii) $\Longrightarrow$ (i), (ii): If X is metrizable and limit point compact, then X is compact. Since a product of compact spaces is compact, X^n is also compact. Thus, the image of X under every continuous function is compact. Since a compact subset of $\mathbb{R}^n$ is bounded, in every metric, the image of X^n under every continuous function

$$f\colon X^n\longrightarrow\mathbb{R}^n$$

is bounded. Taking $n=1$ and $n=2$, we obtain (i) and (ii), respectively.

(i) $\Longrightarrow$ (ii): Suppose X is bounded under every metric that gives the topology of X, d is a metric on X, and $f\colon X\longrightarrow\mathbb{R}$ is a continuous function. We will show that

$$\tilde{d}\colon X\times X\longrightarrow\mathbb{R}, \qquad \tilde{d}(x,y)=d(x,y)+\big|f(x)-f(y)\big|,$$

is also a metric on X that gives the topology of X. Since d and $\tilde{d}$ are both bounded, it then follows that so is f.

First, we check that $\tilde{d}$ is indeed a metric. Since d is a metric,

$$\begin{aligned}
&\tilde{d}(x,y)=d(x,y)+\big|f(x)-f(y)\big|\ge 0+0=0;\\
&\tilde{d}(x,y)=d(x,y)+\big|f(x)-f(y)\big|=0 \iff d(x,y)=0,\ \big|f(x)-f(y)\big|=0 \iff x=y;\\
&\tilde{d}(x,y)=d(x,y)+\big|f(x)-f(y)\big|=d(y,x)+\big|f(y)-f(x)\big|=\tilde{d}(y,x);\\
&\tilde{d}(x,z)=d(x,z)+\big|f(x)-f(z)\big|\le\big(d(x,y)+d(y,z)\big)+\big(\big|f(x)-f(y)\big|+\big|f(y)-f(z)\big|\big)\\
&\qquad=\big(d(x,y)+\big|f(x)-f(y)\big|\big)+\big(d(y,z)+\big|f(y)-f(z)\big|\big)=\tilde{d}(x,y)+\tilde{d}(y,z).
\end{aligned}$$

Since $|f(x)-f(y)|$ is never negative,

$$d(x,y)\le\tilde{d}(x,y)\quad\forall\,x,y\in X \qquad\Longrightarrow\qquad B_{\tilde{d}}(x,\delta)\subset B_d(x,\delta)\quad\forall\,x\in X,\ \delta\in\mathbb{R}.$$

Thus, the topology induced by the metric $\tilde{d}$ is finer (or larger) than the topology induced by the metric d. The latter is the topology of X. On the other hand, since $f:X\longrightarrow\mathbb{R}$ is a continuous function by assumption and the function

$$d\colon X\times X\longrightarrow\mathbb{R}$$

is continuous by Exercise 3a on p126, from PS2, the function

$$\tilde{d}\colon X \times X \longrightarrow \mathbb{R}$$

is also continuous. Thus, the topology of X is finer than the topology of $(X, \tilde{d})$, by Exercise 3b on p126. It follows that $\tilde{d}$ induces the topology of X.

(ii) $\Longrightarrow$ (iii): Suppose X is a metrizable space, every continuous function $f\colon X \longrightarrow \mathbb{R}$ is bounded, and $A \subset X$ has no limit points in X. Let $\phi\colon A \longrightarrow \mathbb{Z}$ be any map. We will show that the map ϕ must be bounded. Thus, A is a finite set, and every infinite subset of A must have a limit point.

Since A has no limit points in X, neither does any subset of A. Thus, every subset of A is closed in X and thus in A. In particular, if $B \subset \mathbb{Z}$ is any (closed) subset, then $\phi^{-1}(B)$ is closed in A. Thus, $\phi\colon A \longrightarrow \mathbb{Z}$ is continuous. Since X is metrizable, X is normal. Since $A \subset X$ is closed, by the Tietze Extension Theorem there exists a continuous function

$$f\colon X \longrightarrow \mathbb{R} \qquad \text{s.t.} \qquad f|_A = \phi.$$

Since f is bounded, so is ϕ, as claimed.