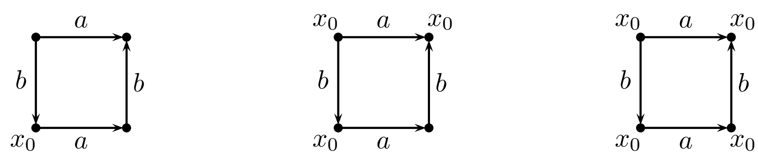
MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 12

Solution to Problem p454, #3

Let $K = X_{aba^{-1}b}$ *be the Klein bottle.*
(a) *Find a representation for the fundamental group of* K.
(b) *Find a double covering map* $p\colon T \longrightarrow K$, *where* T *is the torus. Describe the induced homomorphism of the fundamental groups.*

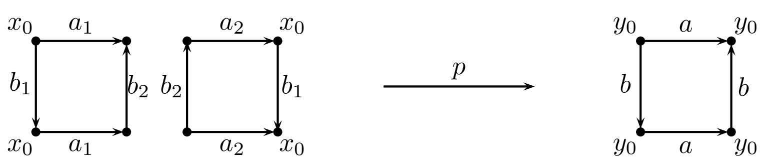
(a) Begin by labeling one of the vertices, say the bottom left, x_0 and determine which of the other vertices are identified with x_0:



Since the edges labeled a are identified, the bottom left and top left vertices are identified, since both are the beginning of where edges labeled a begin. Similarly the bottom left and top right vertices are identified because both are endpoints of segments labeled b. Finally, the top left and bottom right vertices are identified because both are beginnings of edges labeled b. Since all of the vertices of the square are mapped to one point, $\pi_1(X)$ is generated by the two labels, a and b, with the only relation given by the labeling scheme:

$$\pi_1(X) = \big\langle a, b \,|\, aba^{-1}b \big\rangle.$$

(b) Such a covering map can be described by



The diagram on the left represents a torus. The map p takes the first square directly onto the square on the right and flips the second square (along the middle horizontal line) before taking it onto the square on the right. Explicitly, we define

$$p\colon T \longrightarrow K \qquad \text{by} \qquad p\big([s,t]\big) = \begin{cases} [2s, t], & \text{if } s \in [0, 1/2]; \\ [2s-1, 1-t], & \text{if } s \in [1/2, 1]. \end{cases}$$

This map is continuous on the image of each of the two square and agrees on the overlaps (corresponding to b_1 and b_2), since

$$[2(1/2), t]_K = [1, t]_K = [0, 1-t]_K = [2(1/2)-1, 1-t]_K \qquad \text{and}$$
$$[2 \cdot 1 - 1, 1-t]_K = [1, 1-t]_K = [0, t]_K = [2 \cdot 0, t]_K.$$

We now compute the homomorphism $p_*: \pi_1(T, x_0) \longrightarrow \pi_1(K, y_0)$. Its domain is the free abelian generated by the loops $\alpha = a_1 a_2$ and $\beta = b_1$. From the picture,

$$p_* a_1 = a, \quad p_* a_2 = a, \quad p_* b_1 = b \qquad \Longrightarrow \qquad p_* \alpha = a^2, \quad p_* \beta = b.$$

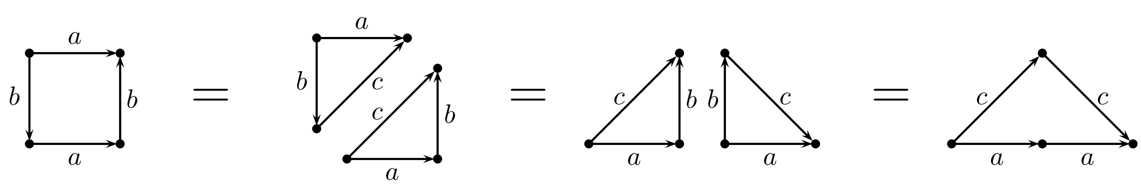
The relation $aba^{-1}b = 1$ in $\pi_1(K, y_0)$ insures that the last two equations induce a homomorphism on $\pi_1(T, x_0)$, i.e.

$$(p_*\alpha)(p_*\beta) = (p_*\beta)(p_*\alpha).$$

Solution to Problem p454, #4

(a) *Show that the Klein bottle J is homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2$.*
(b) *Describe an immersion of $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ in $\mathbb{R}^3$.*

(a) Starting with a diagram representing K, we obtain a diagram representing $\mathbb{R}P^2 \# \mathbb{R}P^2$ through the following sequence of cutting, pasting, and flipping of polygons:



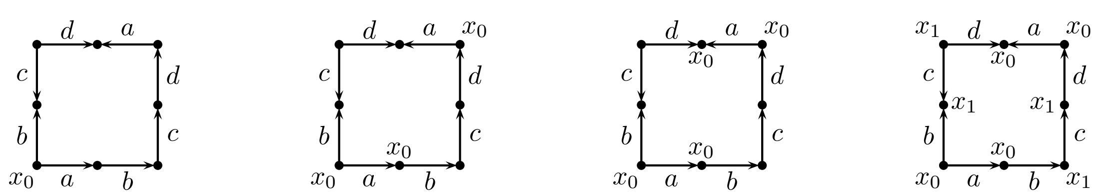
These steps do not change the quotient space.

(b) By part (a), $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 = K \# K$. Thus, an immersion of $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ can be represented by taking two copies of the Klein bottle as pictured in the last diagram of Figure 74.11 on p454, removing a small disk from each of them, and joining the two boundary circles together.

Solution to Problem p457, #4

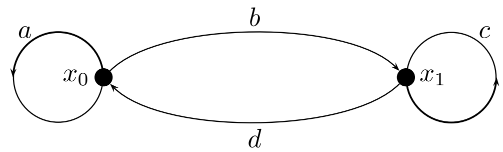
Let $q: P \longrightarrow X$ be the quotient map corresponding to the labeling scheme $abcdad^{-1}cb^{-1}$.
(a) *Show that q does not map all of the vertices of P to the same point of X.*
(b) *Determine the space $A = p(\mathrm{Bd}\, P)$ and compute its fundamental group.*
(c) *Compute $\pi_1(X)$ and $H_1(X)$.*
(d) *Determine which of the surfaces T_n and P_m the surface X is homeomorphic to, assuming it is homeomorphic to one of them.*

(a) We begin by sketching the labeling scheme and marking one of the vertices, e.g. the lower left, x_0:



The first task is to determine which of the vertices of P are identified with x_0 by q. Since the two edges labeled a are identified according to the arrows, the upper right vertex must also be marked x_0 since it is the beginning of the upper line segment labeled a. Similarly, the middle point on the bottom line segment must also be labeled x_0, since this is the beginning of a segment labeled b. This gives us the second diagram. It then follows that the middle vertex on the top segment must be labeled by x_0 as well, since it is the end of segment labeled by a. This gives us the third diagram. We can't get to any of the other four vertices in this. So, we mark one of them, e.g. the lower right, x_1; it will be taken by q to a different point from x_0. We then proceed in the same way as above starting with x_1 and find that the remaining vertices are labeled by x_1 as well; see the last diagram.

(b) We first indicate the image of the vertices under q. It is two point, x_0 and x_1. The edge of P gives paths between these points: a runs from x_0 to x_0, b from x_0 to x_1, c from x_1 to x_1, and d from x_1 to x_0:



This topological space, $A = q(\mathrm{Bd}\, P)$, is homotopy equivalent to the wedge of three circles and $\pi_1(A, x_0)$ is a free group on three generators. It is generated by the loops

$$\alpha = a, \quad \beta = bd, \quad \gamma = bcd, \qquad \text{i.e.} \qquad \pi_1(A, x_0) = \langle \alpha, \beta, \gamma | \cdot \rangle.$$

There are three other natural choices for the last generator: bcb^{-1}, $d^{-1}cd$, and dcb^{-1}. However, they all differ from each other by multiplication by β, on the left and/or on the right.

(c) By the adjoining-a-two-cell theorem, $\pi_1(X, x_0)$ is the quotient of $\pi_1(A, x_0)$ by the least normal subgroup containing $q_*\pi_1(P, x_0)$. This is determined by what happens on one of its generators, i.e. a loop going around the boundary of P once. As can be read off from the first diagram above, its image is

$$abcdad^{-1}cb^{-1} = \alpha\gamma\alpha\beta^{-1}\gamma\beta^{-1}.$$

Thus,

$$\begin{aligned}\pi_1(X, x_0) = \pi_1(A, x_0)/N\big(\alpha\gamma\alpha\beta^{-1}\gamma\beta^{-1}\big) &= \big\langle \alpha, \beta, \gamma | \cdot \big\rangle / N\big(\alpha\gamma\alpha\beta^{-1}\gamma\beta^{-1}\big) \\ &= \big\langle \alpha, \beta, \gamma | N\big(\alpha\gamma\alpha\beta^{-1}\gamma\beta^{-1}\big)\big\rangle.\end{aligned}$$

The group $H_1(X)$ is the abelianization of $\pi_1(X, x_0)$. In order to compute it, we make all generators α, β, γ of $\pi_1(X, x_0)$ commute and change to the additive notation:

$$\begin{aligned}H_1(X) = \pi_1(X, x_0)/\big[\pi_1(X, x_0), \pi_1(X, x_0)\big] &= \big(\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta] \oplus \mathbb{Z}[\gamma]\big)/\mathbb{Z}\big[2(\alpha+\gamma-\beta)\big] \\ &= \big(\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta] \oplus \mathbb{Z}[\alpha+\gamma-\beta]\big)/\mathbb{Z}\big[2(\alpha+\gamma-\beta)\big] \approx \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2.\end{aligned}$$

(d) Since $H_1(X) \approx \mathbb{Z}^2 \oplus \mathbb{Z}_2$, X must be homeomorphic to $P_3 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$. The first homology groups of the other spaces T_n and P_m are different from $H_1(X)$; so, none of them can be homeomorphic to X.

Remark: For the purposes of determining $\pi_1(X, x_0)$ in part (c), it is essential to express generators from $\pi_1(A, x_0)$ in terms of the labels of the polygon(s). The words of labels give relations that then need to be expressed in terms of the chosen generators for $\pi_1(A, x_0)$.

Solution to Problem p470, #1

Let X be a space obtained by pasting the edges of a polygonal region in pairs. Show that X is homeomorphic to exactly one of
(a) S^2*,* T_n *with* $n\geq 1$*,* $\mathbb{R}P^2$*,* K*, and* $T_n\#K$ *with* $n\geq 1$*, where* K *is the Klein bottle;*
(b) S^2*,* T_n *with* $n\geq 1$*,* $\mathbb{R}P^2$*,* K_m *with* $m\geq 1$*, and* $K_m\#\mathbb{R}P^2$ *with* $m\geq 1$*, where* K_m *is the* m*-fold connect sum of* K *with itself.*

Since X is obtained by identifying the edges of a polygon in *pairs*, X corresponds to a regular labeling scheme (i.e. every label appears exactly twice on the edges of the polygon(s)), X is a surface. It follows that X is a compact connected surface and thus must be homeomorphic to exactly one of the model surfaces in the classification theorem: S^2, T_n with $n\geq 1$, and P_m with $m\geq 1$. Thus, we need to check that the lists in (a) and (b) give exactly the same surfaces, without repetitions. Both lists contain S^2, T_n with $n\geq 1$, and $\mathbb{R}P^2$. Thus, we need to check that the remaining surfaces on the lists are in bijection with P_m with $m\geq 2$. Since compact connected surfaces are classified by the first homology group, we need to check that the first homology groups of the remaining surfaces on the list correspond to the first homology groups of $m\geq 2$, i.e. $\mathbb{Z}^{m-1}\oplus\mathbb{Z}_2$.

(b) We need to show that the surfaces K_m and $K_m\#\mathbb{R}P^2$ with $m\geq 1$ are the same as the surfaces P_n with $n\geq 2$. In fact, it is shown directly in p454, #4a, that

$$K=\mathbb{R}P^2\#\mathbb{R}P^2 \qquad\Longrightarrow\qquad K_m=P_{2m} \quad\text{and}\quad K_m\#\mathbb{R}P^2=P_{2m}\#\mathbb{R}P^2=P_{2m+1},$$

as needed.

Alternatively, we can compute the first homology groups of K_m and $K_m\#\mathbb{R}P^2$. Since K is represented by the labeling scheme $aba^{-1}b$ (see Figure 74.11 on p454 in the book), K_m is represented by $\prod_{k=1}^{k=m}(a_kb_ka_k^{-1}b_k)$. Since all vertices in the labeling scheme $aba^{-1}b$ are identified, all vertices in the labeling scheme $\prod_{k=1}^{k=m}(a_kb_ka_k^{-1}b_k)$ are identified. Thus,

$$\pi_1(X)=\big\langle a_1,b_1,\ldots,a_m,b_m\big|\prod_{k=1}^{k=m}(a_kb_ka_k^{-1}b_k)\big\rangle \qquad\Longrightarrow$$

$$\begin{aligned}H_1(X)&=\big(\mathbb{Z}[a_1]\oplus\mathbb{Z}[b_1]\oplus\ldots\oplus\mathbb{Z}[a_m]\oplus\mathbb{Z}[b_m]\big)/\mathbb{Z}\big[2(b_1+\ldots+b_m)\big]\\&=\mathbb{Z}[a_1]\oplus\ldots\oplus\mathbb{Z}[a_m]\oplus\mathbb{Z}[b_1]\oplus\ldots\mathbb{Z}[b_{m-1}]\oplus\mathbb{Z}[b_1+\ldots+b_m]/\mathbb{Z}\big[2(b_1+\ldots+b_m)\big]=\mathbb{Z}^{2m-1}\oplus\mathbb{Z}_2.\end{aligned}$$

Similarly, $K_m\#\mathbb{R}P^2=\mathbb{R}P^2\#K_m$ is represented by $cc\prod_{k=1}^{k=m}(a_kb_ka_k^{-1}b_k)$. Since all vertices in this labeling scheme are mapped to the same point,

$$\pi_1(X)=\big\langle c,a_1,b_1,\ldots,a_m,b_m\big|c^2\prod_{k=1}^{k=m}(a_kb_ka_k^{-1}b_k)\big\rangle \qquad\Longrightarrow$$

$$\begin{aligned}H_1(X)&=\big(\mathbb{Z}[c]\oplus\mathbb{Z}[a_1]\oplus\mathbb{Z}[b_1]\oplus\ldots\oplus\mathbb{Z}[a_m]\oplus\mathbb{Z}[b_m]\big)/\mathbb{Z}\big[2(c+b_1+\ldots+b_m)\big]\\&=\mathbb{Z}[c]\oplus\mathbb{Z}[a_1]\oplus\ldots\oplus\mathbb{Z}[a_m]\oplus\mathbb{Z}[b_1]\oplus\ldots\mathbb{Z}[b_{m-1}]\oplus\mathbb{Z}[c+b_1+\ldots+b_m]/\mathbb{Z}\big[2(c+b_1+\ldots+b_m)\big]\\&=\mathbb{Z}^{2m}\oplus\mathbb{Z}_2.\end{aligned}$$

We conclude that the first homology groups of the spaces K_m and $K_m\#\mathbb{R}P^2$ with $m\geq 1$ correspond to the first homology groups of P_n with $n\geq 2$. Thus, the compact connected surfaces K_m and $K_m\#\mathbb{R}P^2$ with $m\geq 1$ correspond to the compact connected surfaces P_n with $n\geq 2$.