MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 11

Solution to Problem p433, #2

Suppose $\mathcal{U}, V \subset X$ *are open,* $X = \mathcal{U} \cap V$*,* U*,* V*, and* $\mathcal{U} \cap V$ *are path-connected,* $x_0 \in \mathcal{U} \cap V$*, and*

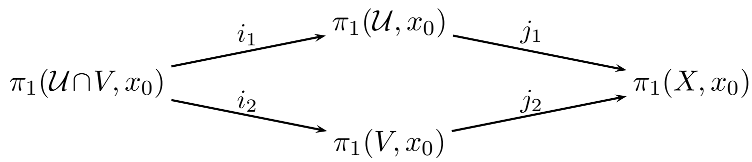

Figure 1: Van Kampen's Theorem Setting

are the homomorphisms induced by inclusions. Suppose in addition that i_2 *is surjective. Let* $M \subset \pi_1(X, x_0)$ *be the least normal subgroup containing* $i_1(\ker i_2)$*.*
(a) *Show that* j_1 *induces a surjective homomorphism*

$$h\colon \pi_1(\mathcal{U}, x_0)/M \longrightarrow \pi_1(X, x_0).$$

(b) *Show that* h *is an isomorphism.*

(a) By (the weak version of) van Kampen's Theorem, the homomorphism

$$j_1 * j_2 \colon \pi_1(\mathcal{U}, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective. Since

$$j_1 \circ i_1 = j_2 \circ i_2 \colon \pi_1(\mathcal{U} \cap V, x_0) \longrightarrow \pi_1(X, x_0),$$

i.e. the diagram in Figure **??** is commutative, and i_2 is surjective in this case,

$$\mathrm{Im}\, j_2 = \mathrm{Im}\, j_2 \circ i_2 = \mathrm{Im}\, j_1 \circ i_1 \subset \mathrm{Im}\, i_1 \subset \pi_1(X, x_0).$$

Thus, the homomorphism j_1 is surjective. In addition, since the diagram in Figure **??** is commutative

$$\ker i_2 \subset \ker i_2 \circ j_2 = \ker i_1 \circ j_1 \qquad \Longrightarrow \qquad i_1(\ker i_2) \subset \ker j_1.$$

Since $\ker j_1$ is a normal subgroup of $\pi_1(\mathcal{U}, x_0)$ and contains $i_1(\ker i_2)$, it must contain M as well. Thus, j_1 induces a homomorphism

$$h\colon \pi_1(\mathcal{U}, x_0)/M \longrightarrow \pi_1(X, x_0).$$

Since j_1 is surjective, so is h.

(b) Define homomorphisms

$$\phi_1 \colon \pi_1(\mathcal{U}, x_0) \longrightarrow \pi_1(\mathcal{U}, x_0)/M \quad \text{and} \quad \phi_2 \colon \pi_1(V, x_0) \longrightarrow \pi_1(\mathcal{U}, x_0)/M \qquad \text{by}$$
$$\phi_1(\alpha) = \alpha M \quad \text{and} \quad \phi_2\big(i_2(\alpha)\big) = \phi_1\big(i_1(\alpha)\big).$$

Since i_2 is surjective and $i_1(\ker i_2) \subset M$, ϕ_2 is well-defined. It is immediate that

$$\phi_1 \circ i_1 = \phi_2 \circ i_2 \colon \pi_1(\mathcal{U} \cap V, x_0) \longrightarrow \pi_1(\mathcal{U}, x_0)/M,$$

i.e. the diagram

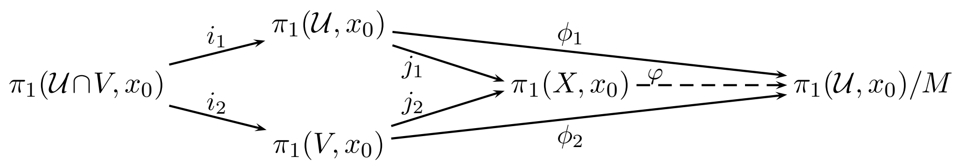

Figure 2: Amalgated Product Setting

of solid lines is commutative. Since Figure **??** is an amalgated product by van Kampen's Theorem, there exists a (unique) homomorphism

$$\varphi\colon \pi_1(X, x_0) \longrightarrow \pi_1(\mathcal{U}, x_0)/M \qquad \text{s.t.} \qquad \phi_1 = \varphi \circ j_1 \quad \text{and} \quad \phi_2 = \varphi \circ j_2.$$

In particular,

$$\varphi\big(h(\alpha M)\big) = \varphi\big(j_1(\alpha)\big) = \phi_1(\alpha) = \alpha M \quad \forall\, \alpha M \in \pi_1(\mathcal{U}, x_0)/M \qquad \Longrightarrow \qquad \varphi \circ h = \mathrm{id}_{\pi_1(\mathcal{U}, x_0)/M}.$$

Thus, h is an injective homomorphism. On the other hand, it is surjective by part (a). We conclude that h is an isomorphism (and its inverse is φ).

Solution to Problem p438, #5

Let $S_n \subset \mathbb{R}^2$ be the circle with center at $(n,0)$ and of radius n. Let Y be the subspace of $\mathbb{R}^2$ consisting of the circles S_n, with $n \in \mathbb{Z}^+$. Denote the common point of the circles by p.

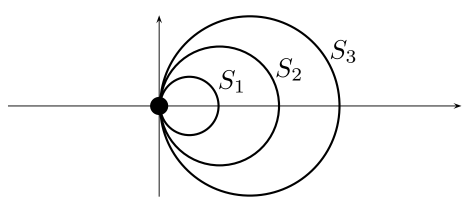

Figure 3: Some Circles S_n

(a) *Show that Y is not homeomorphic to either a countably infinite wedge X of circles or the infinite earring Z of Example 1 on p436.*
(b) *Show that $\pi_1(Y, p)$ is a free abelian group with $\{[f_n]\}$ as a system of free generators, where f_n is a loop representing a generator of $\pi_1(S_n, p)$.*

(a) Since Y is an unbounded (in the standard metric) subset of $\mathbb{R}^2$, Y is not compact and is second countable. Since Z is closed and bounded with respect to the standard metric on $\mathbb{R}^2$, Z is compact. On the other hand, if $q \in X$ is the point common to all of the circles in X, $X - \{q\}$ is homeomorphic to $\mathbb{Z}^+ \times (S^1 - \{q\})$. Since $X - \{q\}$ is not second countable, neither is X. Thus, Y is homeomorphic to neither X not Z.

Remark: In fact, X does not have a countable basis at p. So, X is not even first countable.

(b) For each $n \in \mathbb{Z}^+$, let

$$i_n\colon \pi_1(S_n, p) \longrightarrow \pi_1(Y, p)$$

be the homomorphism induced by the inclusion $(S_n,p)\longrightarrow(Y,p)$. We will show that the homomorphism

$$\prod_{n\in\mathbb{Z}^+} i_n\colon \prod_{n\in\mathbb{Z}^+}\pi_1(S_n,p)\longrightarrow\pi_1(Y,p)$$

is an isomorphism. First, let

$$r_N\colon Y\longrightarrow Y_N\equiv\bigcup_{n=1}^{n=N}S_n$$

be the retraction obtained by collapsing the circles S_n with $n>N$ to the point p. By the existence of such a retraction, the homomorphism

$$j_N\colon \pi_1(Y_N,p)\longrightarrow\pi_1(Y,p)$$

induced by the inclusion $(Y_N,p)\longrightarrow(Y,p)$ is injective. If $n\le N$, let

$$i_{N,n}\colon \pi_1(S_n,p)\longrightarrow\pi_1(Y_N,p)$$

be the homomorphism induced by the inclusion $(S_n,p)\longrightarrow(Y_N,p)$. By Theorem 71.1, the homomorphism

$$\prod_{n=1}^{n=N} i_{N,n}\colon \prod_{n=1}^{n=N}\pi_1(S_n,p)\longrightarrow\pi_1(Y_N,p)$$

is an isomorphism. Thus, the homomorphism

$$\prod_{n=1}^{n=N} i_n = j_N\circ\prod_{n=1}^{n=N} i_{N,n}\colon \prod_{n=1}^{n=N}\pi_1(S_n,p)\longrightarrow\pi_1(Y,p)$$

is injective, and so is $\prod_{n=1}^{\infty} i_n$.

It remains to show that every element $[\alpha]$ of $\pi_1(Y,p)$ lies in the image of the homomorphism j_N for some $N\in\mathbb{Z}^+$. Let

$$\alpha\colon \big(I,\{0,1\}\big)\longrightarrow(Y,p)$$

be a loop in Y based at p. Since $\alpha(I)$ is compact, $\alpha(I)$ is bounded and thus

$$\alpha(I)\subset Y_N^*\equiv Y-\bigcup_{n=N+1}^{\infty}\{(2n,0)\}$$

for some $N\in\mathbb{Z}^+$. Let

$$H\colon Y_N^*\times I\longrightarrow Y_N^*$$

be a deformation retraction of Y_N^* onto Y_N, i.e. a homotopy from $\mathrm{id}_{Y_N^*}$ to $r_N|_{Y_N^*}$ such that $H(x,t)=x$ for all $x\in Y_n$. Such a homotopy is obtained by retracting the open upper and lower semicircles of S_n, with $n>N$, to p. Then, $H\circ\{\alpha\times\mathrm{id}_I\}$ is a path homotopy from the loop α in Y to the loop $r_N\circ\alpha$ in Y_N. In particular,

$$[\alpha]=\big[r_N\circ\alpha\big]\in\pi_1(Y,p)\qquad\text{and}\qquad\big[r_N\circ\alpha\big]\in\operatorname{Im} j_N,$$

as needed.

Solution to Problem p441, #3

Suppose G is a group, $h\in G$, and N is the least normal subgroup of G containing h. Show that if $\pi_1(X)\approx G$ for some (compact) path-connected normal topological space X, then $\pi_1(Y)\approx G$ for some (compact) path-connected normal topological space Y.

Let $p\colon I\longrightarrow S^1$, $p(s)=e^{2\pi is}$, be the usual quotient map. Choose a representative

$$\alpha\colon \big(I,\{0,1\}\big) \longrightarrow (X,x_0)$$

for $h\in\pi_1(X,x_0)$. Since $\alpha(0)=\alpha(1)$, α induces a continuous map $f: S^1\longrightarrow X$ such that $\alpha=f\circ p$, i.e. the diagram

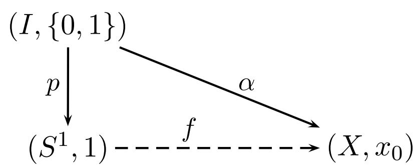

Figure 4: A Commutative Diagram

commutes. Let

$$X_\alpha = \big(X\sqcup B^2\big)/\sim, \qquad x\sim f(x) \;\; \forall\, x\in S^1\subset B^2.$$

Let $q\colon X\sqcup B^2\longrightarrow X_\alpha$ be the quotient map. If X is compact, then so are $X\sqcup B^2$ and thus X_α. Since X is path-connected and B^2 are path-connected, so are $q(X)$ and $q(B^2)$. Since

$$X_\alpha = q(X)\cup q(B^2) \qquad \text{and} \qquad q(X)\cap q(B^2)\neq\emptyset,$$

it follows that X_α is path-connected. It is shown in the next paragraphs that X_α is normal. Finally, by Figure **??**, $f_*\pi_1(S^1,1)\subset\pi_1(X,x_0)$ is generated by $h=[\alpha]$. Since the map

$$q|_{B^2-S^1}\colon B^2-S^1 \longrightarrow X_\alpha$$

is a homeomorphism, Theorem 72.1 implies that

$$\pi_1\big(X_\alpha,q(x_0)\big) \approx \pi_1(X,\alpha)/N.$$

We now show that X_α is normal, i.e. X_α is $T1$ (one-point sets are closed) and disjoint closed sets can be separated by continuous functions. We begin by showing that the map q is closed. If $A\subset X$ is closed, then

$$q^{-1}\big(q(A)\big) = q^{-1}\big(q(A)\big)\cap X\cup q^{-1}\big(q(A)\big)\cap B^2 = q|_X^{-1}\big(q(A)\big)\cup q|_{B^2}^{-1}\big(q(A)\big) = A\cup f^{-1}(A),$$

since $q|_A$ is injective and $q(X)\cap q(B^2-S^1)=\emptyset$. Since f is continuous, $f^{-1}(A)$ is closed in S^1. Since S^1 is closed in B^2, it follows that $f^{-1}(A)$ is closed in B^2 and thus $q^{-1}\big(q(A)\big)$ is closed in $X\sqcup B^2$. Since q is a quotient map, $q(A)$ is then closed in X_α. On the other hand, if $A\subset B^2$, then

$$q^{-1}\big(q(A)\big) = q^{-1}\big(q(A)\big)\cap X\cup q^{-1}\big(q(A)\big)\cap B^2 = q|_X^{-1}\big(q(A)\big)\cup q|_{B^2}^{-1}\big(q(A)\big) = f(A\cap S^1)\cup A.$$

Since A is closed in B^2 and S^1 is compact, $A\cap S^1$ is closed in S^1 and thus compact. It follows that $f(A\cap S^1)$ is a compact subset of X. Since X is Hausdorff, $f(A\cap S^1)$ is a closed subset of X. Thus, $q^{-1}\big(q(A)\big)$ is closed in $X\sqcup B^2$ and $q(A)$ is closed in X_α. We conclude that the quotient map is closed and the space X_α is Hausdorff.

It remains to show that closed subsets of X_α can be separated by continuous functions. First note that the map

$$q|_X : X \longrightarrow q(X) \subset X_\alpha$$

is continuous, bijective, and closed. Thus, it is a homeomorphism. Since X is normal, so is $q(X)$. Suppose that $A, B \subset X_\alpha$ are disjoint closed subsets. Then, $A\cap q(X)$ and $B\cap q(X)$ are disjoint closed subsets of $q(X)$. Since $q(X)$ is normal, by Urysohn Lemma there exists a continuous function

$$g_X : q(X) \longrightarrow [0,1] \qquad \text{s.t.} \qquad g_X\big(A\cap q(X)\big) = \{0\} \quad \text{and} \quad g_X\big(B\cap q(X)\big) = \{1\}.$$

Then,

$$g_X\circ q \colon S^1 \longrightarrow [0,1]$$

is continuous function such that

$$g_X\circ q\big(q^{-1}(A)\cap S^1\big) = \{0\} \quad \text{and} \quad g_X\circ q\big(q^{-1}(B)\cap S^1\big) = \{1\}.$$

Define

$$g\colon S^1 \cup \big(q^{-1}(A)\cap B^2\big) \cup \big(q^{-1}(B)\cap B^2\big) \longrightarrow [0,1] \quad \text{by} \quad g(x) = \begin{cases} g_X\circ q(x), & \text{if } x\in S^1; \\ 0, & \text{if } x\in q^{-1}(A)\cap B^2; \\ 1, & \text{if } x\in q^{-1}(B)\cap B^2. \end{cases}$$

These definitions agree on the overlap and define a continuous function on each of the three closed sets. By the pasting lemma, g is continuous. Since B^2 is normal and

$$S^1 \cup \big(q^{-1}(A)\cap B^2\big) \cup \big(q^{-1}(B)\cap B^2\big) \subset B^2$$

is closed, by Tietze's Extension Theorem g extends to a continuous function

$$h_{B^2} \colon B^2 \longrightarrow [0,1], \qquad \text{i.e.} \qquad h_{B^2}(x) = g(x) = \begin{cases} g_X\circ q(x), & \text{if } x\in S^1; \\ 0, & \text{if } x\in q^{-1}(A)\cap B^2; \\ 1, & \text{if } x\in q^{-1}(B)\cap B^2. \end{cases}$$

Let $h_X = g_X\circ q$. Then, the function

$$h_X\sqcup h_{B^2} \colon X\sqcup B^2 \longrightarrow [0,1]$$

is continuous and

$$\begin{aligned} h_X(f(x)) = g_X\big(q(f(x))\big) = g_X(q(x)) = h_{B_2}(x) \quad \forall\, x\in S^1\subset B^2, \\ h_X\sqcup h_{B^2}\big(q^{-1}(A)\big) = g_X\big(A\cap q(X)\big) \cup g_{B^2}\big(q^{-1}(A)\cap B^2\big) = \{0\}, \qquad \text{and} \\ h_X\sqcup h_{B^2}\big(q^{-1}(B)\big) = g_X\big(B\cap q(X)\big) \cup g_{B^2}\big(q^{-1}(B)\cap B^2\big) = \{1\}. \end{aligned}$$

By the first property, $h_X\sqcup h_{B^2}$ induces a map $h\colon X_\alpha \longrightarrow [0,1]$ such that $h_X\sqcup h_{B^2} = h\circ q$, i.e. the diagram

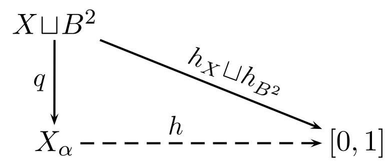

Figure 5: Construction of Separating Map

commutes. The function h is continuous, because q is a quotient map. By the other two properties,

$$h(A) = h_X \sqcup h_{B^2}\big(q^{-1}(A)\big) = \{0\} \qquad \text{and} \qquad h(B) = h_X \sqcup h_{B^2}\big(q^{-1}(B)\big) = \{1\},$$

as needed.

Solution to Problem p445, #2

Show that for every finitely presentable group G, there exists a compact Hausdorff path-connected space X such that $\pi_1(X) \approx G$.

Suppose

$$G = \big\langle \alpha_1, \ldots, \alpha_n | r_1, \ldots, r_n \big\rangle, \qquad \text{i.e.} \qquad G = \mathbb{Z}[\alpha_1] * \ldots * \mathbb{Z}[\alpha_n] / N(r_1, \ldots, r_m),$$

where $N(r_1, \ldots, r_m)$ is the smallest normal subgroup of $\mathbb{Z}[\alpha_1] * \ldots * \mathbb{Z}[\alpha_n]$ containing

$$\{r_1, \ldots, r_m\} \subset \mathbb{Z}[\alpha_1] * \ldots * \mathbb{Z}[\alpha_n].$$

For each $k = 0, \ldots, m$, let

$$H_k = N(r_1, \ldots, r_k), \qquad G_k = G/H_k, \qquad h_k = r_k H_{k-1} \in G_{k-1} \ \text{ if } \ k > 0.$$

We note that the smallest normal subgroup N_k of G_{k-1} containing h_k is

$$H_k H_{k-1} \equiv \bigcup_{h \in H_k} h H_{k-1} \subset G_{k-1}.$$

Thus, $G_k \approx G_{k-1}/N_k$.

Let X_0 be the wedge of n circles. Let p be the point common to all of the circles. By Theorem 71.1,

$$\pi_1(X, p) \approx \mathbb{Z}[\alpha_1] * \ldots * \mathbb{Z}[\alpha_n] = G_0,$$

where α_i is the homotopy class of a loop going around the ith circle once. The space X_0 is compact Hausdorff and path-connected. Suppose $k \in \mathbb{Z}^+$, $k \leq n$, and there exists a compact Hausdorff path-connected space X_{n-1} such that $\pi_1(X_{k-1}) \approx G_{k-1}$. Then, by Problem p441, #3, there exists a compact Hausdorff path-connected space X_k such that

$$\pi_1(X_k) \approx G_{k-1}/N_k \approx G_k.$$

After applying this construction m times, we obtain a compact Hausdorff path-connected space $X \equiv X_m$ such that

$$\pi_1(X) = \pi_1(X_m) \approx G_m \equiv G.$$

Remark: In brief, in order to obtain a compact Hausdorff path-connected space whose fundamental group is G we begin with the wedge of n circles and then make the elements $r_1, \ldots, r_m$ null-homotopic by attaching m disks B^2. The jth disk is attached by wrapping its boundary, S^1, along a representative for r_j, which can be taken to be a path going around some of the circles, possibly multiple times.