MAT 530: Topology&Geometry, I
Fall 2005

Midterm Solutions

Note: These solutions are more detailed than solutions sufficient for full credit.

Problem 1 (5+5 pts)

Let X denote the set $\{a,b,c\}$. The collections

$$\mathcal{T}_1 = \big\{\emptyset, X, \{a\}, \{a,b\}\big\} \qquad and \qquad \mathcal{T}_2 = \big\{\emptyset, X, \{b,c\}\big\}$$

are topologies on X.
(a) *What is the largest topology on X which is smaller (coarser) than both $\mathcal{T}_1$ and $\mathcal{T}_2$?*
(b) *What is the smallest topology on X which is larger (finer) than both $\mathcal{T}_1$ and $\mathcal{T}_2$?*

(a) $\mathcal{T} = \{\emptyset, X\}$, i.e. the trivial topology. By assumption,

$$\mathcal{T} \subset \mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, X\}.$$

Since $\{\emptyset, X\}$ happens to be a topology on X, this is the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.
Note: The intersection of any collection of topologies on a set is again a topology.

(b) $\mathcal{T} = \big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\big\}$. By assumption,

$$\mathcal{T} \supset \mathcal{T}_1 \cup \mathcal{T}_2 = \big\{\emptyset, X, \{a\}, \{a,b\}, \{b,c\}\big\}.$$

Since $\mathcal{T}$ is a topology and contains $\{a,b\}$ and $\{b,c\}$, $\mathcal{T}$ must also contain their intersection, i.e. $\{b\}$. Thus,

$$\mathcal{T} \supset \big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\big\}.$$

Since the collection on the right is closed under (finite) intersections and (arbitrary) unions of its elements, it is a topology on X. Thus, this is the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$.

Problem 2 (20 pts)

Show that the subset

$$A = (0,2)^\omega \equiv \prod_{k\in\mathbb{Z}^+} (0,2)$$

of $\mathbb{R}^\omega$ *is not open in the uniform topology on* $\mathbb{R}^\omega$.

Let $\bar{\rho}$ denote the uniform metric on $\mathbb{R}^\omega$. Let

$$\mathbf{x} = \big(1/n\big)_{n\in\mathbb{Z}^+} \in A.$$

It is sufficient to show that no ball $B_{\bar{\rho}}(\mathbf{x},\delta)$ centered $\mathbf{x}$ is contained in A. Suppose $\delta > 0$. Choose $n \in \mathbb{Z}^+$ such that $1/n < \delta$. Then,

$$\mathbf{y} \equiv \big(1, 1/2, \ldots, 1/n, 0, 0, \ldots\big) \in B_{\bar{\rho}}(\mathbf{x},\delta),$$

since

$$\bar{\rho}(\mathbf{x},\mathbf{y}) \equiv \sup\big\{ \min(|x_k - y_k|, 1) \colon k\in\mathbb{Z}^+ \big\} = 1/(n+1) < \delta.$$

However, $y \notin A$, since the $(n+1)$st coordinate of $\mathbf{y}$ does not lie in $(0,2)$

Problem 3 (20 pts)

Suppose J *is a set and* X_α *is a compact Hausdorff space for each* $\alpha \in J$. *Show that the space* $\prod_{\alpha\in J} X_\alpha$ *is normal in the product topology.*

Since X_α is Hausdorff for every $\alpha \in J$, $\prod_{\alpha\in J} X_\alpha$ is also Hausdorff. Since X_α is compact for every $\alpha \in J$, $\prod_{\alpha\in J} X_\alpha$ is also compact (in the product topology), by the Tychonoff Theorem. Since $\prod_{\alpha\in J} X_\alpha$ is compact and Hausdorff, it is normal.
Note: Since X_α is compact Hausdorff, X_α is normal for every $\alpha \in J$. However, since the product of a collection of normal spaces may not be normal, it does not follow that $\prod_{\alpha\in J} X_\alpha$ is normal. Thus, the order of the argument matters here.

Problem 4 (20 pts)

Suppose that X *is a topological space and* Y *is a compact topological space. Show that the projection map* $\pi_1 \colon X \times Y \longrightarrow X$ *is closed.*

Let A be a closed subset of $X \times Y$. We show that $\pi_1(A)$ is closed by showing that $X - \pi_1(A)$ is open. Suppose $x \in X - \pi_1(A)$. Then

$$\{x\} \times Y = \pi_1^{-1}(x) \subset X \times Y - \pi_1^{-1}\big(\pi_1(A)\big) \subset X \times Y - A.$$

Since the slice $\{x\} \times Y$ is contained in the open subset $X \times Y - A$ of $X \times Y$ and Y is compact, by the Tube Lemma there exists an open subset $\mathcal{U}$ of X such that

$$\{x\} \times Y \subset \mathcal{U} \times Y \subset X \times Y - A \quad\Longrightarrow\quad A \subset X \times Y - \mathcal{U} \times Y = (X - U) \times Y \quad\Longrightarrow\quad \pi_1(A) \subset X - U.$$

Thus, $\mathcal{U}$ is an open neighborhood of x in X which is contained in $X - \pi_1(A)$.

Problem 5 (15+15 pts)

Suppose X is a paracompact Hausdorff space and $(\mathcal{U}_\alpha)_{\alpha\in J}$ is an indexed collection of open subsets of X whose union covers X.

(a) *Show that there exists a locally finite indexed collection $(V_\alpha)_{\alpha\in J}$ of open subsets of X whose union covers X such that $\bar{V}_\alpha \subset \mathcal{U}_\alpha$ for all $\alpha\in J$;*

(b) *Show that there exists a partition of unity $(\phi_\alpha)_{\alpha\in J}$ subordinate to $(\mathcal{U}_\alpha)_{\alpha\in J}$.*

(a) Since X is paracompact and Hausdorff, X is normal. Three approaches to (a) are described below. They all use the Axiom of Choice (or the Well-Ordering Theorem), explicitly and implicitly.

Approach 1: Let

$$\mathcal{A} = \big\{\mathcal{U}\subset X \text{ open}\colon \bar{\mathcal{U}}\subset\mathcal{U}_\alpha \text{ for some } \alpha\in J\big\}.$$

Since $(\mathcal{U}_\alpha)_{\alpha\in J}$ covers X *and* X is regular, $\mathcal{A}$ also covers X. Thus, $\mathcal{A}$ is an open cover of X. Since X is paracompact, $\mathcal{A}$ has a locally finite open refinement $\mathcal{B}$ covering X. In particular, for every $V\in\mathcal{B}$ there exists $\mathcal{U}\in\mathcal{A}$ such that $V\subset\mathcal{U}$. Since for every $\mathcal{U}\in\mathcal{A}$, there exists $\alpha\in J$ such that $\bar{\mathcal{U}}\subset\mathcal{U}_\alpha$, it follows that for every $V\in\mathcal{B}$ there exists $f(V)\in J$ such that $\bar{V}\subset\mathcal{U}_{f(V)}$. For every $\alpha\in J$, let

$$V_\alpha = \bigcup_{f(V)=\alpha} V.$$

Since the collection $\{V\in\mathcal{B}\colon f(V)=\alpha\}\subset\mathcal{B}$ is locally finite and $\bar{V}\subset\mathcal{U}_{f(V)}$ for all $V\in\mathcal{B}$,

$$\bar{V}_\alpha \equiv \overline{\bigcup_{f(V)=\alpha} V} = \bigcup_{f(V)=\alpha} \bar{V} \subset \mathcal{U}_\alpha.$$

Since the collection $\mathcal{B}$ is an open cover of X, so is the indexed collection $(V_\alpha)_{\alpha\in J}$. Since $\mathcal{B}$ is locally finite, so is $(V_\alpha)_{\alpha\in J}$. In fact, if W is any subset of X, then

$$\big\{\alpha\in J\colon W\cap V_\alpha\neq\emptyset\big\} = \big\{f(V)\colon V\in\mathcal{B};\ W\cap V\neq\emptyset\big\}.$$

Approach 2: Well-order the set J. Since X is paracompact, there exists an indexed locally finite open collection $(W_\alpha)_{\alpha\in J}$ that refines $(\mathcal{U}_\alpha)_{\alpha\in J}$, i.e. $W_\alpha\subset U_\alpha$ for all $\alpha\in J$, and covers X; see below. We will now shrink the sets W_α. Suppose $\alpha\in J$ and for every $\beta<\alpha$, we have constructed an open subset V_β of X such that $\bar{V}_\beta\subset W_\beta$ and

$$X = \bigcup_{\beta<\alpha} V_\beta \cup \bigcup_{\beta\geq\alpha} W_\beta.$$

Thus,

$$X - \bigcup_{\beta<\alpha} V_\beta - \bigcup_{\beta>\alpha} W_\beta \subset W_\alpha.$$

Since X is normal, there exists an open subset V_α of X such that $\bar{V}_\alpha \subset W_\alpha$ and

$$X - \bigcup_{\beta<\alpha} V_\beta - \bigcup_{\beta>\alpha} W_\beta \subset V_\alpha \qquad \Longrightarrow \qquad X = \bigcup_{\beta\le\alpha} V_\beta \cup \bigcup_{\beta>\alpha} W_\beta.$$

Thus, we can construct inductively an indexed collection $\{V_\alpha\}_{\alpha\in J}$ of open subset of X such that for all $\alpha\in J$

$$\bar{V}_\alpha \subset W_\alpha \subset V_\alpha \qquad \text{and} \qquad X = \bigcup_{\beta\le\alpha} V_\beta \cup \bigcup_{\beta>\alpha} W_\beta.$$

Since $(W_\alpha)_{\alpha\in J}$ is locally finite, so is $(V_\alpha)_{\alpha\in J}$. It remains to check that $(V_\alpha)_{\alpha\in J}$ covers X. Given $x\in X$, let

$$J_x = \{\alpha\in J\colon x\in W_\alpha\}.$$

Since $(W_\alpha)_{\alpha\in J}$ is a locally finite cover of X, J_x is a finite non-empty subset of J. Let α be the largest element of J_x. If

$$x \in X - \bigcup_{\beta<\alpha} V_\beta - \bigcup_{\beta>\alpha} W_\beta,$$

then $x\in V_\alpha$. On the other hand, if

$$x \notin X - \bigcup_{\beta<\alpha} V_\beta - \bigcup_{\beta>\alpha} W_\beta,$$

then $x\in V_\beta$ for some $\beta<\alpha$, since $x\notin W_\beta$ for all $\beta>\alpha$.

Note: The second sentence of the previous paragraph is not what the definition of paracompactness says, but the two statements are equivalent. If X is paracompact, the open cover

$$\mathcal{A} = \{\mathcal{U}_\alpha\colon \alpha\in J\}$$

has a locally finite open refinement $\mathcal{B}$ that covers X. In particular, for every $V\in\mathcal{B}$ there exists $f(V)\in J$ such that $V\subset U_{f(V)}$. Let

$$W_\alpha = \bigcup_{f(V)=\alpha} V \subset \mathcal{U}_\alpha.$$

Then, $(W_\alpha)_{\alpha\in J}$ is an open cover of X, because $\mathcal{B}$ is. Similarly to end of *Approach 1*, $(W_\alpha)_{\alpha\in J}$ is locally finite, because $\mathcal{B}$ is.

Approach 3: Since X is regular and paracompact, there exists an indexed locally finite closed collection $(C_\alpha)_{\alpha\in J}$ that refines $(\mathcal{U}_\alpha)_{\alpha\in J}$, i.e. $C_\alpha\subset U_\alpha$ for all $\alpha\in J$, and covers X; see below. Since X is normal, for every $\alpha\in J$ there exists an open subset W_α of X such that $C_\alpha\subset W_\alpha$ and $\overline{W}_\alpha\subset\mathcal{U}_\alpha$. Since $(C_\alpha)_{\alpha\in J}$ covers X, $(W_\alpha)_{\alpha\in J}$ is an open cover of X. Since X is paracompact, $(W_\alpha)_{\alpha\in J}$ has a locally finite open refinement $(V_\alpha)_{\alpha\in J}$ that covers X; see the note above. Since $V_\alpha\subset W_\alpha$ and $\overline{W}_\alpha\subset\mathcal{U}_\alpha$, $\bar{V}_\alpha\subset\mathcal{U}_\alpha$ for all $\alpha\in J$, as needed.

Note: By the equivalence-of-covering-conditions Lemma 41.3 for regular spaces, the open cover

$$\mathcal{A} = \{\mathcal{U}_\alpha\colon \alpha\in J\}$$

has a locally finite closed refinement $\mathcal{B}$ that covers X. In particular, for every $C\in\mathcal{B}$ there exists $f(C)\in J$ such that $C\subset U_{f(C)}$. Let

$$C_\alpha = \bigcup_{f(C)=\alpha} C \subset \mathcal{U}_\alpha.$$

Similarly to the previous note, $(C_\alpha)_{\alpha\in J}$ is an indexed locally finite cover of X. Since $\mathcal{B}$ is a locally finite closed collection,

$$\bar{C}_\alpha = \overline{\bigcup_{f(C)=\alpha} C} = \bigcup_{f(C)=\alpha} \bar{C} = \bigcup_{f(C)=\alpha} C = C_\alpha.$$

Thus, $(C_\alpha)_{\alpha\in J}$ is a closed collection.

(b) By (a), there exist indexed locally finite collections $(V_\alpha)_{\alpha\in J}$ and $(W_\alpha)_{\alpha\in J}$ that cover X such that

$$\overline{W}_\alpha \subset V_\alpha \quad \text{and} \quad \bar{V}_\alpha \subset \mathcal{U}_\alpha \qquad \forall\, \alpha \in J.$$

Since X is normal, by the Urysohn Lemma for every $\alpha\in J$ there exists a continuous function

$$f_\alpha\colon X \longrightarrow [0,1] \qquad \text{s.t.} \qquad f_\alpha\big(\overline{W}_\alpha\big) = \{1\} \quad \text{and} \quad f_\alpha\big(X - V_\alpha\big) = \{0\}.$$

Since $(V_\alpha)_{\alpha\in J}$ is point finite, for every x

$$\Phi(x) = \sum_{\alpha\in J} f_\alpha(x)$$

is well-defined, being the sum of a finite collection of nonzero numbers. Since $(V_\alpha)_{\alpha\in J}$ is locally finite,

$$\Phi\colon X \longrightarrow \mathbb{R}$$

is continuous, since on all sufficiently small opens sets Φ is the sum of a finitely collection of nonzero functions. Since $(W_\alpha)_{\alpha\in J}$ covers X,

$$\Phi(x) \geq 1 \qquad \forall\, x \in X.$$

Thus, for every $\alpha\in J$, the function

$$\varphi_\alpha \equiv f_\alpha/\Phi : X \longrightarrow [0,1]$$

is continuous. Furthermore, for all $x\in X$

$$\sum_{\alpha\in J}\phi_\alpha(x) = \sum_{\alpha\in J}\big(f_\alpha(x)/\Phi(x)\big) = \Big(\sum_{\alpha\in J}(f_\alpha(x)\Big)/\Phi(x) = \Phi(x)/\Phi(x) = 1.$$