Topology MAT 530 Homework 5

1. Prove that the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by a continuous map $f : (X, x_0) \to (Y, y_0)$, is a group homomorphsm.

2. Prove that the fundamental group of the sphere S^n is trivial for $n > 1$.

3. Let A be a subspace of a topological space X and suppose that $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. Such map is called a *retraction* of X onto A. If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

4. Let G be a topological group with operation $\cdot$ and the identity element e. Let $\Omega(G, e)$ denote the set of all loops in G based at e. If $f, g \in \Omega(G, e)$, then define a loop $f \otimes g$ be the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

Show that $\Omega(G, e)$ is a group with respect to $\otimes$ Show that $\otimes$ induces a group operation on $\pi_1(G, e)$.

5. In the notation of Problem 4, prove that $\otimes$ coincides with the usual group operation on $\pi_1(G, e)$. Prove that $\pi_1(G, e)$ is commutative.

6. A subspace A of X is called a *deformation retract* of X if the identity self-map of X is homotopic to some retraction of X onto A so that during the homotopy every point of A remains fixed. Prove that if A is a path connected deformation retract of X, then X is also path connected and $\pi_1(X) = \pi_1(A)$.