Topology MAT 530 Homework 3

1. Give an example of a bounded metric space that is not totally bounded.

2. Prove that two metrics d and d' on the same set X define the same topology if and only if there exist constants $C > 0$ and $C' < \infty$ such that

$$Cd(x, y) \leq d'(x, y) \leq C'd'(x, y)$$

for all $x, y \in X$.

3. Prove that for almost all continuous functions $f : \mathbb{R} \to \mathbb{R}$ (in the sense of the Baire category, i.e. the set of such functions is the intersection of countably many open dense sets), the values of f at rational points are irrational.

4. Let X be a topological space and Y a metric space. Prove that the evaluation map

$$C(X, Y) \times X \to Y, \quad (f, x) \mapsto f(x)$$

is continuous.

5. Define a topology on the space of all maps $f : X \to Y$ such that the convergence in this topology is exactly the point-wise convergence.

6. Give an example of a complete metric space that admits a nested sequence of balls with empty intersection. *Hint:* consider the *Sierpinski metric* on the set of positive integers:

$$d(i, j) = 1 + 1/(i + j).$$

Prove that the sets $\{N, N + 1, N + 2, \dots\}$ are balls.